

Trial Examination 2016

SPECIALIST MATHEMATICS

Written Examination 2 - SOLUTIONS

SECTION A

Question	Answer	Question	Answer
1	A	11	B
2	D	12	C
3	B	13	E
4	A	14	E
5	E	15	D
6	E	16	B
7	E	17	D
8	C	18	C
9	C	19	B
10	A	20	A

Question 1

$$(2i)^3 - 5(2i)^2 + 4(2i) - mi = 0$$

$$20 - mi = 0$$

$$m = \frac{20}{i} = -20i$$

Answer is A

Question 2

Can begin by looking for a term which involves either $\bar{z} + z$ or $z\bar{z}$

$$\frac{1}{z} + \frac{1}{\bar{z}} = \frac{\bar{z} + z}{z\bar{z}} = \frac{x - iy + x + iy}{(x^2 + y^2)} = \frac{2x}{(x^2 + y^2)}$$

All other terms involve i

Answer is D

Question 3

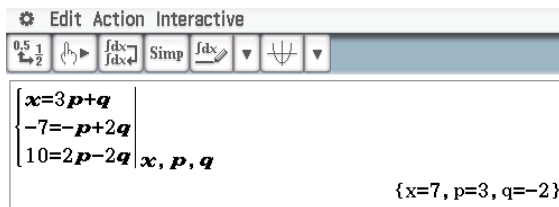
To be linearly dependent can be expressed as $\underline{c} = q\underline{a} + p\underline{b}$

Therefore using CAS solve

$$x = 3p + q$$

$$-7 = -p + 2q$$

$$10 = 2p - 2q$$



this will give $p = 3, q = -2$ and $x = 7$

Answer is B

Question 4

$$|\underline{p}| = 4 \text{ therefore } 4 + x^2 + 9 = 16, x = \pm\sqrt{3}$$

$$\underline{p} \cdot \underline{q} = -8 + \sqrt{3}x - 3y = 0, y = \frac{\sqrt{3}x - 8}{3}$$

$$x = \sqrt{3}, y = -\frac{5}{3} \text{ and } x = -\sqrt{3}, y = -\frac{11}{3}$$

Answer is A

Question 5

The particles meet if the $\underline{i}$, $\underline{j}$ and $\underline{k}$ components are all exactly the same at the same time.

Need to solve

$$3 = t + 1 \Rightarrow t = 2$$

$$\text{and } 2t - 6 = 4 \Rightarrow t = 5$$

$$\text{and } t^2 - 7t = -10 \Rightarrow t = 2, 5$$

Since there is no value of t in common for all of the components, they will never collide.

Answer is E

Question 6

$$a = v \frac{dv}{dx} \text{ and } \frac{dv}{dx} = \frac{1}{2}(1-2x^2)^{\frac{1}{2}} \times (-4x)$$

$$a = \frac{1}{3}(1-2x^2)^{\frac{3}{2}} \times \frac{1}{2}(1-2x^2)^{\frac{1}{2}} \times (-4x) = -\frac{2x}{3}(1-2x^2)^2$$

Answer is E**Question 7**

$$\dot{\mathbf{r}}(t) = 2(1+3t) \times 3\mathbf{i} - \frac{18}{2}t^{-\frac{1}{2}}\mathbf{k}$$

$$\text{When } t = 3 \quad |\dot{\mathbf{r}}(t)| = \sqrt{\left(60^2 + \left(\frac{9}{\sqrt{3}}\right)^2\right)} = 60.22$$

Answer is E**Question 8**

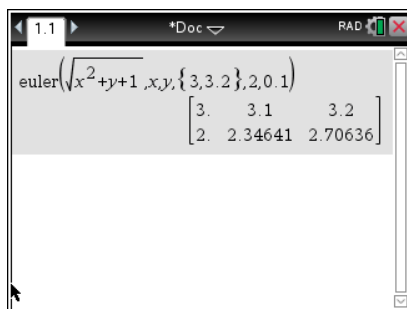
$$u = 3x + 2, \quad u - 3 = 3x - 1, \quad \frac{du}{dx} = 3$$

When $x = 0, u = 2$ and $x = 2, u = 8$

$$\text{Therefore } \int_0^2 (3x-1)\sqrt{(3x+2)}dx = \frac{1}{3} \int_2^8 (u-3)\sqrt{u}du$$

The screenshot shows a software interface with a toolbar at the top containing icons for 'Edit', 'Action', and 'Interactive'. Below the toolbar, there are two integration problems displayed in a list. The first problem is $\int_0^2 (3x-1)(3x+2)^{0.5} dx$ and the second is $\int_2^8 (1/3)(u-3)u^{0.5} du$. To the right of each problem is its solution, $\frac{36\sqrt{2}}{5}$.

