

Question 1 (3 marks)

$$\begin{aligned} & \int_2^3 (x+1)\sqrt{3-x} dx \\ &= \int_1^0 (4-u)\sqrt{u} \times -1 \frac{du}{dx} dx \\ &= \int_0^1 \left(4u^{\frac{1}{2}} - u^{\frac{3}{2}} \right) du \quad \text{(1 mark)} \\ &= \left[4u^{\frac{3}{2}} \times \frac{2}{3} - u^{\frac{5}{2}} \times \frac{2}{5} \right]_0^1 \quad \text{(1 mark)} \\ &= \left\{ \left(4 \times \frac{2}{3} - \frac{2}{5} \right) - (0-0) \right\} \\ &= \frac{8}{3} - \frac{2}{5} \\ &= \frac{40-6}{15} \\ &= \frac{34}{15} \end{aligned}$$

Let $u = 3 - x$

$$\frac{du}{dx} = -1$$

So $x = 3 - u$

and $x+1 = 4-u$

Also, $x=3, u=0$

and $x=2, u=1$

(1 mark)

Question 2 (3 marks)

$$s = \sqrt{4} = 2$$

sample size = 100

sample mean $\bar{x} = \frac{2150}{100} = 21.5$ mL **(1 mark)**

95% confidence interval $\approx \left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}} \right)$ (formula sheet)

$$= \left(21.5 - 2 \times \frac{2}{10}, 21.5 + 2 \times \frac{2}{10} \right) \quad \text{(1 mark)}$$

$$= (21.5 - 0.4, 21.5 + 0.4)$$

$$= (21.1, 21.9)$$

(1 mark)

Question 3 (4 marks)

a. $\vec{OA} = \underline{i} - 2\underline{j} + 2\underline{k}$

$$\vec{OB} = 2\underline{i} + \underline{j} - 3\underline{k}$$

$$\begin{aligned}\vec{AB} &= \vec{AO} + \vec{OB} \\ &= -\underline{i} + 2\underline{j} - 2\underline{k} + 2\underline{i} + \underline{j} - 3\underline{k} \\ &= \underline{i} + 3\underline{j} - 5\underline{k}\end{aligned}$$

(1 mark)

b.
$$\begin{aligned}\cos(\theta) &= \frac{\vec{AO} \cdot \vec{AB}}{|\vec{AO}| |\vec{AB}|} \\ &= \frac{15}{3\sqrt{35}} \\ &= \frac{5}{\sqrt{35}}\end{aligned}$$

(1 mark)

- c. Vectors $\vec{OA}$, $\vec{OB}$ and $\vec{OC}$ are linearly dependent if $\alpha\vec{OA} + \beta\vec{OB} = \vec{OC}$ where α and β are real numbers.

We require

$$\alpha(\underline{i} - 2\underline{j} + 2\underline{k}) + \beta(2\underline{i} + \underline{j} - 3\underline{k}) = m\underline{j} + 7\underline{k} \quad \textbf{(1 mark)}$$

So $\alpha + 2\beta = 0$

$$\alpha = -2\beta \quad \text{--- (1)}$$

and $-2\alpha + \beta = m \quad \text{--- (2)}$

and $2\alpha - 3\beta = 7 \quad \text{--- (3)}$

Put (1) into (3)

$$-4\beta - 3\beta = 7$$

$$\beta = -1$$

In (1) $\alpha = 2$

In (2) $-4 - 1 = m$

$$m = -5$$

(1 mark)

Question 4 (3 marks)

$$z^3 = -27i$$

Let $z = r\text{cis}(\theta)$

So $z^3 = r^3\text{cis}(3\theta)$

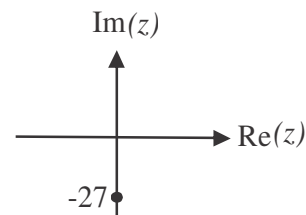
Also $-27i = 27\text{cis}\left(-\frac{\pi}{2}\right)$ **(1 mark)**

So $r^3\text{cis}(3\theta) = 27\text{cis}\left(-\frac{\pi}{2}\right)$

$$r^3 = 27, \quad r = 3$$

$$3\theta = -\frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\theta = -\frac{\pi}{6} + \frac{2k\pi}{3} \quad \textbf{(1 mark)}$$



S	A
T	C

For $k = 0$, $\theta = -\frac{\pi}{6}$ So $z = 3\text{cis}\left(-\frac{\pi}{6}\right)$ $= 3\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$	For $k = 1$, $\theta = -\frac{\pi}{6} + \frac{2\pi}{3}$ $= \frac{\pi}{2}$ So $z = 3\text{cis}\left(\frac{\pi}{2}\right)$ $= 3(0 + i)$ $= 3i$	For $k = 2$, $\theta = -\frac{\pi}{6} + \frac{4\pi}{3}$ $= \frac{7\pi}{6}$ So $z = 3\text{cis}\left(\frac{7\pi}{6}\right)$ $= 3\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$ $= -3\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$
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The three solutions are $z = 3\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$, $z = 3i$ and $z = -3\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$. **(1 mark)**

