Neap:

Trial Examination 2016

VCE Specialist Mathematics Units 3&4

Written Examination 1

Suggested Solutions

Question 1 (2 marks)

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2(x)}{1+\tan(x)}dx = [\log_e(1+\tan(x))]_0^{\frac{\pi}{4}}$$ M1

$$= \log_e(2)$$ A1

Question 2 (3 marks)

a. Using $\underset{\sim}{p} = m\underset{\sim}{v}$ with $m = 0.25$ and $\underset{\sim}{v} = (4t^3 - 4t)\underset{\sim}{i} + (12t^2 - 4t^3)\underset{\sim}{j}$.

So $\underset{\sim}{p} = (t^3 - t)\underset{\sim}{i} + (3t^2 - t^3)\underset{\sim}{j}$. A1

b. Using $\underset{\sim}{F} = m\underset{\sim}{a}$ with $m = 0.25$ and $\underset{\sim}{a} = (12t^2 - 4)\underset{\sim}{i} + (24t - 12t^2)\underset{\sim}{j}$. M1

$\underset{\sim}{F} = (3t^2 - 1)\underset{\sim}{i} + (6t - 3t^2)\underset{\sim}{j}$

$3t^2 - 1 = 0 \Rightarrow t = \frac{1}{\sqrt{3}}$ (seconds) (since $t \geq 0$) A1

Question 3 (3 marks)

$$\cos\left(\frac{x}{2}\right) = \sin\left(\frac{x}{4}\right)$$

$$1 - 2\sin^2\left(\frac{x}{4}\right) = \sin\left(\frac{x}{4}\right)$$

$$2\sin^2\left(\frac{x}{4}\right) + \sin\left(\frac{x}{4}\right) - 1 = 0,\ 0 \leq \frac{x}{4} \leq \pi$$ M1

$$\left(2\sin\left(\frac{x}{4}\right) - 1\right)\left(\sin\left(\frac{x}{4}\right) + 1\right) = 0$$

$$\sin\left(\frac{x}{4}\right) = -1, \frac{1}{2}$$

As $0 \leq \sin\left(\frac{x}{4}\right) \leq 1$, $\sin\left(\frac{x}{4}\right) = \frac{1}{2}$. A1

$$\frac{x}{4} = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\therefore x = \frac{2\pi}{3}, \frac{10\pi}{3}$$ A1

Question 4 (6 marks)

a. By the conjugate root theorem, $z = 1 + \sqrt{3}i$ is a root. A1

So $(z-(1+\sqrt{3}i))(z-(1-\sqrt{3}i)) = z^2 - 2z + 4$.

$$\therefore z^3 + z^2 + bz + 12 = (z^2 - 2z + 4)(z - m)$$ M1

By equating coefficients, $-4m = 12 \Rightarrow m = -3$.

So the roots are -3, $1 \pm \sqrt{3}i$. A1

b. $1 - \sqrt{3}i = 2\text{cis}\left(-\frac{\pi}{3}\right)$ M1

$(1-\sqrt{3}i)^k \in R$ for $k\left(-\frac{\pi}{3}\right) = n\pi, n \in Z$

So $k = -3n \Rightarrow k_{\min} = 3$. A1

$$(1-\sqrt{3}i)^3 = 2^3\text{cis}(-\pi)$$
$$= -8$$ A1

Question 5 (4 marks)

a. $\text{E}(\bar{X}) = \text{E}(X)$

$= 30$ A1

b. $\text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$

$= \frac{7}{\sqrt{100}}$

$= \frac{7}{10}$ A1

c. $\text{E}(\bar{X}) = 30$ A1

Note: This is unchanged as it is independent of n.

$\text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$

$= \frac{7}{\sqrt{400}}$

$= \frac{7}{20}$

$\text{sd}(\bar{X})$ is reduced by a factor of 2. A1

Question 6 (3 marks)

Let $y = \cos^{-1}(u)$ and so $u = 2x^{-1}$.

$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$ and $\frac{du}{dx} = -2x^{-2} = \frac{-2}{x^2}$ M1

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{-1}{\sqrt{1-\left(\frac{2}{x}\right)^2}} \times \frac{-2}{x^2}$$ A1

$$= \frac{2}{x^2\sqrt{\frac{x^2-4}{x^2}}}$$

$\frac{dy}{dx} = \frac{2}{|x|\sqrt{x^2-4}}$ but $\sqrt{x^2} = |x|$, and since $x > 2$, $\sqrt{x^2} = x$. So $\frac{dy}{dx} = \frac{2}{x\sqrt{x^2-4}}$. A1

Note: The final A1 should only be awarded if the second-last line (or equivalent) is shown.

Question 7 (9 marks)

a. $\underset{\sim}{r}'(t) = \int -g\underset{\sim}{j}\,dt$

$= -gt\underset{\sim}{j} + \underset{\sim}{c}$ M1

When $t = 0$, $\dot{\underset{\sim}{r}}(0) = V\cos(\theta)\underset{\sim}{i} + V\sin(\theta)\underset{\sim}{j}$ and so $\underset{\sim}{c} = V\cos(\theta)\underset{\sim}{i} + V\sin(\theta)\underset{\sim}{j}$.

