



Units 3 and 4 Specialist Maths: Exam 1

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Any questions?

Check the Engage website for updated solutions, then email practiceexams@ee.org.au.

Marks allocated are indicated by a number in square brackets, for example, [1] indicates that the line is worth one mark.

Question 1a

The box is accelerating downward at 2.8 m/s^2 , in one dimensional motion. Taking the downward direction to be positive, the net force acting on the box can be written as:

$$F = ma = mg - N \quad [1]$$

(where mg is the force due to gravity, and N is the reaction force)

Rearranging:

$$\begin{aligned} m &= \frac{N}{(g - a)} \\ &= \frac{N}{9.8 - 2.8} \\ &= \frac{N}{7} \end{aligned}$$

The scale reads a value proportional to the reaction force N . Specifically, it outputs:

$$m': N = m'g \quad [1]$$

Hence:

$$\begin{aligned} m &= \frac{N}{7} \\ &= \frac{m'g}{7} \\ &= \frac{35g}{7} \\ &= 5g \\ &= 49 \text{ kg} \quad [1] \end{aligned}$$

Question 1b

Now taking the upwards direction to be positive:

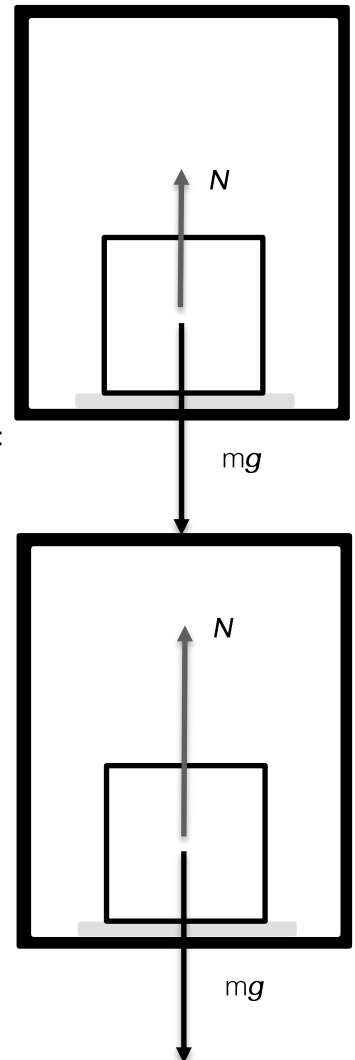
$$F = ma = N - mg \quad [1]$$

Rearranging:

$$\begin{aligned} N &= ma + mg \\ &= m(a + g) \end{aligned}$$

Thus the output on the scale is given:

$$\begin{aligned} m' &= \frac{N}{g} \\ &= \frac{m(a + g)}{g} \\ &= \frac{m \times \frac{3g}{2}}{g} \text{ since } a = 4.9 = \frac{g}{2} \\ &= \frac{3m}{2} \\ &= 73.5 \text{ kg} \quad [1] \end{aligned}$$



Question 2

Method 1:

This method is a variation on completing the square.

$$z^4 + 9z^2 + 64 = 0$$

$$z^4 + 16z^2 + 8^2 - 7z^2 = 0 \quad [1]$$

$$(z^2 + 8)^2 = 7z^2$$

$$z^2 + 8 = \pm z\sqrt{7} \quad [1]$$

$$\Rightarrow z^2 + z\sqrt{7} + 8 = 0 \text{ or } z^2 - z\sqrt{7} + 8 = 0 \quad [1]$$

Each equation can then be solved by the quadratic formula, yielding:

$$z = \frac{-\sqrt{7} \pm \sqrt{7 - 4 \times 1 \times 8}}{2 \times 1} \text{ or } z = \frac{\sqrt{7} \pm \sqrt{7 - 4 \times 1 \times 8}}{2 \times 1} = \frac{\sqrt{7} \pm 5i}{2} \quad [1]$$

$$z = \frac{-\sqrt{7} \pm 5i}{2} \text{ or } z = \frac{\sqrt{7} \pm 5i}{2}$$

So the solutions are:

$$z_1 = \frac{-\sqrt{7} - 5i}{2} \quad z_2 = \frac{-\sqrt{7} + 5i}{2} \quad z_3 = \frac{\sqrt{7} - 5i}{2} \quad z_4 = \frac{\sqrt{7} + 5i}{2} \quad [1]$$