Question 5 (3 marks)

a. $|\underline{b}| = \sqrt{4+4+1} = 3$

$$\underline{\hat{b}} = \frac{1}{3}(2\underline{i} + 2\underline{j} - \underline{k}) \quad \textbf{(1 mark)}$$

b. scalar resolute of $\underline{a}$ in the direction of $\underline{b}$

$$= \underline{a} \cdot \underline{\hat{b}}$$

$$= \frac{1}{3}(5 \times 2 + 6 \times 2 + 4 \times -1) \quad \text{(using part a.)}$$

$$= 6 \quad \text{(Note, the answer should be a scalar!)} \quad \textbf{(1 mark)}$$

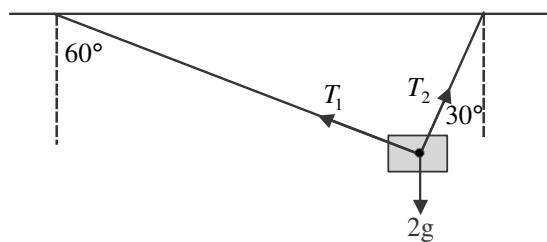
c. vector resolute of $\underline{a}$ perpendicular to $\underline{b}$

$$= \underline{a} - (\underline{a} \cdot \underline{\hat{b}})\underline{\hat{b}}$$

$$= 5\underline{i} + 6\underline{j} + 4\underline{k} - 6 \times \frac{1}{3}(2\underline{i} + 2\underline{j} - \underline{k}) \quad \text{(from parts a. and b.)}$$

$$= 5\underline{i} + 6\underline{j} + 4\underline{k} - 4\underline{i} - 4\underline{j} + 2\underline{k}$$

$$= \underline{i} + 2\underline{j} + 6\underline{k} \quad \text{(Note, the answer should be a vector!)} \quad \textbf{(1 mark)}$$

Question 6 (4 marks)**a.****(1 mark)****b.** Resolving horizontally:

$$T_1 \sin(60^\circ) = T_2 \sin(30^\circ)$$

$$\frac{\sqrt{3}}{2} T_1 = \frac{1}{2} T_2$$

$$T_2 = \sqrt{3} T_1 \quad - (A)$$

(1 mark)

Resolving vertically:

$$T_1 \cos(60^\circ) + T_2 \cos(30^\circ) = 2g$$

(1 mark)

$$\frac{1}{2} T_1 + \frac{\sqrt{3}}{2} T_2 = 2g$$

$$\frac{1}{2} T_1 + \frac{3}{2} T_1 = 2g$$

$$T_1 = g$$

$$\text{In (A) } T_2 = \sqrt{3}g$$

(1 mark)**Question 7** (4 marks)

$$\arccos(x) + \arcsin(x) = \frac{y}{2}$$

$$\frac{-1}{\sqrt{1-x^2}} + \frac{dy}{dx} \arcsin(x) + \frac{y}{\sqrt{1-x^2}} = \frac{1}{2} \frac{dy}{dx}$$

(1 mark)At the point $(0, \pi)$, we have

$$-1 + \frac{dy}{dx} \times 0 + \pi = \frac{1}{2} \frac{dy}{dx}$$

$$\frac{dy}{dx} = 2(\pi - 1)$$

(1 mark)Gradient of perpendicular line is therefore $\frac{-1}{2(\pi-1)}$ or $\frac{1}{2(1-\pi)}$.**(1 mark)**

Required equation is

$$y - \pi = \frac{1}{2(1-\pi)} x$$

$$y = \frac{x}{2(1-\pi)} + \pi$$

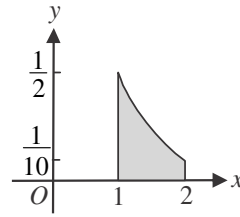
(1 mark)

Question 8 (4 marks)

Do a quick sketch of $y = \frac{1}{x(x^2+1)}$
in the vicinity of $x=1$ and $x=2$.

When $x=1$, $y = \frac{1}{2}$

When $x=2$, $y = \frac{1}{10}$



We know that between $x=1$ and $x=2$, the graph is above the x -axis.

$$\text{area} = \int_1^2 \frac{1}{x(x^2+1)} dx$$

(1 mark)

$$\begin{aligned} \text{Let } \frac{1}{x(x^2+1)} &\equiv \frac{A}{x} + \frac{Bx+C}{x^2+1} \\ &= \frac{A(x^2+1) + (Bx+C)x}{x(x^2+1)} \end{aligned}$$