Answer is C

Question 9

$$x_0 = 3, y_0 = 2$$

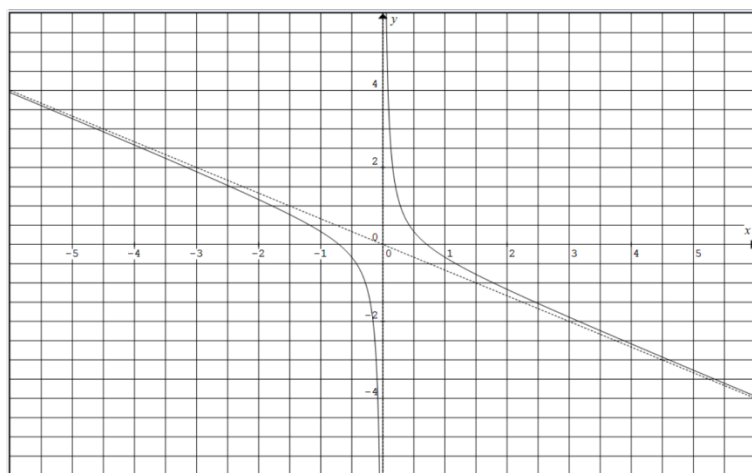
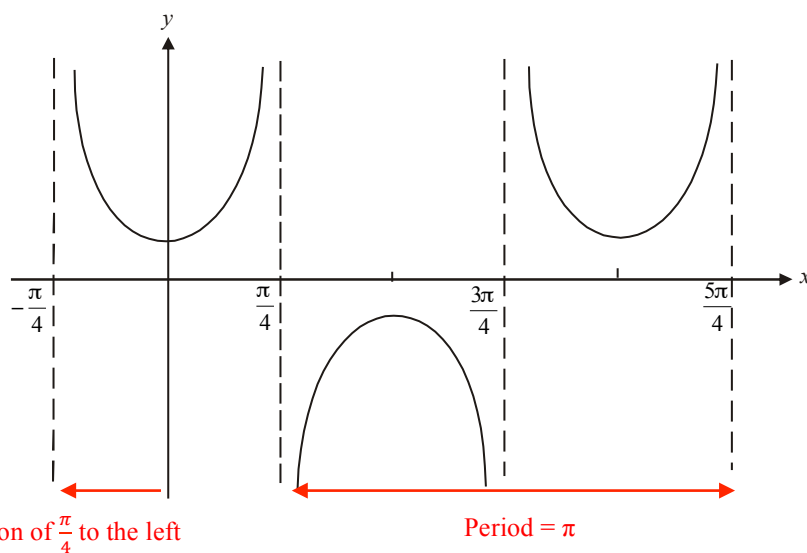
$$y_1 = 2 + 0.1\sqrt{(3^2 + 2 + 1)}$$

$$y_2 = y_1 + 0.1\sqrt{(3.1^2 + y_1 + 1)} \\ = 2.7064$$

Answer is C**Question 10**

$$y = \frac{1 - 2x^2}{3x} = -\frac{2}{3}x + \frac{1}{3x}$$

Therefore asymptotes are $y = -\frac{2}{3}x$ and $x = 0$

**Answer is A****Question 11**

period = π therefore $\pi = \frac{2\pi}{a}$, $a = 2$

Translation of $\frac{\pi}{4}$ to the left, so function is $y = \operatorname{cosec}(2(x + \frac{\pi}{4}))$ therefore $b = -\frac{\pi}{4}$

Answer is B

Question 12

$$x^2 + 2px + y^2 + 1 = 0$$

$$(x + p)^2 - p^2 + y^2 + 1 = 0$$

$$(x + p)^2 + y^2 = p^2 - 1$$

To be a circle we require $p^2 - 1 > 0$, $p^2 > 1$ so the only correct option $|p| > 1$

Answer is C

Question 13

The graph has the form $y = \frac{k}{(x^2 + a)}$ where $k, a \in \mathbb{R}$

The graph goes through the points $(0, 4)$ and $(1, 2)$, therefore

$$\frac{k}{a} = 4 \text{ and } 2 = \frac{k}{a + 1} \text{ solving these gives } a = 1, k = 4$$

Thus $y = \frac{4}{(x^2 + 1)}$ and $\int y dx = 4 \tan^{-1}(x) + c$

Alternatively, this could be done by inputting each function and plotting the gradient graph to see which matches.

