SPECIALIST MATHEMATICS

Units 3 and 4 – Written Examination 2

TSSM™
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2016 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: **C**

Explanation:

By CAS we get

$$\text{propFrac}\left(\frac{-2\cdot x^4+x^3-2\cdot x^2+5\cdot x-1}{x^3+x-2}\right) \qquad \frac{1}{x^3+x-2}-2\cdot x+1$$

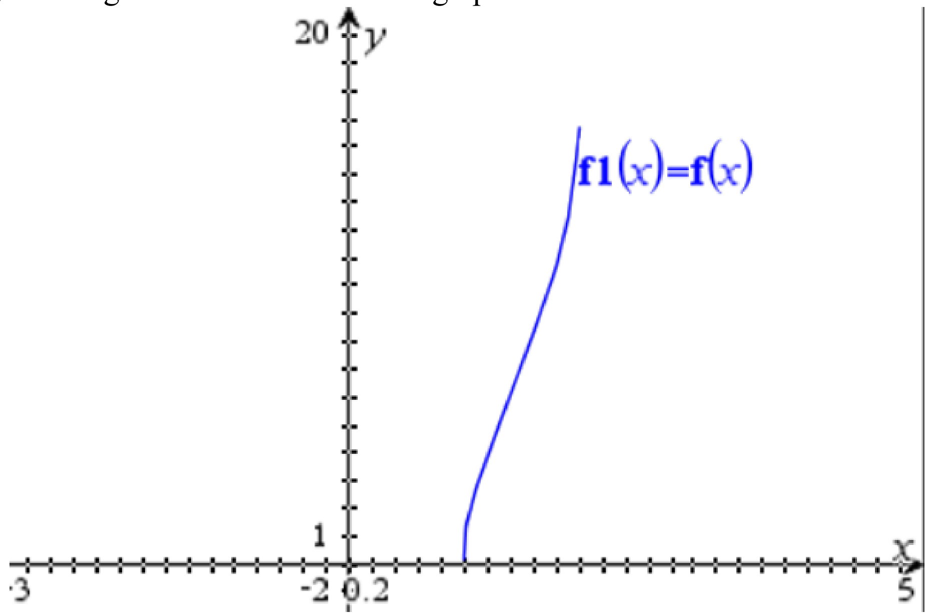
Let $x^3+x-2=0$. Then $x^3+x-2=(x-1)(x^2+x+2)=0 \Rightarrow x=1$.
Therefore the function has asymptotes $y=-2x+1$ and $x=1$.

Question 2

Answer: E

Explanation
Finding the domain: $-1 \le 3-2x \le 1 \Rightarrow 1 \le x \le 2$
The range of $3+x$ is $[4,5]$ and the range of $\arccos(3-2x)$ is $[0,\pi]$. Hence the range of $f(x)$ is $[0,5\pi]$.

Alternatively the range can be found from the graph in CAS



Question 3

Answer: A

Explanation:
Results can be found from CAS

Define $z=3\cdot\cos\left(\frac{\pi}{3}\right)\cdot i-3\cdot\sin\left(\frac{\pi}{3}\right)$	*Done*
$\lvert z\rvert$	3
$\text{angle}(z)$	$\frac{5\cdot\pi}{6}$

Question 4

Answer: C

Explanation:
z_1, z_2 and z_3 are the roots of the polynomial $P(z)=z^3+bz^2+cz+d \Rightarrow$
$P(z)=(z-z_1)(z-z_2)(z-z_3)=z^3-(z_1+z_2+z_3)z+(z_1z_2+z_2z_3+z_3z_1)z-z_1z_2z_3 \Rightarrow$
$z_1z_2+z_2z_3+z_3z_1=c,\ \ z_1z_2z_3=-d$

Question 5

Answer: D

Explanation:

$(5+5i)^n - \left(2\sqrt{3}+6i\right)^n = \left(5\sqrt{2}\right)^n cis\left(\frac{n\pi}{4}\right) - \left(4\sqrt{3}\right)^n cis\left(\frac{n\pi}{3}\right)$ is a positive number ⇒

Both $\frac{n\pi}{4}$ and $\frac{n\pi}{3}$ must be even multiple of π⇒ n is a multiple of 24. Hence D is the correct answer.

