

SPECIALIST MATHEMATICS

Units 3 and 4 – Written Examination 1



2016 Trial Examination

SOLUTIONS

Question 1 (4 marks)

Denote the mass and speed of a toy car by m and v separately. Then
 $m \sim N(200, 3^2)$, $v \sim N(80, 4^2)$

and

Exerted force $P = 4m$

Resistant force $R = 3v$

The resultant force $F = 4m - 3v$

a.

$$E(F) = 4E(m) - 3E(R) = 4 \times 200 - 3 \times 80 = 560 \text{ N}$$

1 mark

$$\text{Var}(F) = 4^2 \text{Var}(m) + (-3)^2 \text{Var}(v) = 288$$

1 mark

$$\sigma(F) = \sqrt{288} = 12\sqrt{2}$$

b.

$$\text{The standard deviation of the sample mean} = \frac{\sigma}{\sqrt{n}} = \frac{12\sqrt{2}}{11}.$$

1 mark

Question 2 (3 marks)

P is the intersection of the lines AD and CB $\Rightarrow$

$$\overrightarrow{CP} = x\overrightarrow{CB} \text{ and } \overrightarrow{DP} = y\overrightarrow{DA}$$

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OC} + \overrightarrow{CP} \\ &= \overrightarrow{OC} + x\overrightarrow{CB} \\ &= \overrightarrow{OC} + x\overrightarrow{CO} + x\overrightarrow{OB} \\ &= -3\tilde{a} + 3x\tilde{a} + x\tilde{b} \\ &= (3x - 3)\tilde{a} + x\tilde{b}\end{aligned}$$

Also

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OD} + \overrightarrow{DP} \\ &= \overrightarrow{OD} + y\overrightarrow{DA} \\ &= \overrightarrow{OD} + y\overrightarrow{DO} + y\overrightarrow{OA} \\ &= 4\tilde{b} - 4y\tilde{b} + y\tilde{a}\end{aligned}$$

1 mark

Therefore

$$3x - 3 = y \text{ and } x = 4 - 4y$$

1 mark

Solve them simultaneously, $x = \frac{16}{13}$, $y = \frac{9}{13}$.

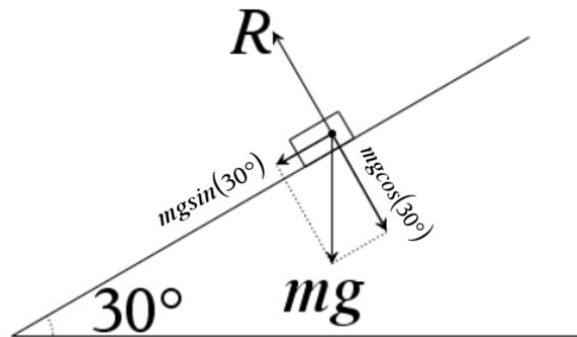
Hence

$$\overrightarrow{OP} = \frac{9}{13}\tilde{a} + \frac{16}{13}\tilde{b}$$

1 mark

Question 3 (3marks)

All forces are labelled in the diagram below.



The equations of the motion are

$$R = mg\cos(30^\circ), \quad mg\sin(30^\circ) = ma$$

1 mark

Therefore the acceleration

$$a = \frac{1}{2}g$$

1 mark

The travelled distance in $\frac{10}{\sqrt{g}}$ seconds is

$$s = ut + \frac{1}{2}at^2 = \frac{1}{2} \times \frac{1}{2}g \times \left(\frac{10}{\sqrt{g}}\right)^2 = 25 \text{ m}$$

1 mark

Question 4 (3 marks)

The velocity

$$\vec{v}(t) = \frac{\sqrt{3}}{4}t\vec{i} + \left(t^2 - \frac{7}{2}\right)\vec{j} + \frac{\sqrt{3}}{2}t\vec{k}$$

$$\vec{v}(2) = \frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j} + \sqrt{3}\vec{k}$$

1 mark

Then

$$\vec{v}(2) \cdot \vec{k} = \sqrt{3} \quad \text{and} \quad |\vec{v}(2)| = 2$$

1 mark

If θ is the angle between $\vec{v}(2)$ and $\vec{k}$ then

$$\cos(\theta) = \frac{\vec{v}(2) \cdot \vec{k}}{|\vec{v}(2)|} = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30^\circ$$

Therefore the angle between $\vec{v}(2)$ and the horizontal direction is $90^\circ - 30^\circ = 60^\circ$.

1 mark

Question 5 (6 marks)**a.** $z = -2 + i$ is a solution $\Rightarrow z = -2 - i$ is also a solution.

1 mark

Hence

$$z^4 + 5z^3 + az^2 + bz + c = (z + 3)(z + 2 - i)(z + 2 + i)(z - r)$$

1 mark

for a real number $r \in \mathbb{R}$.When expanding the right hand side the coefficient of z^3 is

$$(3 + 2 - i + 2 + i - r) \text{ or } (7 - r)$$

1 mark

Hence

$$3 + 2 - i + 2 + i - r = 5$$

Therefore

$$r = 2$$

1 mark

b. $z^4 + 5z^3 + az^2 + bz + c$

$$= (z + 3)(z + 2 - i)(z + 2 + i)(z - 2)$$

1 mark

$$= (z + 3)(z^2 + 4z + 5)(z - 2)$$

$$= (z^2 + z - 6)(z^2 + 4z + 5)$$

$$= z^4 + 5z^3 + 3z^2 - 19z - 30$$

Therefore

$$a = 3, \quad b = -19, \quad c = -30$$

1 mark

Question 6 (4 marks)

Let V be the volume of the water.