True iff $1 \equiv A(x^2+1) + (Bx+C)x$

Put $x=0$, $1 = A$ so $A=1$

Put $x=1$, $1 = 2 + B + C$

$$B + C = -1 \quad - (1)$$

Put $x=-1$, $1 = 2 + B - C$

$$B - C = -1 \quad - (2)$$

$$(1)+(2) \quad 2B = -2$$

$$B = -1$$

In (1) $C = 0$

$$\text{So area} = \int_1^2 \left(\frac{1}{x} - \frac{x}{x^2+1} \right) dx$$

(1 mark)

$$= \left[\log_e |x| - \frac{1}{2} \log_e |x^2+1| \right]_1^2$$

(1 mark)

$$= \left(\log_e(2) - \frac{1}{2} \log_e(5) \right) - \left(\log_e(1) - \frac{1}{2} \log_e(2) \right)$$

$$= \log_e(2) - \log_e(\sqrt{5}) + \log_e(\sqrt{2})$$

$$= \log_e \left(\frac{2\sqrt{2}}{\sqrt{5}} \right)$$

$$= \log_e \left(\frac{2\sqrt{10}}{5} \right) \text{ square units.}$$

(1 mark)

Question 9 (5 marks)

$$\frac{\sqrt{x^2-1}}{x} \frac{dy}{dx} = 4 + y^2$$

$$\int \frac{1}{4+y^2} dy = \int \frac{x}{\sqrt{x^2-1}} dx \quad (\text{separation of variables}) \quad (1 \text{ mark})$$

$$\frac{1}{2} \int \frac{2}{4+y^2} dy = \int u^{-\frac{1}{2}} \times \frac{1}{2} \frac{du}{dx} dx \quad \left| \quad \begin{array}{l} \text{where } u = x^2 - 1 \\ \frac{du}{dx} = 2x \end{array} \right.$$

$$\frac{1}{2} \arctan\left(\frac{y}{2}\right) + c_1 = \frac{1}{2} \int u^{-\frac{1}{2}} du$$

$$\frac{1}{2} \arctan\left(\frac{y}{2}\right) + c_1 = \frac{1}{2} u^{\frac{1}{2}} \times 2 + c_2$$

$$\frac{1}{2} \arctan\left(\frac{y}{2}\right) = \sqrt{x^2-1} + c \quad \text{where } c = c_2 - c_1$$

(1 mark) – left side

(1 mark) – right side

Given $y(1) = 2$,

$$\frac{1}{2} \arctan(1) = c$$

$$c = \frac{1}{2} \times \frac{\pi}{4}$$

$$c = \frac{\pi}{8}$$

(1 mark)

$$\text{So } \frac{1}{2} \arctan\left(\frac{y}{2}\right) = \sqrt{x^2-1} + \frac{\pi}{8}.$$

$$\arctan\left(\frac{y}{2}\right) = 2\sqrt{x^2-1} + \frac{\pi}{4}$$

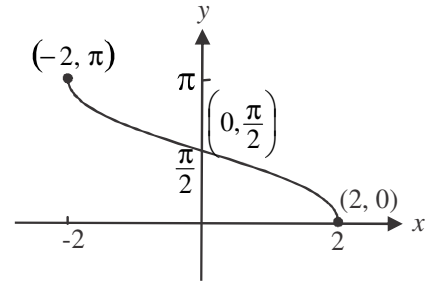
$$\tan\left(2\sqrt{x^2-1} + \frac{\pi}{4}\right) = \frac{y}{2}$$

$$\text{The solution is } y = 2 \tan\left(2\sqrt{x^2-1} + \frac{\pi}{4}\right).$$

(1 mark)

Question 10 (7 marks)

- a. The graph of $y = \arccos\left(\frac{x}{2}\right)$ is obtained when the graph of $y = \arccos(x)$ is dilated by a factor of 2 from the y-axis.



(1 mark) – correct shape

(1 mark) – correctly labelled intercept and endpoints

- b. i. $f(x) = \arccos\left(\frac{x}{2}\right)$
 Let $y = \arccos\left(\frac{x}{2}\right)$
 Swap x and y for inverse
 $x = \arccos\left(\frac{y}{2}\right)$

$$\cos(x) = \frac{y}{2}$$

$$y = 2\cos(x)$$

So $f^{-1}(x) = 2\cos(x)$ as required.

(1 mark)

- ii. From the graph in part a., $r_f = [0, \pi]$.
 Since $d_{f^{-1}} = r_f$
 then $d_{f^{-1}} = [0, \pi]$.

(1 mark)

- c. Do a quick sketch.
 The region to be rotated is shaded.

$$\text{volume} = \int_0^{\frac{\pi}{2}} \pi y^2 dx$$

$$= \pi \int_0^{\frac{\pi}{2}} 4\cos^2(x) dx \quad (1 \text{ mark})$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \frac{1}{2} (\cos(2x) + 1) dx$$

$$= 2\pi \left[\frac{1}{2} \sin(2x) + x \right]_0^{\frac{\pi}{2}}$$

$$= 2\pi \left\{ \left(\frac{1}{2} \times 0 + \frac{\pi}{2} \right) - \left(\frac{1}{2} \sin(0) + 0 \right) \right\}$$

$$= 2\pi \times \frac{\pi}{2}$$

$$= \pi^2 \text{ cubic units}$$

(1 mark)

(1 mark)