Question 6

Answer: B

Explanation:

Differentiate both sides of the equation $9x^2 + 25y^2 = 225$:

$18x + 50yy' = 0 \Rightarrow y' = -\frac{9x}{25y}$

When $x = 4,\ y = \pm\frac{9}{5}$. $\therefore y' = -\frac{9x}{25y} = \pm\frac{4}{5}$

Hence the product of the gradients is $-\frac{16}{25}$.

Question 7

Answer: E

Explanation:

$A = \frac{1}{2}bh, \frac{db}{dt} = 1$cm/min, $\frac{dh}{dt} = -2$ cm/min ⇒

$\frac{dA}{dt} = \frac{1}{2}\left(\frac{db}{dt}h + b\frac{dh}{dt}\right) = \frac{1}{2}(1 \times h + b \times (-2)) = \frac{1}{2}(18 - 10) = 4\ cm^2/\text{min}$

Question 8

Answer: D

Explanation:

Let $u = \cos(x)$. Then $dx = -\frac{du}{\sin(x)}$. $When\ x = \frac{\pi}{6}, u = \frac{\sqrt{3}}{2}; x = \frac{\pi}{2}, u = 0$

$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sin(x)}{\cos^2(x) + 4\cos(x) + 3} dx = -\int_{\frac{\sqrt{3}}{2}}^{0} \frac{1}{u^2 + 4u + 3} dx = -\frac{1}{2}\int_{\frac{\sqrt{3}}{2}}^{0}\left(\frac{1}{u+1} - \frac{1}{u+3}\right) dx = \frac{1}{2}\int_{0}^{\frac{\sqrt{3}}{2}}\left(\frac{1}{u+1} - \frac{1}{u+3}\right)dx$

Question 9

Answer: C

Explanation:
Using CAS

$$\text{Define } f(x)=(x-2)^{\frac{3}{2}} \qquad \textit{Done}$$

$$\text{Define } df(x)=\frac{d}{dx}(f(x)) \qquad \textit{Done}$$

$$\int_{3}^{10} \sqrt{1+(df(x))^2}\, dx \qquad 22.803$$

Question 10

Answer: C

Explanation:
Using CAS

$$\text{deSolve}(y''-7\cdot y'+10=0,x,y)$$

$$y=c2\cdot e^{7\cdot x}+\frac{10\cdot x}{7}+c1+\frac{10}{49}$$

$$\text{deSolve}(y''+6\cdot y'+9=0,x,y) \qquad y=c3\cdot e^{-6\cdot x}+c4-\frac{3\cdot x}{2}+\frac{1}{4}$$

$$\text{deSolve}(y''-7\cdot y'+10\cdot y=0,x,y) \qquad y=c6\cdot e^{5\cdot x}+c5\cdot e^{2\cdot x}$$

Question 11

Answer: A

Explanation:
Formula: $y_{i+1} = y_i + h \times y_i', \;\; h = 0.5$

i	x_i	y_i'	y_i
0	1	-4	3
1	1.5	2	$3 + 0.5 \times (-4) = 1$
2	2		$1 + 0.5 \times 2 = 2$

Question 12

Answer: A

Explanation:
A useful mathematical model for setting up differential equations of dynamic systems

$$\frac{dx}{dt} = R_{in} \times C_{in} - R_{out} \times C_{out}$$

where R_{in} and R_{out} are the flowing in and flowing out rate; C_{in} and C_{out} are the concentrations of the solutions which are flowing in and flowing out respectively.
Therefore

$$\frac{dx}{dt} = 6 \times 20 - 8 \times \frac{x}{60 + (8-6)t} = 120 - \frac{4x}{30 + t}$$

Question 13

Answer: E

Explanation:

Look at the slope field in CAS for each of the differential equations.