Answer is E

Question 14

Range of $\cos^{-1} \theta$ is $[0, \pi]$, therefore the range of $f(x) = 4 \cos^{-1}(3x + 1) + \frac{\pi}{2}$ is

$$[4 \times 0 + \frac{\pi}{2}, 4 \times \pi + \frac{\pi}{2}] = [\frac{\pi}{2}, \frac{9\pi}{2}]$$

Answer is E

Question 15

When $t = 3$, $\underline{F}_1 = \underline{i} - 3\underline{j}$, $\underline{F}_2 = 2\underline{i} + \underline{j}$, $\underline{F}_3 = -7\underline{i} + 5\underline{j}$ therefore the resultant force $\underline{F} = -4\underline{i} + 3\underline{j}$

$$a = \frac{|\underline{F}|}{m} = \frac{\sqrt{(4^2 + 3^2)}}{5} = 1$$

Answer is D**Question 16**

Equations of motion:

$$m_1 g - T = \frac{m_1 g}{2} \quad \text{equation 1}$$

$$T - m_2 g = \frac{m_2 g}{2} \quad \text{equation 2}$$

Adding the equations gives:-

$$g(m_1 - m_2) = \frac{g}{2}(m_1 + m_2)$$

$$\frac{g}{2}m_1 = \frac{3g}{2}m_2 \quad \text{therefore} \quad \frac{m_1}{m_2} = 3$$

Answer is B**Question 17**

Vertical component of motion has $u = 30\sin(60^\circ) = 15\sqrt{3}\text{ms}^{-1}$, $a = -g\text{ms}^{-2}$

Solve $v = at + u$ when $v = 0$ to find the time to reach its maximum height

$$0 = -gt + 15\sqrt{3}, \quad t = \frac{15\sqrt{3}}{g} \quad \text{therefore the time to reach the ground} \quad t = 2 \times \frac{15\sqrt{3}}{g} = \frac{30\sqrt{3}}{g} \text{ s}$$

Answer is D

(Note, this is equivalent to using the formulae for time to travel $T = \frac{2V\sin(\alpha)}{g}$,

$$T = \frac{2 \times 30 \times \sin(60)}{g} = \frac{30\sqrt{3}}{g})$$

An alternative method

$$\underline{a} = -g \underline{j}$$

$$\therefore \underline{v} = 30 \cos(60^\circ) \underline{i} + (30 \sin(60^\circ) - gt) \underline{j} = 15 \underline{i} + (15\sqrt{3} - gt) \underline{j}$$

$$\therefore \underline{r} = 15t \underline{i} + \left(15\sqrt{3}t - \frac{1}{2}gt^2\right) \underline{j}$$

Ball at ground level when $15\sqrt{3}t - \frac{1}{2}gt^2 = 0$

Solving $t = 0, \frac{30\sqrt{3}}{g}$ So returns to ground when $t = \frac{30\sqrt{3}}{g}$

Answer is D

Question 18

$$E(T) = 3E(X) - E(Y) = 3(3.6) - 12.3 = -1.5$$

$$\text{VAR}(T) = 3^2 \text{VAR}(X) + \text{VAR}(Y) = 3^2 \times 0.68^2 + 5.1^2$$

$$\text{Sd}(T) = \sqrt{\text{VAR}(T)} = \sqrt{(3^2 \times 0.68^2 + 5.1^2)} = 5.23$$

