

Trial Examination 2016

VCE Specialist Mathematics Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E

11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E

Question 1 **C**

From $y = \frac{x}{2} - \frac{3}{x}$, the graph has two straight-line asymptotes, namely $x = 0$ and $y = \frac{x}{2}$.

$$\frac{dy}{dx} = \frac{x^2 + 6}{2x^2} \left(\frac{dy}{dx} = \frac{1}{2} + \frac{3}{x^2} \right)$$

The equation $\frac{dy}{dx} = 0$ has no real solutions and so the graph has no stationary points.

Question 2 **E**

Consider the graphs of $y_1 = |4x - 1|$ and $y_2 = \left| 4\left(\frac{x}{4}\right) - 1 \right|$.

The graph of y_1 has been dilated by a factor of 4 from the y-axis to form the graph of y_2 .

For example, the point (1, 3) is transformed to (4, 3).

Question 3 **E**

The turning points of the graph of $y = \sec(x)$ occur at $x = 0, \pi$ and 2π for $0 \leq x \leq 2\pi$.

Hence the turning points for the graph of $y = \sec\left(x + \frac{\pi}{3}\right)$ occur at $x = \pi - \frac{\pi}{3}$ and $x = 2\pi - \frac{\pi}{3}$.

That is, $x = \frac{2\pi}{3}, \frac{5\pi}{3}$.

Question 4 **B**

The repeated factor $(x - 1)^2$ must be re-expressed as the sum of two fractions. Hence we can disregard options **C**, **D** and **E**.

The quadratic factor $(x^2 + 16)$ must be re-expressed as a fraction with a linear factor in the numerator. Hence we can disregard option **A**.

Question 5 **A**

$$uv = 6a \operatorname{cis}\left(\frac{\pi}{5} + b\right)$$

Equating the two expressions for uv we obtain $6a \operatorname{cis}\left(\frac{\pi}{5} + b\right) = 48 \operatorname{cis}\left(\frac{\pi}{12}\right)$.

$$6a = 48 \Rightarrow a = 8 \text{ and } \frac{\pi}{5} + b = \frac{\pi}{12} \Rightarrow -\frac{7\pi}{60}$$

Question 6 **C**

$$i\bar{z} - iz = 2$$

Substituting for $\bar{z}$ and z we obtain $i(x - yi) - i(x + yi) = 2$.

$$-2i^2y = 2 \Rightarrow y = 1$$

Question 7 **C**

Options **A**, **B**, **D** and **E** all represent circles. Option **C** represents an ellipse with foci (2, 0) and (−2, 0).

Question 8 B

$$y^2 - xy = -4$$

$$2y \frac{dy}{dx} - \left(y + x \frac{dy}{dx} \right) = 0$$

$$(2y - x) \frac{dy}{dx} = y$$

$$\frac{dy}{dx} = \frac{y}{2y - x} \text{ (this can be obtained directly with CAS)}$$

$$\frac{dy}{dx} = 0 \Rightarrow y = 0$$

If $y^2 - xy = -4$, $y = 0 \Rightarrow \text{LHS} = 0$.

So $\text{LHS} \neq \text{RHS}$ and there are no stationary points.

For a tangent parallel to the y -axis to exist, we require $2y - x = 0$.

Substituting $y = \frac{x}{2}$ into $y^2 - xy = -4$ gives $\left(\frac{x}{2}\right)^2 - x\left(\frac{x}{2}\right) = -4$.

Solving $\left(\frac{x}{2}\right)^2 - x\left(\frac{x}{2}\right) = -4$ for x gives $x = \pm 4$; that is, 2 tangents.

Question 9 B

$$f(x) = 3x^5 - 5x^3$$

So $f'(x) = 15x^4 - 15x^2$ and $f''(x) = 60x^3 - 30x$.

$$f''(x) = 30x(2x^2 - 1)$$

The graph of f is concave up for values of x such that $f''(x) > 0$.

Solving $f''(x) > 0$ for x gives $-\frac{\sqrt{2}}{2} < x < 0$ or $x > \frac{\sqrt{2}}{2}$.

Question 10 C

Solving $\sqrt{6x+4} = 2x$ for x with $x > 0$ gives $x = 2$.

$$\begin{aligned} V &= \pi \int_0^2 (\sqrt{6x+4})^2 - (2x)^2 dx \\ &= \pi \int_0^2 (6x+4-4x^2) dx \end{aligned}$$

Question 11 D

$$\frac{dV}{dt} = -2$$

$$V = \frac{1}{2} \times 2 \times 3 \times 5$$

$$= 15 \text{ (cubic metres)}$$

So when the trough is one-quarter full, $V = 3.75$ (cubic metres).

$$\frac{b}{h} = \frac{2}{3} \Rightarrow b = \frac{2h}{3}$$

Substituting $b = \frac{2h}{3}$ into $V = \frac{5bh}{2}$ gives $V = \frac{5h^2}{3}$.

