



Trial Examination 2016

VCE Specialist Mathematics Units 3&4

Written Examination 1

Suggested Solutions

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Question 1 (2 marks)

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2(x)}{1 + \tan(x)} dx = [\log_e(1 + \tan(x))]_0^{\frac{\pi}{4}} \quad \text{M1}$$

$$= \log_e(2) \quad \text{A1}$$

Question 2 (3 marks)

a. Using $\underline{p} = m\underline{v}$ with $m = 0.25$ and $\underline{v} = (4t^3 - 4t)\underline{i} + (12t^2 - 4t^3)\underline{j}$.

So $\underline{p} = (t^3 - t)\underline{i} + (3t^2 - t^3)\underline{j}$. A1

b. Using $\underline{F} = m\underline{a}$ with $m = 0.25$ and $\underline{a} = (12t^2 - 4)\underline{i} + (24t - 12t^2)\underline{j}$. M1

$$\underline{F} = (3t^2 - 1)\underline{i} + (6t - 3t^2)\underline{j}$$

$$3t^2 - 1 = 0 \Rightarrow t = \frac{1}{\sqrt{3}} \text{ (seconds) (since } t \geq 0)$$
 A1

Question 3 (3 marks)

$$\cos\left(\frac{x}{2}\right) = \sin\left(\frac{x}{4}\right)$$

$$1 - 2\sin^2\left(\frac{x}{4}\right) = \sin\left(\frac{x}{4}\right)$$

$$2\sin^2\left(\frac{x}{4}\right) + \sin\left(\frac{x}{4}\right) - 1 = 0, 0 \leq \frac{x}{4} \leq \pi \quad \text{M1}$$

$$\left(2\sin\left(\frac{x}{4}\right) - 1\right)\left(\sin\left(\frac{x}{4}\right) + 1\right) = 0$$

$$\sin\left(\frac{x}{4}\right) = -1, \frac{1}{2}$$

As $0 \leq \sin\left(\frac{x}{4}\right) \leq 1$, $\sin\left(\frac{x}{4}\right) = \frac{1}{2}$. A1

$$\frac{x}{4} = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\therefore x = \frac{2\pi}{3}, \frac{10\pi}{3} \quad \text{A1}$$

Question 4 (6 marks)

- a. By the conjugate root theorem, $z = 1 + \sqrt{3}i$ is a root.

A1

$$\text{So } (z - (1 + \sqrt{3}i))(z - (1 - \sqrt{3}i)) = z^2 - 2z + 4.$$

$$\therefore z^3 + z^2 + bz + 12 = (z^2 - 2z + 4)(z - m)$$

M1

By equating coefficients, $-4m = 12 \Rightarrow m = -3$.

So the roots are $-3, 1 \pm \sqrt{3}i$.

A1

- b. $1 - \sqrt{3}i = 2\text{cis}\left(-\frac{\pi}{3}\right)$

M1

$$(1 - \sqrt{3}i)^k \in R \text{ for } k\left(-\frac{\pi}{3}\right) = n\pi, n \in Z$$

$$\text{So } k = -3n \Rightarrow k_{\min} = 3.$$

A1

$$\begin{aligned}(1 - \sqrt{3}i)^3 &= 2^3 \text{cis}(-\pi) \\ &= -8\end{aligned}$$

A1

Question 5 (4 marks)

- a. $E(\bar{X}) = E(X)$
 $= 30$

A1

- b. $\text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$
 $= \frac{7}{\sqrt{100}}$
 $= \frac{7}{10}$

A1

- c. $E(\bar{X}) = 30$

A1

Note: This is unchanged as it is independent of n .

$$\begin{aligned}\text{sd}(\bar{X}) &= \frac{\sigma}{\sqrt{n}} \\ &= \frac{7}{\sqrt{400}} \\ &= \frac{7}{20}\end{aligned}$$

$\text{sd}(\bar{X})$ is reduced by a factor of 2.

A1

Question 6 (3 marks)

Let $y = \cos^{-1}(u)$ and so $u = 2x^{-1}$.

$$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}} \text{ and } \frac{du}{dx} = -2x^{-2} = \frac{-2}{x^2} \quad \text{M1}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{-1}{\sqrt{1-\left(\frac{2}{x}\right)^2}} \times \frac{-2}{x^2} \quad \text{A1} \end{aligned}$$

$$= \frac{2}{x^2 \sqrt{\frac{x^2-4}{x^2}}}$$

$$\frac{dy}{dx} = \frac{2}{|x|\sqrt{x^2-4}} \text{ but } \sqrt{x^2} = |x|, \text{ and since } x > 2, \sqrt{x^2} = x. \text{ So } \frac{dy}{dx} = \frac{2}{x\sqrt{x^2-4}}. \quad \text{A1}$$

Note: The final A1 should only be awarded if the second-last line (or equivalent) is shown.