So $\underset{\sim}{r}'(t) = V\cos(\theta)\underset{\sim}{i} + (V\sin(\theta) - gt)\underset{\sim}{j}$. A1

$\underset{\sim}{r}(t) = \int V\cos(\theta)\underset{\sim}{i} + (V\sin(\theta) - gt)\underset{\sim}{j}\,dt$

$= V\cos(\theta)t\underset{\sim}{i} + \left(V\sin(\theta)t - \frac{1}{2}gt^2\right)\underset{\sim}{j} + \underset{\sim}{d}$ M1

When $t = 0$, $\underset{\sim}{r}(0) = h\underset{\sim}{j}$ and so $\underset{\sim}{d} = h\underset{\sim}{j}$.

So $\underset{\sim}{r}(t) = V\cos(\theta)t\underset{\sim}{i} + \left(V\sin(\theta)t - \frac{1}{2}gt^2 + h\right)\underset{\sim}{j}$. A1

Note: The final A1 should only be awarded if the second last line is shown.

b. Parametric equations are $x = V\cos(\theta)t$ and $y = V\sin(\theta)t - \frac{1}{2}gt^2 + h$.

Substitute $t = \frac{x}{V\cos(\theta)}$ into $y = V\sin(\theta)t - \frac{1}{2}gt^2 + h$. M1

$$y = \tan(\theta)x - \frac{gx^2}{2V^2\cos^2(\theta)} + h$$

$$= h + \tan(\theta)x - \frac{g\sec^2(\theta)}{2V^2}x^2$$ A1

Note: The final A1 should only be awarded if the second-last line is shown.

c. Consider $y = h + \tan(\theta)x - \frac{gx^2}{2v^2}\sec^2(\theta)$.

Projectile fired horizontally:

$\theta = 0$, $V = U$, $y = 0$

$$0 = h - \frac{gx^2}{2U^2} \Rightarrow h = \frac{gx^2}{2U^2} \quad \ldots(\mathbf{1})$$ A1

Projectile fired at an angle of elevation of $\tan^{-1}(3)$:

$$0 = h + 3x - \frac{10gx^2}{2U^2} \quad \ldots(\mathbf{2})$$

Substituting $h = \frac{gx^2}{2U^2}$ into **(2)** gives $3x - \frac{9gx^2}{2U^2} = 0$. M1

Solving gives $x = \frac{2U^2}{3g}$ (metres). A1

Question 8 (3 marks)

$\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$ where $m, n \in R\backslash\{0\}$ for linear dependence.

$\underset{\sim}{i} + 6\underset{\sim}{j} + 10\underset{\sim}{k} = m(\underset{\sim}{i} + 2\underset{\sim}{j}) + n(-\underset{\sim}{i} + 5\underset{\sim}{k})$ A1

Equating $\underset{\sim}{i}$ components: $m - n = 1$

Equating $\underset{\sim}{j}$ components: $2m = 6 \Rightarrow m = 3$

Equating $\underset{\sim}{k}$ components: $5n = 10 \Rightarrow n = 2$ M1

Note: Award M1 for attempting to form three equations.

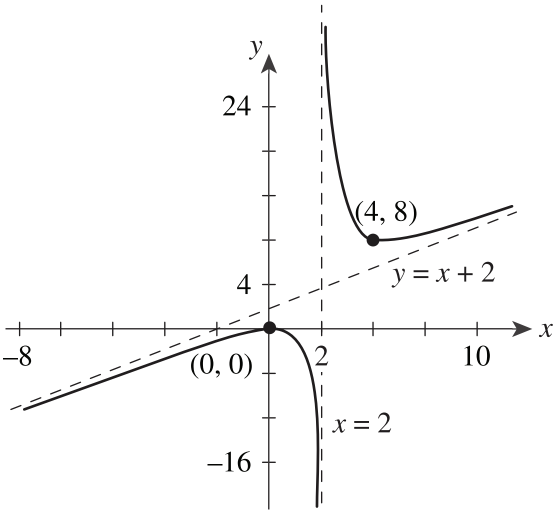
$m = 3$ and $n = 2$ satisfy $m - n = 1$ and so $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent. A1

Question 9 (7 marks)

a. Re-expressing $y = \dfrac{x^2}{x-2}$ to give $y = x + 2 + \dfrac{4}{x-2}$. M1

Asymptotes are $y = x + 2$ and $x = 2$. A1

Stationary points are $(4, 8)$ and $(0, 0)$. A1



correct shape A1

b. $x^2 = p(x^2 - 4) \Rightarrow \dfrac{x^2}{x-2} = p(x+2)$ M1

Due to the asymptote $y = x + 2$, the maximum value of p is 1, A1
and we also require $p > 0$ (gradient of the line steeper than the horizontal line joining (–2, 0) and (0, 0)).

So the required values are $0 < p \leq 1$. A1