Answer is A

Question 19

The standard deviation of the sample mean will be $\frac{\mu}{2\sqrt{n}}$

To achieve a width of 0.2μ for 95% confidence interval it is necessary to solve

$$1.96 \frac{\mu}{2\sqrt{n}} \leq 0.1\mu$$

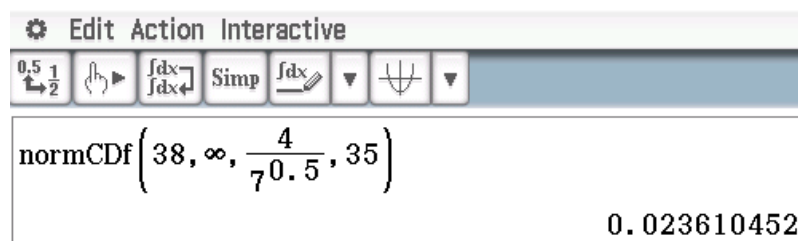
$$\sqrt{n} \geq \frac{1.96}{0.2}, \quad n \geq 96.04$$

Answer is B

Question 20

The probability required is $\Pr(\bar{X} \geq 38)$ where $\bar{X} : N\left(35, \frac{4^2}{7}\right)$ as there are 7 days in the week.

$$\Pr(\bar{X} \geq 38) = 0.0236$$



(Note as 38 is larger than the mean of 35, you can eliminate answers with probabilities greater than 0.5 straight away)

Answer is A

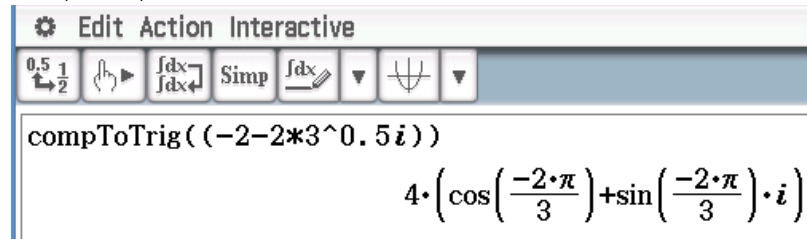
SECTION B

Question 1 (12 marks)

a. i. $r = \sqrt{((-2)^2 + (-2\sqrt{3})^2)} = \sqrt{16}, \quad \theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{-2}\right) = \tan^{-1}(\sqrt{3})$ [A1]

$$w^4 = 4\text{cis}\left(-\frac{2\pi}{3}\right)$$

Or using CAS



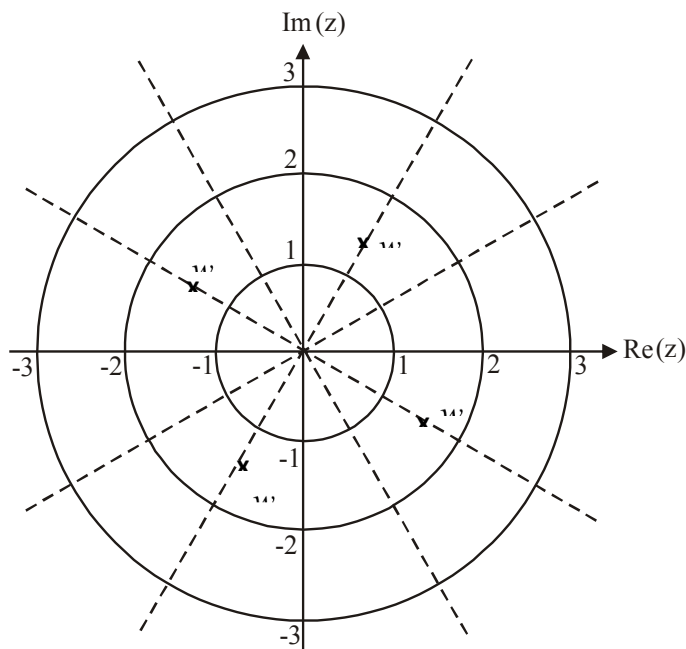
compToTrig((-2-2*3^0.5*i))
 $4 \cdot \left(\cos\left(-\frac{2\pi}{3}\right) + \sin\left(-\frac{2\pi}{3}\right) \cdot i \right)$

ii. $w = 4^{\frac{1}{4}} \text{cis}\left(-\frac{2\pi}{(3 \times 4)} + \frac{2\pi k}{4}\right), \quad k = 0, 1, 2, 3$

$$w_1 = \sqrt{2} \text{cis}\left(-\frac{2\pi}{12}\right), w_2 = \sqrt{2} \text{cis}\left(\frac{4\pi}{12}\right), w_3 = \sqrt{2} \text{cis}\left(\frac{10\pi}{12}\right), w_4 = \sqrt{2} \text{cis}\left(\frac{16\pi}{12}\right)$$

$$w_1 = \sqrt{2} \text{cis}\left(-\frac{\pi}{6}\right), w_2 = \sqrt{2} \text{cis}\left(\frac{\pi}{3}\right), w_3 = \sqrt{2} \text{cis}\left(\frac{5\pi}{6}\right), w_4 = \sqrt{2} \text{cis}\left(-\frac{2\pi}{3}\right)$$