Solving $\frac{15}{4} = \frac{5h^2}{3}$ for h with $h > 0$ gives $h = \frac{3}{2}$.

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$-2 = \frac{10}{3} \times \frac{3}{2} \times \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -\frac{2}{5} \text{ (metres/minute)}$$

Question 12 E

Let the arc length be L .

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_a^b \sqrt{1 + (\sec^2(x))^2} dx \text{ as } \frac{dy}{dx} = \sec^2(x)$$

$$= \int_a^b \sqrt{1 + \sec^4(x)} dx$$

Question 13 D

When $t = 0$, $T = 15$.

The differential equation is $\frac{dT}{dt} = -k(T - 3)$.

Question 14 A

Let d be the distance travelled by the particle.

$$d = \frac{1}{2}(2 + 3)(10) + \frac{1}{2}(1)(10)$$

So the distance travelled is 30 metres.

Let s be the displacement of the particle from its starting point.

$$s = \frac{1}{2}(2 + 3)(10) - \frac{1}{2}(1)(10)$$

So the displacement from the starting position is 20 metres.

Question 15 D

A unit vector in the direction of $-\underline{\hat{i}} - 4\underline{\hat{j}} + 5\underline{\hat{k}}$ is $\frac{1}{\sqrt{42}}(-\underline{\hat{i}} - 4\underline{\hat{j}} + 5\underline{\hat{k}})$.

Hence a vector with a magnitude of 7 parallel to $-\underline{\hat{i}} - 4\underline{\hat{j}} + 5\underline{\hat{k}}$ is $\frac{7}{\sqrt{42}}(-\underline{\hat{i}} - 4\underline{\hat{j}} + 5\underline{\hat{k}})$.

As $\frac{7}{\sqrt{42}} = \frac{\sqrt{42}}{6}$, the required vector is $\frac{\sqrt{42}}{6}(-\underline{\hat{i}} - 4\underline{\hat{j}} + 5\underline{\hat{k}})$.

Question 16 B

$$\overrightarrow{AM} = \underline{\hat{i}} + \frac{1}{2}\underline{\hat{j}} \text{ and } \overrightarrow{AN} = \frac{1}{3}\underline{\hat{i}} + \underline{\hat{j}}$$

Let θ be the required angle.

$$\theta = \cos^{-1} \left(\frac{\frac{1}{3} + \frac{1}{2}}{\frac{\sqrt{5}}{2} \times \frac{\sqrt{10}}{3}} \right)$$

So the angle between $\overrightarrow{AM}$ and $\overrightarrow{AN}$ is 45° .

Question 17 A

Resolving forces parallel and perpendicular to the plane we obtain $R = 40\cos(20^\circ)$ and $F = 40\sin(20^\circ)$.

Question 18 E

Let R be the resistance to the motion of the trailer.

$$1600 - 400 - R = 1200 \times 8$$

So $R = 240$ (newtons).

Question 19 A

Let W be the total weight of the 12 raspberries.

$$E(W) = 12 \times 10 = 120 \text{ and } \text{var}(W) = 12 \times 1.5^2$$

So $W \sim N(120, 12 \times 1.5^2)$ and $\Pr(W > 130) = 0.0271$.

Question 20 B

The p -value for a two-sided H_1 is twice the value for a one-sided H_1 .

So the new p -value is $2 \times 0.007 = 0.014$.

As $0.014 > 0.01(\alpha)$, Emily should not reject H_0 .

SECTION B**Question 1** (10 marks)

a. $v \frac{dv}{dx} = \frac{1}{2500}(10\,000 + v^2)$ A1

Attempting to solve this differential equation either by CAS or by hand with $v(0) = 0$. M1

$$v^2 = 10\,000e^{\frac{x}{1250}} - 10\,000$$
 A1

$$v = 100\sqrt{e^{\frac{x}{1250}} - 1} \text{ (since } v > 0 \text{)}$$
 A1

b. When $x = 900$ (m), $v = 102.7$ (m/s). A1

As $102.7 > 80$, the aeroplane will take off successfully. A1

c. i. $v = 100 \tan\left(\frac{t}{25}\right) \Rightarrow \frac{dv}{dt} = 4 \sec^2\left(\frac{t}{25}\right)$ A1

$$\frac{dv}{dt} = \frac{1}{2500}\left(10\,000 + 10\,000 \tan^2\left(\frac{t}{25}\right)\right)$$
 M1