$V = \frac{1}{3}\pi r^2 h$ where r is the radius of water surface.

$$\frac{r}{4} = \frac{h}{8} \Rightarrow r = \frac{1}{2}h$$

$$\therefore V = \frac{1}{12}\pi h^3 \Rightarrow \frac{dV}{dh} = \frac{1}{4}\pi h^2$$

1 mark

$$\Rightarrow \frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = -\frac{\pi(h^2-1)}{8} \times \frac{4}{\pi h^2} = -\frac{h^2-1}{2h^2}$$

1 mark

$$\Rightarrow t = -\int_8^2 \frac{2h^2}{h^2-1} dh$$

1 mark

$$= -\int_8^2 \left(2 + \frac{1}{h-1} - \frac{1}{h+1}\right) dh$$

$$= -\left[2h + \ln\left(\frac{h-1}{h+1}\right)\right]_8^2$$

$$= 12 + \ln\left(\frac{7}{3}\right)$$

1 mark

Question 7 (3+3=6 marks)

a. $\frac{dy}{dx} = \frac{e^x - e^{-x}}{2}$

1 mark

The required arc length

$$L = \int_{-1}^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_{-1}^1 \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2} dx$$

1 mark

$$= \int_{-1}^1 \sqrt{1 + \frac{e^{2x} - 2 + e^{-2x}}{4}} dx$$

$$= \int_{-1}^1 \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}} dx$$

$$= \int_{-1}^1 \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2} dx$$

$$= \int_{-1}^1 \frac{e^x + e^{-x}}{2} dx$$

$$= \left[\frac{e^x - e^{-x}}{2}\right]_{-1}^1 = e - \frac{1}{e}$$

1 mark

b. The required volume

$$V = \pi \int_{-1}^1 (f(x))^2 dx = \pi \int_{-1}^1 \left(\frac{e^x + e^{-x}}{2}\right)^2 dx = \pi \int_{-1}^1 \frac{e^{2x} + 2 + e^{-2x}}{4} dx$$

1 mark

$$= \pi \left[\frac{e^{2x} + 4x - e^{-2x}}{8}\right]_{-1}^1$$

1 mark

$$= \frac{\pi(e^2 + 4 - e^{-2})}{4}$$

1 mark

Question 8 (6 marks)

a. $\arctan(x)$ is an increasing only function $\Rightarrow$

$$x^2 < 2x + 15 \Rightarrow$$

$$(x + 3)(x - 5) < 0 \Rightarrow$$

$$-3 < x < 5$$

1 mark

1 mark

1 mark

b. $\sqrt{3} \cos(2x) + \sin(2x) = \sqrt{2}, 0 \leq x \leq \pi \Rightarrow$

$$2\left(\frac{\sqrt{3}}{2} \cos(2x) + \frac{1}{2} \sin(2x)\right) = \sqrt{2}, 0 \leq x \leq \pi \Rightarrow$$

1 mark

$$2\left(\cos\left(\frac{\pi}{6}\right) \cos(2x) + \sin\left(\frac{\pi}{6}\right) \sin(2x)\right) = \sqrt{2}, 0 \leq x \leq \pi \Rightarrow$$

$$\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}, -\frac{\pi}{6} \leq 2x - \frac{\pi}{6} \leq 2\pi - \frac{\pi}{6} \Rightarrow$$

1 mark

$$2x - \frac{\pi}{6} = \frac{\pi}{4}, 2\pi - \frac{\pi}{6} \Rightarrow$$

$$x = \frac{5\pi}{12}, \frac{23\pi}{12}$$

1 mark

Question 9 (5 marks)

$$\frac{dy}{dx} = \frac{1+y^2}{(2+e^x)y} \Rightarrow \frac{y}{1+y^2} dy = \frac{1}{2+e^x} dx \Rightarrow$$

$$\int \frac{y}{1+y^2} dy = \int \frac{1}{2+e^x} dx \Rightarrow$$

1 mark

$$\frac{1}{2} \int \frac{1}{1+y^2} d(1+y^2) = \int \frac{e^{-x}}{2e^{-x}+1} dx \Rightarrow$$

$$\frac{1}{2} \ln(1+y^2) = -\frac{1}{2} \int \frac{1}{2e^{-x}+1} d(2e^{-x}+1) \Rightarrow$$

1 mark

$$\frac{1}{2} \ln(1+y^2) = -\frac{1}{2} \ln(2e^{-x}+1) + c$$

1 mark

Substitute $x = 0, y = 0$ and solve for $c, c = \frac{1}{2} \ln(3)$

Therefore

$$\ln(1+y^2) = \ln\left(\frac{3}{2e^{-x}+1}\right) = \ln\left(\frac{3e^x}{2+e^x}\right)$$

1 mark

Hence

$$y = \sqrt{\frac{2e^x-2}{e^x+2}}$$

1 mark