Question 14

Answer: E

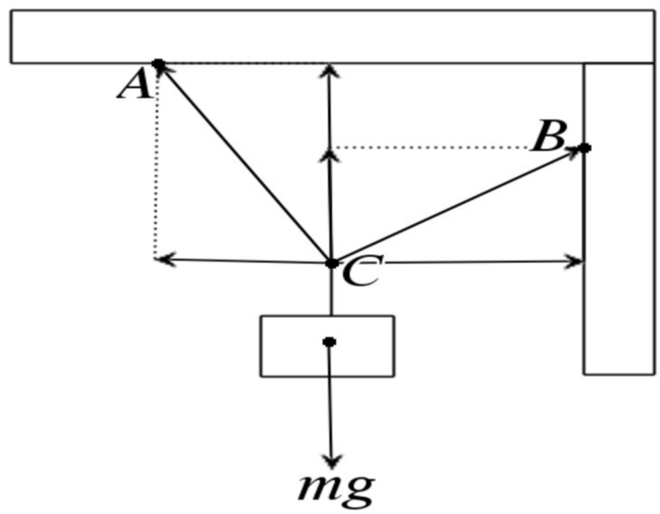
Explanation:
Using CAS

Define $v=\begin{bmatrix}4\\3\\2\end{bmatrix}$	*Done*
Define $u=\begin{bmatrix}3\\1\\6\end{bmatrix}$	*Done*
Define $vh=\text{unitV}(v)$	*Done*
$u-\text{dotP}(u,vh)\cdot vh$	$\begin{bmatrix}\frac{-21}{29}\\ \frac{-52}{29}\\ \frac{120}{29}\end{bmatrix}$

Question 15

Answer: A

Explanation:



All forces acting on the block and their components in the horizontal and vertical directions are labelled in the in the diagram. The motion equations are

$$F_{AC}\cos(30^\circ) + F_{BC}\cos(60^\circ) = mg,\ F_{AC}\sin(30^\circ) = F_{BC}\sin(60^\circ)$$

Solving by CAS

$$F_{AC} = \frac{\sqrt{3}}{2}mg,\ F_{BC} = \frac{1}{2}mg$$

Question 16

Answer: A

Explanation:

Substitute $s = 0, u = 29.4, a = -9.8$ into $s = ut + \frac{1}{2}at^2$ solve for t, $t = 0, 6$

Question 17

Answer: B

Explanation:

$\frac{dx}{dv} = \frac{dx}{dt} \times \frac{dt}{dv} = \frac{v}{v \times sec^2(2v)} = \cos^2(2v) \Rightarrow$

$x\left(\frac{\pi}{12}\right) - x(0) = \int_0^{\frac{\pi}{12}} \cos^2(2v)\,dv \Rightarrow x\left(\frac{\pi}{12}\right) = \int_0^{\frac{\pi}{12}} \cos^2(2v)\,dv + 5$

Question 18

Answer: D

Explanation:
Let X_1 and X_2 be the scores of the assignment and the test. Then

$$E(X_1) = 85, std(X_1) = 10, E(X_2) = 62, std(X_2) = 8$$

Therefore

$$E(0.4X_1 + 0.6X_2) = 0.4E(X_1) + 0.6E(X_2) = 0.4 \times 85 + 0.6 \times 62 = 71.2$$

$$std(0.4X_1 + 0.6X_2) = \sqrt{\left(0.4std(X_1)\right)^2 + \left(0.6std(X_2)\right)^2} = 6.2482$$

Question 19

Answer: C

Explanation:
Using CAS

zInterval 1.5,3.2,200,0.95: *stat.results*

$$\begin{bmatrix} \text{"Title"} & \text{"z Interval"} \\ \text{"CLower"} & 2.99211 \\ \text{"CUpper"} & 3.40789 \\ \text{"}\bar{x}\text{"} & 3.2 \\ \text{"ME"} & 0.207886 \\ \text{"n"} & 200. \\ \text{"}\sigma\text{"} & 1.5 \end{bmatrix}$$