Method 2:

$$z^4 + 9z^2 + 64 = 0$$

$$\text{Let } a = z^2$$

$$a^2 + 9a + 64 = 0$$

$$\left(a + \frac{9}{2}\right)^2 - \frac{81}{4} + \frac{256}{4} = 0$$

$$\left(a + \frac{9}{2}\right)^2 = -\frac{175}{4}$$

$$\therefore a = z^2 = \frac{-9 \pm 5\sqrt{-7}}{2} \quad [1]$$

$$\Rightarrow i^2 z^2 = \frac{9 \pm 5\sqrt{-7}}{2}$$

$$\Rightarrow iz = \pm \sqrt{\frac{9 \pm 5\sqrt{-7}}{2}}$$

$$= \pm \frac{\sqrt{18 \pm 10\sqrt{-7}}}{2} \quad [1]$$

The problem now is that the real and imaginary terms under the square root cannot be separated.

Suppose:

$$18 \pm 10\sqrt{-7} = (A \pm B)^2$$

$$9 + X \pm 10\sqrt{-7} + 9 - X = (A \pm B)^2$$

$$(\sqrt{9+X})^2 \pm 10\sqrt{-7} + (\sqrt{9-X})^2 = A^2 \pm 2AB + B^2$$

Now take:

$$A = \sqrt{9 + X}$$

$$B = \sqrt{9 - X}$$

And choose X such that:

$$2AB = 10\sqrt{-7}$$

$$\sqrt{(9 + X)(9 - X)} = 5\sqrt{-7}$$

$$\Rightarrow 81 - X^2 = 25(-7)$$

$$X^2 = 256$$

$$\Rightarrow X = 16 \text{ (sign is irrelevant because of symmetry in A, B)} \quad [1]$$

Hence:

$$18 \pm 10\sqrt{-7} = (\sqrt{9 + 16} \pm \sqrt{9 - 16})^2$$

$$= (5 \pm \sqrt{7}i)^2 \quad [1]$$

And:

$$iz = \pm \frac{\sqrt{18 \pm 10\sqrt{-7}}}{2}$$

$$= \pm \frac{\sqrt{(5 \pm \sqrt{7}i)^2}}{2}$$

$$= \pm \frac{(5 \pm \sqrt{7}i)}{2}$$

$$\Rightarrow z = -i \cdot iz$$

$$= \pm \frac{(\sqrt{7} \pm 5i)}{2}$$

Giving the solutions:

$$z_1 = \frac{-\sqrt{7} - 5i}{2} \quad z_2 = \frac{-\sqrt{7} + 5i}{2} \quad z_3 = \frac{\sqrt{7} - 5i}{2} \quad z_4 = \frac{\sqrt{7} + 5i}{2} \quad [1]$$

Question 3a

Method 1: By implicit differentiation:

Take the natural logarithm of both sides to eliminate the power.

$$y = x^x$$

$$\Rightarrow \ln(y) = x \ln(x) \quad \left[\frac{1}{2} \right]$$

$$\frac{d}{dy} \ln(y) \times \frac{dy}{dx} = \frac{d}{dx} (x \ln(x)) \quad \left[\frac{1}{2} \right]$$

$$\frac{1}{y} \frac{dy}{dx} = x \times \frac{1}{x} + \ln(x) = 1 + \ln(x)$$

$$\frac{dy}{dx} = y + y \times \ln(x) = x^x + x^x \cdot \ln(x) \quad [1]$$

Method 2:

$$\begin{aligned}
 y &= x^x \\
 &= e^{\ln(x^x)} \quad \left[\frac{1}{2} \right] \\
 &= e^u \text{ where } u = x \ln x \\
 \Rightarrow \frac{dy}{dx} &= \frac{d(e^u)}{du} \times \frac{d(x \ln x)}{dx} \quad \left[\frac{1}{2} \right] \\
 &= e^u \left(x \left(\frac{1}{x} \right) + (1) \ln(x) \right) \\
 &= x^x (1 + \ln(x)) \quad [1]
 \end{aligned}$$

Question 3b i

$$f(x) \in \mathbb{R}^- \cup \{0\} \quad [1]$$

This can be worked out from a rough sketch of $y = -x$ and $y = \arctan(x)$. They have opposite signs regardless of x .