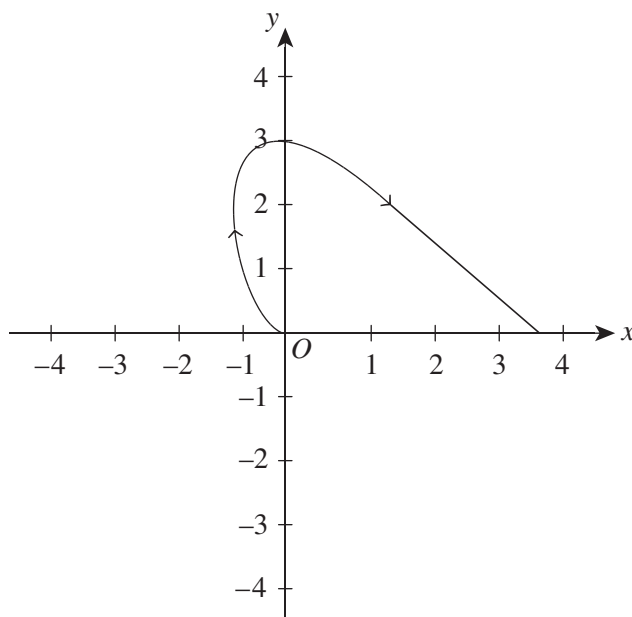
$$= 4\left(1 + \tan^2\left(\frac{t}{25}\right)\right)$$

$$= 4 \sec^2\left(\frac{t}{25}\right), \text{ so } v = 100 \tan\left(\frac{t}{25}\right) \text{ is a solution to the differential equation.}$$
 A1

ii. This model suggests that $v \rightarrow \infty$ for a finite time value $t \rightarrow \frac{25\pi}{2}$. A1

Question 2 (10 marks)

a.



correct shape A1

correct direction of motion indicated A1

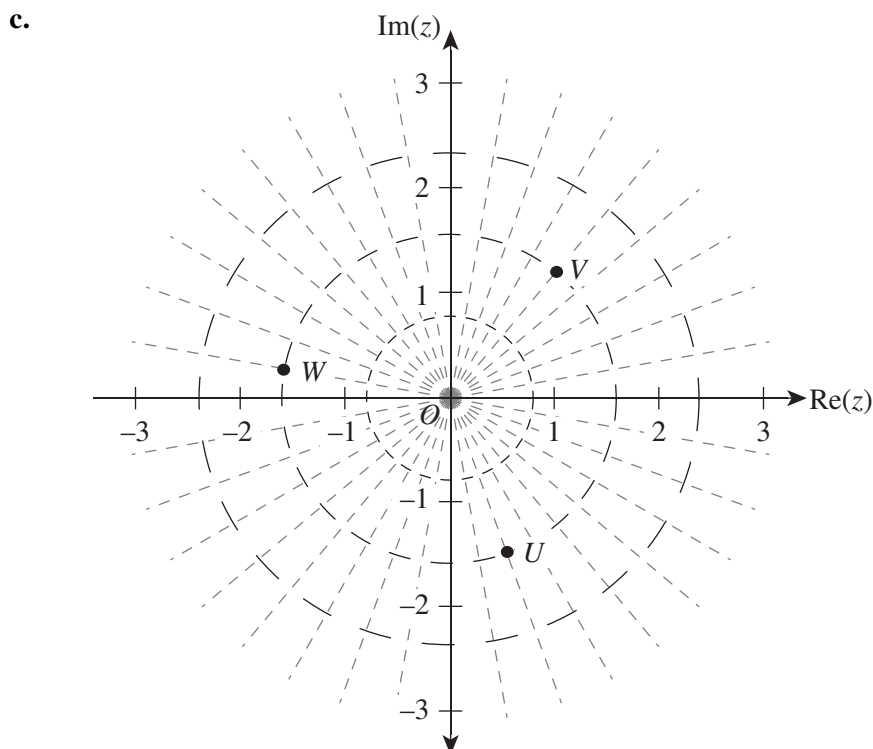
- b. i. Solving $\frac{t^2}{2} - \log_e(1+t) = 0$ for t M1
 we obtain $t = 1.29$ (s) (correct to two decimal places). A1
- ii. $|\tilde{r}'(1.285\dots)| = \sqrt{(x'(1.285\dots))^2 + (y'(1.285\dots))^2}$ A1
 Attempting to evaluate. M1
 speed = 1.2 (m/s) (correct to one decimal place) A1
- c. Let the distance travelled be d metres.
- $$d = \int_0^\pi \sqrt{(x'(t))^2 + (y'(t))^2} dt$$
- A1
-
- Attempting to evaluate. M1
-
- total distance = 7.8 (m) (correct to one decimal place) A1