Question 20

Answer: C

Explanation:

zTest 2,0.1,1.95,20,-1: *stat.results*

$$\begin{bmatrix} \text{"Title"} & \text{"z Test"} \\ \text{"Alternate Hyp"} & \text{"}\mu < \mu0\text{"} \\ \text{"z"} & -2.23607 \\ \text{"PVal"} & 0.012674 \\ \text{"}\bar{x}\text{"} & 1.95 \\ \text{"n"} & 20. \\ \text{"}\sigma\text{"} & 0.1 \end{bmatrix}$$

SECTION 2: Extended Response questions

Question 1 (9 marks)

a.

i. Let $X_1, X_2, X_3, X_4\ X_5, X_6$ be the study scores of English, Specialist Maths, Maths Methods, Chemistry, Physics and LOTE. Let X be the aggregate score.
Then

$$X = X_1 + X_2 + X_3 + X_4 + 0.1X_5 + 0.1\,X_6$$

Therefore

$E(X) = E(X_1) + E(X_2) + E(X_3) + E(X_4) + 0.1E(X_5) + 0.1E\,(X_6)$ 1 mark

$= 32.1 + 42.6 + 37.8 + 35.5 + 0.1 \times 34.1 + 0.1 \times 35$

$= 154.91$ 1 mark

ii.

$std(X) =$

$\sqrt{(std(X_1))^2 + (\,std(X_2))^2 + (std(X_3))^2 + (std(X_4))^2 + (0.1std(X_5))^2 + (0.1std\,(X_6))^2}$

$= \sqrt{7.1^2 + 8.6^2 + 7.6^2 + 8^2 + (0.1 \times 8.4)^2 + (0.1 \times 9)^2}$ 1 mark

≈ 15.74 1 mark

b. $E(\bar{X}) = E(X) = 154.91$ 1 mark

$std(\bar{X}) = \frac{std(X)}{\sqrt{n}} = \frac{15.74}{\sqrt{30}} = 2.87$ 1 mark

c. **i.** Using CAS for a z-test

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zTest 154.91,15.74,160,30,1: stat.results
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$$\begin{bmatrix} \text{"Title"} & \text{"z Test"} \\ \text{"Alternate Hyp"} & \text{"}\mu > \mu0\text{"} \\ \text{"z"} & 1.77122 \\ \text{"PVal"} & 0.038262 \\ \text{"}\bar{x}\text{"} & 160. \\ \text{"n"} & 30. \\ \text{"}\sigma\text{"} & 15.74 \end{bmatrix}$$

1 mark

Therefore the required p-value is 0.0383. 1 mark

ii. Since the p −value is less than 0.05, there is sufficient evidence to conclude that the students in this specific area perform better in those subjects.

1 mark

Question 2 (12 marks)

a. The determinant of the component matrix of vectors $\overrightarrow{OA}, \overrightarrow{OC}$ and $\overrightarrow{OG}$

$\det\left(\begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & -2 \\ 0 & 3 & 2 \end{bmatrix}\right) = 35 \neq 0.$ 1 mark

Therefore $\overrightarrow{OA}, \overrightarrow{OC}$ and $\overrightarrow{OG}$ are linearly independent. 1 mark

Alternatively, let

$$x\overrightarrow{OA}, +y\overrightarrow{OC} + z\overrightarrow{OG} = 0$$

Then

$$\begin{cases} 2x + 3y = 0 \\ -x + 2y + 3z = 0 \\ x - 2y + 2k = 0 \end{cases}$$ 1 mark

Solve by CAS, getting the unique solution $x = 0, y = 0$ and $z = 0$.

Hence $\overrightarrow{OA}, \overrightarrow{OC}$ and $\overrightarrow{OG}$ are linearly independent. 1 mark

b. Let (x, y, z) be the coordinates of B.