Question 3b ii

By the product rule:

$$\frac{d}{dx}(x \cos^{-1}(x)) = \cos^{-1}(x) - \frac{x}{\sqrt{1-x^2}} \quad [1]$$

Question 3b iii

This is done through anti-differentiation by recognition. The use of the above answer is required for 3 marks. Any other method will be awarded 2 marks.

$$\text{Area} = \int_{\frac{1}{2}}^1 \cos^{-1} x \, dx \quad \left[\frac{1}{2} \right]$$

To evaluate the integral, utilising the answer to part (ii), one may integrate by recognition:

$$\begin{aligned}
 \frac{d}{dx}(x \cos^{-1}(x)) &= \cos^{-1}(x) - \frac{x}{\sqrt{1-x^2}} \\
 \Rightarrow \cos^{-1}(x) &= \frac{d}{dx}(x \cos^{-1}(x)) + \frac{x}{\sqrt{1-x^2}} \\
 \therefore \int \cos^{-1}(x) \, dx &= x \cos^{-1}(x) + \int \frac{x}{\sqrt{1-x^2}} \, dx \quad [1]
 \end{aligned}$$

So:

$$\begin{aligned}
 \text{Area} &= \int_{\frac{1}{2}}^1 \cos^{-1} x \, dx \\
 &= [x \cos^{-1}(x)]_{\left(\frac{1}{2}\right)}^1 + \int_{\left(\frac{1}{2}\right)}^1 \frac{x}{\sqrt{1-x^2}} \, dx \\
 &= -\frac{\pi}{6} + \int_{\left(\frac{1}{2}\right)}^1 \frac{x}{\sqrt{1-x^2}} \, dx
 \end{aligned}$$

The integral $\int \frac{x}{\sqrt{1-x^2}} \, dx$ can be evaluated by substitution:

$$\text{Let } u = 1 - x^2 \Rightarrow \frac{du}{dx} = -2x.$$

$$\text{Therefore, } x = -\frac{1}{2} \cdot \frac{du}{dx}.$$

Hence, we have:

$$\int \frac{x}{\sqrt{1-x^2}} dx = \int -\frac{1}{2\sqrt{u}} du = -\sqrt{u} \quad \left[\frac{1}{2} \right]$$

And so:

$$\begin{aligned} \text{Area} &= -\frac{\pi}{6} + \int_{(\frac{1}{2})}^1 \frac{x}{\sqrt{1-x^2}} dx \\ &= -\frac{\pi}{6} - \int_{(\frac{3}{4})}^0 \frac{1}{2\sqrt{u}} du \\ &= -\frac{\pi}{6} - [\sqrt{u}]_{(\frac{3}{4})}^0 \\ &= -\frac{\pi}{6} + \frac{\sqrt{3}}{2} \text{ square units} \quad [1] \end{aligned}$$

Question 4a

$$\begin{aligned} \overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} \\ &= \overrightarrow{AB} + \overrightarrow{AD} \text{ since } \overrightarrow{AD} = \overrightarrow{BC} \\ &= \mathbf{a} + \mathbf{d} \quad [1] \end{aligned}$$

Question 4b

Since M is midpoint of AC, we have:

$$\overrightarrow{AM} = \overrightarrow{MC} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{d}$$

To show that B, M, and D are collinear, it must be shown that $\overrightarrow{DM} = k\overrightarrow{DB}$, that is, that these vectors are parallel.

$$\begin{aligned} \overrightarrow{DB} &= \overrightarrow{DA} + \overrightarrow{AB} \\ &= \mathbf{a} - \mathbf{d} \end{aligned}$$

$$\begin{aligned} \overrightarrow{DM} &= \overrightarrow{DA} + \overrightarrow{AM} \\ &= -\mathbf{d} + \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{d} \\ &= \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{d} \\ &= \frac{1}{2}\overrightarrow{DB} \end{aligned}$$

(Completion up to this stage is awarded [1]. Note that the expression for $\overrightarrow{DM}$ cannot be obtained by claiming $\overrightarrow{DM}$ equal $\frac{1}{2}\overrightarrow{DB}$, since it has not been proven that M is midpoint of $\overrightarrow{DB}$.)