Question 7 (9 marks)

$$\begin{aligned} \text{a. } \underline{\underline{r}}'(t) &= \int -g \underline{\underline{j}} dt \\ &= -gt \underline{\underline{j}} + \underline{\underline{c}} \quad \text{M1} \end{aligned}$$

When $t = 0$, $\underline{\underline{r}}'(0) = V \cos(\theta) \underline{\underline{i}} + V \sin(\theta) \underline{\underline{j}}$ and so $\underline{\underline{c}} = V \cos(\theta) \underline{\underline{i}} + V \sin(\theta) \underline{\underline{j}}$.

$$\text{So } \underline{\underline{r}}'(t) = V \cos(\theta) \underline{\underline{i}} + (V \sin(\theta) - gt) \underline{\underline{j}}. \quad \text{A1}$$

$$\begin{aligned} \underline{\underline{r}}(t) &= \int V \cos(\theta) \underline{\underline{i}} + (V \sin(\theta) - gt) \underline{\underline{j}} dt \\ &= V \cos(\theta) t \underline{\underline{i}} + \left(V \sin(\theta) t - \frac{1}{2} g t^2 \right) \underline{\underline{j}} + \underline{\underline{d}} \quad \text{M1} \end{aligned}$$

When $t = 0$, $\underline{\underline{r}}(0) = h \underline{\underline{j}}$ and so $\underline{\underline{d}} = h \underline{\underline{j}}$.

$$\text{So } \underline{\underline{r}}(t) = V \cos(\theta) t \underline{\underline{i}} + \left(V \sin(\theta) t - \frac{1}{2} g t^2 + h \right) \underline{\underline{j}}. \quad \text{A1}$$

Note: The final A1 should only be awarded if the second last line is shown.

- b. Parametric equations are $x = V\cos(\theta)t$ and $y = V\sin(\theta)t - \frac{1}{2}gt^2 + h$.

Substitute $t = \frac{x}{V\cos(\theta)}$ into $y = V\sin(\theta)t - \frac{1}{2}gt^2 + h$. M1

$$\begin{aligned} y &= \tan(\theta)x - \frac{gx^2}{2V^2\cos^2(\theta)} + h \\ &= h + \tan(\theta)x - \frac{g\sec^2(\theta)}{2V^2}x^2 \end{aligned} \quad \text{A1}$$

Note: The final A1 should only be awarded if the second-last line is shown.

- c. Consider $y = h + \tan(\theta)x - \frac{gx^2}{2v^2}\sec^2(\theta)$.

Projectile fired horizontally:

$$\theta = 0, V = U, y = 0$$

$$0 = h - \frac{gx^2}{2U^2} \Rightarrow h = \frac{gx^2}{2U^2} \quad \dots (1) \quad \text{A1}$$

Projectile fired at an angle of elevation of $\tan^{-1}(3)$:

$$0 = h + 3x - \frac{10gx^2}{2U^2} \quad \dots (2)$$

$$\text{Substituting } h = \frac{gx^2}{2U^2} \text{ into (2) gives } 3x - \frac{9gx^2}{2U^2} = 0. \quad \text{M1}$$

$$\text{Solving gives } x = \frac{2U^2}{3g} \text{ (metres).} \quad \text{A1}$$

Question 8 (3 marks)

$\underline{c} = m\underline{a} + n\underline{b}$ where $m, n \in \mathbb{R} \setminus \{0\}$ for linear dependence.

$$\underline{i} + 6\underline{j} + 10\underline{k} = m(\underline{i} + 2\underline{j}) + n(-\underline{i} + 5\underline{k}) \quad \text{A1}$$

Equating $\underline{i}$ components: $m - n = 1$

Equating $\underline{j}$ components: $2m = 6 \Rightarrow m = 3$

Equating $\underline{k}$ components: $5n = 10 \Rightarrow n = 2$ M1

Note: Award M1 for attempting to form three equations.

$m = 3$ and $n = 2$ satisfy $m - n = 1$ and so $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly dependent. A1

Question 9 (7 marks)

a. Re-expressing $y = \frac{x^2}{x-2}$ to give $y = x + 2 + \frac{4}{x-2}$.

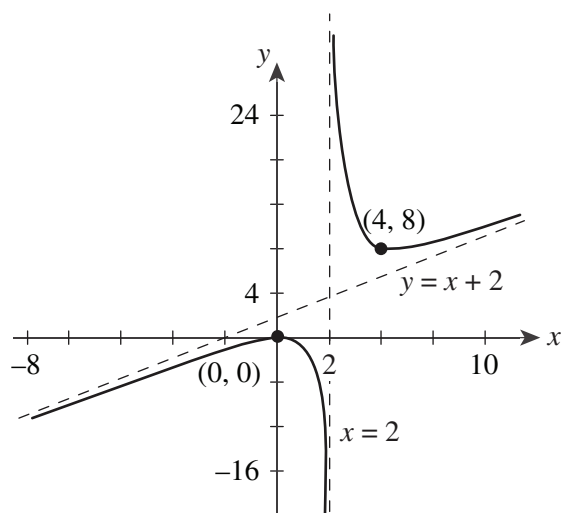
M1

Asymptotes are $y = x + 2$ and $x = 2$.

A1

Stationary points are $(4, 8)$ and $(0, 0)$.

A1

*correct shape* A1

b. $x^2 = p(x^2 - 4) \Rightarrow \frac{x^2}{x-2} = p(x+2)$

M1

Due to the asymptote $y = x + 2$, the maximum value of p is 1,
and we also require $p > 0$ (gradient of the line steeper than the horizontal line joining $(-2, 0)$ and $(0, 0)$).

A1

So the required values are $0 < p \leq 1$.

A1