$OABC$ is a parallelogram⇒

$\overrightarrow{OC} = \overrightarrow{AB} = (x-2)\underset{\sim}{i} + (y+1)\underset{\sim}{j} + (z-1)\underset{\sim}{k} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k} \Rightarrow$ 1 mark

$x = 5, y = 1, z = -1.$ 1 mark

c. **i.** Let $\theta = \angle AOC$. Then $\cos(\theta) = \frac{\overrightarrow{OA}\cdot\overrightarrow{OC}}{|\overrightarrow{OA}||\overrightarrow{OC}|} = \frac{2}{\sqrt{6\times 17}} = \frac{2}{\sqrt{102}}$ 1 mark

Therefore $\sin(\theta) = \sqrt{1 - \left(\frac{2}{\sqrt{102}}\right)^2} = \frac{7}{\sqrt{51}}$ 1 mark

ii. The area of the parallelogram $OABC$

$Area = |\overrightarrow{OA}||\overrightarrow{OC}|\sin(\theta) = \sqrt{102} \times \frac{7}{\sqrt{51}} = 7\sqrt{2}.$ 1 mark

d. $\underset{\sim}{n} \cdot \overrightarrow{OA} = \left(\underset{\sim}{j} + \underset{\sim}{k}\right) \cdot \left(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) = -1 + 1 = 0$ 1 mark

$\underset{\sim}{n} \cdot \overrightarrow{OC} = \left(\underset{\sim}{j} + \underset{\sim}{k}\right) \cdot \left(3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}\right) = 2 - 2 = 0$ 1 mark

Therefore $\underset{\sim}{n}$ is perpendicular to vectors $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

e. The height of the parallelepiped *OABCGDEF* is the magnitude of the resolute of the vector $\overrightarrow{OG}$ in the direction of $\underset{\sim}{n}$. 1 mark

Therefore

$$h = \left|\overrightarrow{OG} \cdot \underset{\sim}{\hat{n}}\right| = \left(3\underset{\sim}{j} + 2\underset{\sim}{k}\right) \cdot \frac{1}{\sqrt{2}}\left(\underset{\sim}{j} + \underset{\sim}{k}\right) = \frac{5}{\sqrt{2}}$$ 1 mark

f. The volume $V = 7\sqrt{2} \times \frac{5}{\sqrt{2}} = 35$ 1 mark

Question 3 (12 marks)

a.

i. $32\left(-\frac{1}{2}\right)^5 + 1 = -1 + 1 = 0 \Rightarrow\ z = -\frac{1}{2}$ is a solution of the equation
$32z^5 + 1 = 0$ 1 mark

ii. $32z^5 + 1 = 0 \Rightarrow z^5 = \frac{1}{32}cis(\pi) \Rightarrow$

$$z = \frac{1}{2}cis\left(\frac{\pi}{5}\right), \frac{1}{2}cis\left(\frac{\pi}{5} + \frac{2\pi}{5}\right), \frac{1}{2}cis\left(\frac{\pi}{5} - \frac{2\pi}{5}\right), \frac{1}{2}cis\left(\frac{\pi}{5} + \frac{4\pi}{5}\right), \frac{1}{2}cis\left(\frac{\pi}{5} - \frac{4\pi}{5}\right)$$

$$\Rightarrow z = \frac{1}{2}cis\left(\frac{\pi}{5}\right), \frac{1}{2}cis\left(\frac{3\pi}{5}\right), \frac{1}{2}cis\left(-\frac{\pi}{5}\right), \frac{1}{2}cis(\pi), \frac{1}{2}cis\left(-\frac{3\pi}{5}\right).$$

2 marks

b.

i. $P\left(-\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2}\left(1 + 2\sqrt{3} + 2i\right) + m + i = 0 \Rightarrow$ 1 mark