Question 3 (12 marks)

a. $z = \sqrt{12} \operatorname{cis}\left(\frac{5\pi}{6}\right)$ (or $z = 2\sqrt{3} \operatorname{cis}\left(\frac{5\pi}{6}\right)$) A1 A1

b. $v = 12^{\frac{1}{6}} \operatorname{cis}\left(\frac{5\pi}{18}\right)$ M1 A1

$u = 12^{\frac{1}{6}} \operatorname{cis}\left(-\frac{7\pi}{18}\right)$ and $w = 12^{\frac{1}{6}} \operatorname{cis}\left(\frac{17\pi}{18}\right)$ A1 A1



V plotted correctly A1
U and W plotted correctly A1

- d. UOV , VOW and UOW are three congruent triangles with $|u| = |v| = |w| = 12^{\frac{1}{6}}$
and $\angle UOV = \angle VOW = \angle UOW = \frac{2\pi}{3}$.

Let the area be A .

$$A = 3 \left(\frac{1}{2} \right) \left(12^{\frac{1}{6}} \right) \left(12^{\frac{1}{6}} \right) \sin \left(\frac{2\pi}{3} \right) \quad \text{M1 A1 A1}$$

$$= \frac{3\sqrt{3}}{4} \left(12^{\frac{1}{3}} \right) \quad \text{A1}$$

Question 4 (11 marks)

- a. Let X be the total weight of the five tangerines.
 $E(X) = 5 \times 200 = 1000$ and $\text{var}(X) = 5 \times 100 = 500$. A1
 $\Pr(875 < X < 975) = 0.1318$ (correct to four decimal places) A1
- b. Let $Y = T - 3M$, where T is the weight of a random tangerine and M is the weight of a random mandarin. M1
 $E(Y) = 200 - 3 \times 75 = -25$ and $\text{var}(Y) = 10^2 + 9 \times 3^2 = 181$. A1
 $\Pr(Y > 0) = 0.0316$ (correct to four decimal places) A1
- c. $H_0: \mu = 100$ versus $H_1: \mu \neq 100$ A1
- d. If H_0 is true, then $\bar{X} \sim N\left(100, \frac{5^2}{15}\right)$. A1
 $p\text{-value} = 2\Pr(\bar{X} \leq 97 | \mu = 100)$ M1
 So $p\text{-value} = 0.0201$ (correct to four decimal places). A1
Note: Award full marks if the correct p-value is stated.
- e. As $0.0201 < 0.05(\alpha)$, we should reject H_0 in favour of H_1 . A1
 We have enough evidence to conclude that the mean weights are not 100 grams. A1

Question 5 (9 marks)

- a. Use of Euler's method (formula or program). M1
 $f(6.1) \approx f(6) + (0.1)f'(6)$
 $= 12.0586\dots$
 $f(6.2) \approx f(6.1) + (0.1)f'(6.1)$
 $= 12.1180$ (correct to four decimal places) A1
- b. i. $f''(x) = \frac{(x-4)e^{\frac{x}{2}} - 4}{2x^3}$ A1

- ii. Since $f''(x)$ is positive on the interval $[6, 6.2]$, A1
 the graph of f is concave up on the interval $[6, 6.2]$. A1
 Hence the tangent lines (Euler approximation lines) are below the actual graph of f . A1

c. $f(6.2) - f(6) = \int_6^{6.2} f'(x) dx$ M1 A1

$f(6.2) = 12.1189$ (correct to four decimal places) A1

Question 6 (8 marks)

a. $(8 - 6)g = (8 + 6)a$ and so $a = 1.4 \text{ (m/s}^2\text{)}$. M1 A1

- b. Let x be the distance travelled by the particle at A .

$$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = 1.4 \Rightarrow \frac{1}{2} v^2 = 1.4x + c$$
 M1 A1

When $x = 0$, $v = 0$ and so $c = 0$; that is, $v^2 = 2.8x$. A1

When $x = 3$, $v = \sqrt{8.4} \text{ (m/s)}$. A1

c. $\frac{dv}{dt} = 1.4 \Rightarrow v = 1.4t + c$

When $t = 0$, $v = 0$ and so $c = 0$; that is $v = 1.4t$. M1

When $v = \sqrt{8.4}$ we have $t = \frac{\sqrt{8.4}}{1.4} = 2.07 \text{ (s)}$ (correct to two decimal places). A1

Note: Constant acceleration formulae are no longer part of the syllabus.