Hence, we have shown $\overrightarrow{DM} = \frac{1}{2}\overrightarrow{DB}$, as required [1].

Question 5

$$3\sin^2(\theta) + \cos^2(\theta) + \sqrt{3}\sin(2\theta) = 4$$

$$3\sin^2(\theta) + \cos^2(\theta) + 2\sqrt{3}\sin(\theta)\cos(\theta) = 4 \quad [1]$$

$$(\sqrt{3}\sin(\theta) + \cos(\theta))^2 = 4$$

$$\sqrt{3}\sin(\theta) + \cos(\theta) = 2 \quad (1) \text{ or } \sqrt{3}\sin(\theta) + \cos(\theta) = -2 \quad (2) \quad [1]$$

Solving (1):

$$\sqrt{3}\sin(\theta) + \cos(\theta) = 2$$

$$\frac{\sqrt{3}}{2}\sin(\theta) + \frac{1}{2}\cos(\theta) = 1$$

$$\cos\left(\frac{\pi}{6}\right)\sin(\theta) + \sin\left(\frac{\pi}{6}\right)\cos(\theta) = 1$$

$$\Rightarrow \sin\left(\frac{\pi}{6} + \theta\right) = 1 \text{ by the compound angle formula} \quad [1]$$

$$\Rightarrow \theta = \frac{\pi}{3} \quad [1]$$

Solving (2):

By the same method, obtain:

$$\sin\left(\frac{\pi}{6} + \theta\right) = -1$$

$$\Rightarrow \theta = \frac{-2\pi}{3} \quad [1]$$

Deduct [1] if only (I) or (II) is solved, failing to acknowledge ± 2 .

Alternatively:

$$3\sin^2(\theta) + \cos^2(\theta) + \sqrt{3}\sin(2\theta) = 4$$

$$\Rightarrow 2\sin^2(\theta) + \sqrt{3}\sin(2\theta) = 3 \text{ since } \sin^2(\theta) + \cos^2(\theta) = 1 \quad [1]$$

$$-(1 - 2\sin^2(\theta)) + \sqrt{3}\sin(2\theta) = 2$$

$$\Rightarrow -\cos(2\theta) + \sqrt{3}\sin(2\theta) = 2 \text{ since } 1 - 2\sin^2(\theta) = \cos(2\theta) \quad [1]$$

$$\left(-\frac{1}{2}\right)\cos(2\theta) + \left(\frac{\sqrt{3}}{2}\right)\sin(2\theta) = 1$$

$$\sin\left(\frac{-\pi}{6}\right)\cos(2\theta) + \cos\left(-\frac{\pi}{6}\right)\sin(2\theta) = 1$$

$$\Rightarrow \sin\left(2\theta - \frac{\pi}{6}\right) = 1 \text{ by the compound angle formula} \quad [1]$$

Hence:

$$2\theta - \frac{\pi}{6} = -\frac{3\pi}{2}, \frac{\pi}{2} \text{ only, since } \theta \in [-\pi, \pi] \Rightarrow \left(2\theta - \frac{\pi}{6}\right) \in \left[-\frac{11\pi}{6}, \frac{13\pi}{6}\right] \quad [1]$$

$$\Rightarrow \theta = -\frac{2\pi}{3}, \frac{\pi}{3} \quad [1]$$

Question 6

$$a = \frac{d}{dx} \left(\frac{1}{2} v^2 \right) = 2x^2 + 1 \quad [1]$$

Hence, using the initial condition:

$$\begin{aligned} \int_2^v d \left(\frac{v^2}{2} \right) &= \int_1^x (2x^2 + 1) dx \\ \frac{v^2}{2} - 2 &= \left[\frac{2x^3}{3} + x \right]_1^x = \left[\frac{2x^3}{3} + x \right]_1^0 \text{ at } x = 0 \\ \frac{v^2}{2} &= 2 - \frac{2}{3} - 1 \\ &= \frac{1}{3} \\ \therefore v &= \pm \frac{\sqrt{6}}{3} \text{ ms}^{-1} \end{aligned}$$