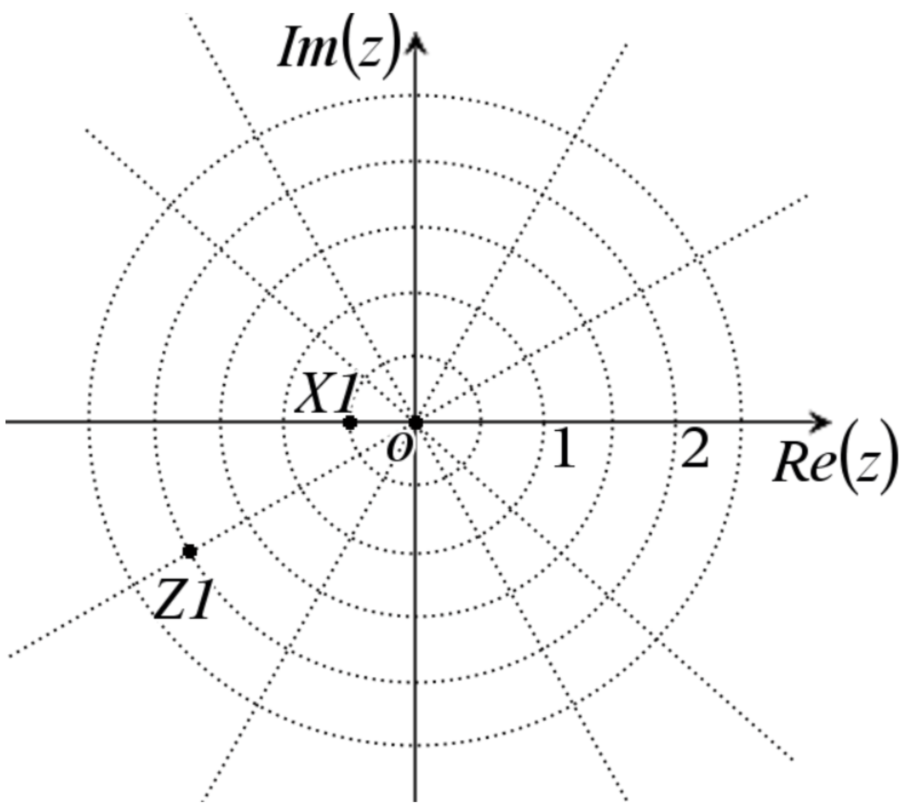
$\frac{1}{2} - \frac{1}{2} - \sqrt{3} - i + m + i = 0 \Rightarrow m = \sqrt{3}$ 1 mark

ii. $P(z) = 2z^2 + \left(1 + 2\sqrt{3} + 2i\right)z + \sqrt{3} + i = (2z+1)[z + \left(\sqrt{3} + i\right)]$

1 mark

$\therefore z = -\left(\sqrt{3} + i\right)$ 1 mark

c. The positions of X_1 and Z_1 are labelled in the diagram below.



1 mark for each point

d. Note that any complex number can be written as a vector.

Let α be the angle between Z_1X_1 and the positive direction of the real axis. Let β be the angle between Z_1O and the positive direction of the real axis.

Then $\theta = \alpha - \beta$.

$\overrightarrow{Z_1O} = \sqrt{3}\underset{\sim}{i} + \underset{\sim}{j} \Rightarrow \tan(\beta) = \frac{1}{\sqrt{3}}$ 1 mark

$\overrightarrow{Z_1X_1} = \overrightarrow{OX_1} - \overrightarrow{OZ_1} = -\frac{1}{2}\underset{\sim}{i} + \left(\sqrt{3}\underset{\sim}{i} + \underset{\sim}{j}\right) = (\sqrt{3} - \frac{1}{2})\underset{\sim}{i} + \underset{\sim}{j}$

$\Rightarrow \tan(\alpha) = \frac{1}{\sqrt{3}-\frac{1}{2}} = \frac{4\sqrt{3}+2}{11}$ 1 mark