Question 7a i

$$E(4X - 2Y + 6) = 4E(X) - 2E(Y) + 6 = 44 \quad [1]$$

Question 7a ii

$$\begin{aligned} \text{Var}(4X - 2Y + 6) &= 4^2 \text{Var}(X) + 2^2 \text{Var}(Y) \text{ since the distributions are independent} \\ &= 100 \quad [1]. \end{aligned}$$

Hence, $sd = 10$.

Question 7b

The standard error for 95% confidence interval is $z \frac{s}{\sqrt{n}} = 1.96 \times \frac{\sqrt{256}}{\sqrt{64}} = 3.92 \quad [1]$

The 95% CI is $[173 - 3.92, 173 + 3.92] = [169.08, 176.92]$.

$176 \in 95\% \text{ CI}$. Hence, the prediction is **not rejected**. [1] (saying the prediction is **accepted** is not acceptable).

Let H be the height of a given individual.

$H \sim N(\mu, 256)$ where μ is the true mean of the population

$$\bar{H} = \frac{1}{n} \sum H_i \sim N \left(\mu, \frac{256}{n} \right) \text{ since it is a sum of identical independent normal random variables}$$

Thus,

$$Z = \frac{\bar{H} - \mu}{\left(\frac{16}{\sqrt{n}} \right)} \sim N(0,1)$$

$$\Pr(-1.96 < Z < 1.96) = 0.95$$

$$\Rightarrow Z \in (-1.96, 1.96) \text{ with 95\% confidence} \quad \left[\frac{1}{2} \right]$$

$$\Rightarrow \frac{\bar{H} - \mu}{\left(\frac{16}{\sqrt{n}} \right)} \in (-1.96, 1.96)$$

$$\Rightarrow \mu \in \left(\bar{H} - 1.96 \left(\frac{16}{\sqrt{n}} \right), \bar{H} + 1.96 \left(\frac{16}{\sqrt{n}} \right) \right)$$

Substituting the given numbers:

$$\mu \in \left(173 - 1.96 \left(\frac{16}{8} \right), 173 + 1.96 \left(\frac{16}{8} \right) \right)$$

$$\mu \in (169.08, 176.92) \text{ with 95\% confidence} \quad [1/2]$$

Since the prediction, 176cm is an element of the 95% confidence interval, the prediction cannot be rejected at the 0.05 significance level [1].

Question 8a

$$x = k \tan(t) - 1$$

$$\therefore \tan(t) = \frac{x+1}{k}$$

$$y = 3 \sec(t) - 8$$

$$\therefore \sec(t) = \frac{y+8}{3}$$

From the trigonometric identity:

$$\sec^2(t) - \tan^2(t) = 1$$

$$\Rightarrow \frac{(y+8)^2}{9} - \frac{(x+1)^2}{k^2} = 1$$

Question 8b

Implicit differentiation is used. The use of chain rule for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ is also accepted. Working needed.

$$\frac{2(y+8)}{9} \times \frac{dy}{dx} - \frac{2(x+1)}{k^2} = 0 \quad [1]$$

$$\therefore \frac{dy}{dx} = \frac{9(x+1)}{(y+8)k^2} \quad [1]$$

Question 8c

The normal has a gradient of $-\frac{2}{3}$. Hence, the gradient of the path at point $x = 2$ equals $\frac{3}{2}$, so we have at, $x = 2$:

$$\frac{dy}{dx} = \frac{9(x+1)}{(y+8)k^2} = \frac{9(2+1)}{(y+8)k^2} = \frac{3}{2} \quad (1)$$

The point at $x = 2$ also satisfies the equation of the normal:

$$3y + 2(2) = -2 \Rightarrow y = -2.$$

Sub $y = -2$ into (1), we have:

$$\frac{27}{6k^2} = \frac{3}{2}$$

$$\therefore k^2 = 3$$

$$\therefore k = \sqrt{3} \text{ (only, since } k > 0)$$