Therefore

$$\tan(\theta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

$$= \frac{\frac{4\sqrt{3}+2}{11} - \frac{1}{\sqrt{3}}}{1 + \frac{4\sqrt{3}+2}{11} \times \frac{1}{\sqrt{3}}}$$

$$= \frac{8+\sqrt{3}}{61}$$ 1 mark

Question 4 (13 marks)

Let $x(t) = 2 + e^t, y(t) = t^2 + t$

a. $x = 2 + e^t \Rightarrow t = \log_e(x-2) \Rightarrow$ 1 mark

$y = (\log_e(x-2))^2 + \log_e(x-2)$ 1 mark

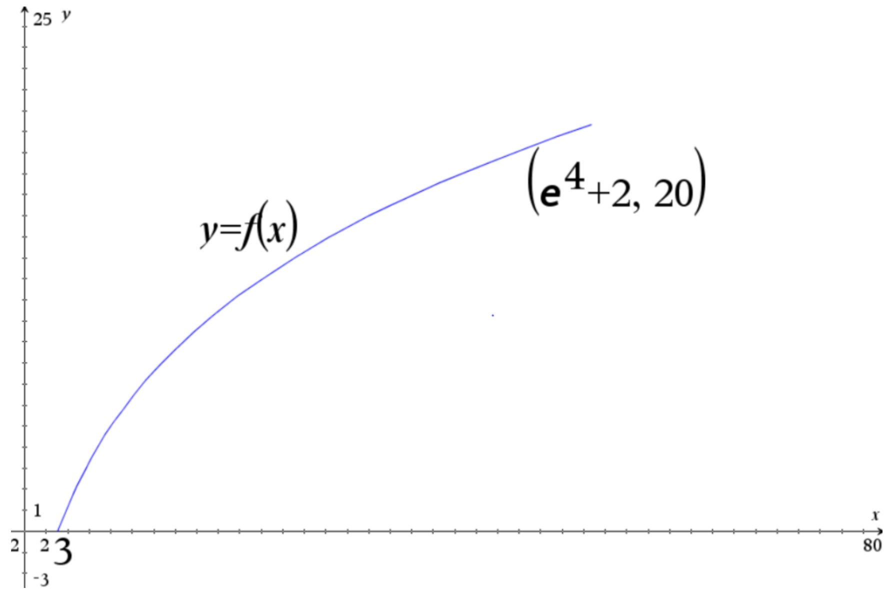
b. Arc length formula $L = \int_{t_1}^{t_2} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$

$x'(t) = e^t, \quad y'(t) = 2t + 1$ 1 mark

$L = \int_0^4 \sqrt{(e^t)^2 + (2t+1)^2}\, dt$ 1 mark

≈ 58.35

c. The graph is shown below



Correct shape 1 mark
Correct coordinates of end points 1 mark

d. Use CAS to solve

$$\text{deSolve}\left((x-2)\cdot y'-2\cdot \ln(x-2)-1=0,x,y\right)$$

$$y=(\ln(x-2))^2+\ln(x-2)+\boldsymbol{c1}+\frac{1}{4}$$

1 mark

Substitute $x = 3, y = 0$,

$0 = (\log_e(3-2))^2 + \log_e(3-2) + c_1 + \frac{1}{4}$

$c_1 = -\dfrac{1}{4}$

Therefore $y = (\log_e(x-2))^2 + \log_e(x-2)$ is the solution of the differential equation

$$(x-2)\frac{dy}{dx} - 2\log_e(x-2) - 1 = 0$$

at $(3, 0)$.

1mark

e. **i.**Using CAS

$$\text{Define } g(x)=(x-2)\cdot(\ln(x-2))^2-(x-2)\cdot\ln(x-2) \qquad \textit{Done}$$

$$\frac{d}{dx}(g(x)) \qquad (\ln(x-2))^2+\ln(x-2)-1$$

1mark

ii. i.e., $\frac{d}{dx}\big(g(x)\big) = f(x) - 1$.

The required area

$$A = \int_3^6 f(x)\,dx = \int_3^6 \left(\tfrac{d}{dx}\big(g(x)\big) + 1\right) dx$$

1mark

$$= [g(x) + x]_3^6 = g(6) - g(3) + 3$$

1mark

iii. The required volume

$V = \pi \int_3^6 \big(f(x)\big)^2\,dx = \pi \int_3^6 ((\log_e(x-2))^2 + \log_e(x-2))^2\,dx$ 1 mark

$= 36.56$ 1 mark

Question 5 (14 marks)

a. $\underset{\sim}{F} + \underset{\sim}{R} = m\underset{\sim}{a} \Rightarrow \underset{\sim}{a} = \frac{\underset{\sim}{F}+\underset{\sim}{R}}{m} = \frac{8\underset{\sim}{i}+6\underset{\sim}{j}}{2} = 4\underset{\sim}{i} + 3\underset{\sim}{j}.$

1 mark

b. $\underset{\sim}{v}(t) = \int_0^t 4\underset{\sim}{i} + 3\underset{\sim}{j}\,dt = 4t\underset{\sim}{i} + 3t\underset{\sim}{j}$ 1 mark

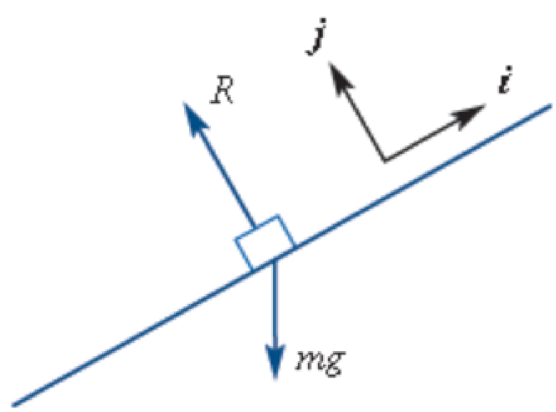
$\underset{\sim}{x}(t) = \int_0^t 4t\underset{\sim}{i} + 3t\underset{\sim}{j}\,dt = 2t^2\underset{\sim}{i} + \frac{3}{2}t^2\underset{\sim}{j}$ 1 mark

c. Solve $\left|\underset{\sim}{x}(t)\right| = \sqrt{(2t^2)^2 + \left(\frac{3}{2}t^2\right)^2} = 30$ for t, $t = 2\sqrt{3}$. 1 mark

The speed $u = \left|\underset{\sim}{v}\left(2\sqrt{3}\right)\right| = 10\sqrt{3} = 17.32$ m/s

1 mark

d. i.



2 marks

ii. $v^2 - u^2 = 2as \Rightarrow v^2 = u^2 + 2as = \left(10\sqrt{3}\right)^2 - 2 \times \frac{1}{2} \times 9.8 \times 24$ 1 mark

$\therefore v = 8.05\ sm^{-1}$ 1 mark

e. The height of the ramp end from the ground is $h = 24\sin(30°) = 12$ m.
The vertical component of the speed: $8.05\sin(30°) = 4.025\ ms^{-1}$. 1 mark
Substitute $u = -4.025, a = 9.8$ and $s = 12$ into $v^2 - u^2 = 2as$.
Solve for v, $v = \sqrt{(-4.025)^2 + 2 * 9.8 * 12} = 15.86\ ms^{-1}$ 1 mark

f. The resultant force $F = mg - R = 2 \times 9.8 - \frac{1}{4}v - \frac{3}{4} \times 9.8 - 4.25 = 8 - \frac{1}{4}v$.
The acceleration $a = \frac{dv}{dt} = \frac{F}{m} = 4 - \frac{1}{8}v$

$\therefore \quad \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = \frac{vdv}{dx} = \frac{32-v}{8}$ 1 mark

Solve for v by CAS,

$x = -256\log_e|32 - v| - 8v + c$ 1 mark

Substitute $v = 15.86$, $x = 0$ solve for c, $c = 838.89$
Hence,

$x = 838.89 - 256\log_e|32 - v| - 8v$ 1 mark