$\frac{1}{2}$ mark each correct root rounded down [A2]



All roots correctly plotted and labelled. Equally spaced by $\frac{\pi}{2}$ around a circle radius $\sqrt{2}$ [A1]

$$\text{b. } w^4 = -\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - b\right)} \text{ and } w^4 = -2 - 2\sqrt{3}i$$

$$\text{Therefore } -\frac{a}{2} = -2 \text{ and } \frac{a^2}{4} - b = -12 \quad [\text{M1}]$$

$$a = 4, b = 16 \quad [\text{A1}]$$

Alternate method

$$\text{Substituting } w = -2 - 2\sqrt{3}i \text{ into } w^8 + aw^4 + b = 0 \text{ gives } -8 + 8\sqrt{3}i + a(-2 - 2\sqrt{3}i) + b = 0$$

$$\text{Equating real and imaginary parts gives } -8 - 2a + b = 0 \text{ and } 8\sqrt{3} - 2\sqrt{3}a = 0 \quad [\text{M1}]$$

$$\text{Solving simultaneously to get } a=4 \text{ and } b=16 \quad [\text{A1}]$$

$$\text{c. i. } (\bar{w})^4 = (\overline{w^4}) = -2 + 2\sqrt{3}i \text{ and } v^2 = -2i$$

$$\text{therefore } \frac{v^2}{(\bar{w})^4} = \frac{-2i}{(-2 + 2\sqrt{3}i)} = \frac{1}{4}(-\sqrt{3} + i) \quad [\text{A1}]$$

ii.

$$\text{Arg}(\bar{u}) = \frac{\pi}{6} \text{ therefore } \text{Arg}(u) = -\frac{\pi}{6} \text{ and } \text{Arg}(u^3) = -\frac{\pi}{2}$$

$$\text{Arg}\left(\frac{v^2}{(\bar{w})^4}\right) = \frac{5\pi}{6} \text{ therefore } \text{Arg}\left(\frac{v^2}{(\bar{w})^4}\right) = -\frac{5\pi}{6}$$

$$\text{Arg}\left(\left(\frac{v^2}{(\bar{w})^4}\right)u^3\right) = \text{Arg}\left(\frac{v^2}{(\bar{w})^4}\right) + \text{Arg}(u^3) = -\frac{5\pi}{6} - \frac{\pi}{2} = -\frac{8\pi}{6} \quad [\text{H1}]$$

$$= \frac{2\pi}{3} \quad [\text{A1}]$$

$$\text{d. } \sqrt{((x+2)^2 + (y+2\sqrt{3})^2)} = 2\sqrt{((x+2)^2 + (y-2\sqrt{3})^2)} \quad [\text{M1}]$$

$$(x+2)^2 + \left(y - \frac{10\sqrt{3}}{3}\right)^2 = \frac{64}{3} \quad [\text{A1}]$$

$$\text{Centre } \left(-2, \frac{10\sqrt{3}}{3}\right) \text{ and radius } \frac{8\sqrt{3}}{3} \quad [\text{A1}]$$

Question 2 (11 marks)

a. $\frac{dx}{dt} = -2\sin(t)$ and $\frac{dy}{dt} = 6\cos(2t)$ [M1]

$$\frac{dy}{dx} = \frac{6(1-2\sin^2(t))}{-2\sin(t)} = -\frac{3}{\sin(t)} + \frac{6\sin^2(t)}{\sin(t)} = -3\operatorname{cosec}(t) + 6\sin(t)$$
 [M1]

$$p = -3, q = 6$$

b. When $t = \frac{\pi}{6}$ $x = \sqrt{3}, y = \frac{3\sqrt{3}}{2}$ and $\frac{dy}{dx} = -3$ [A1]

Equation of the normal to the curve is $y - \frac{3\sqrt{3}}{2} = \frac{1}{3}(x - \sqrt{3})$

$$y = \frac{1}{3}x + \frac{7\sqrt{3}}{6}$$
 [A1]

c. $y = 6\sin(t)\cos(t)$

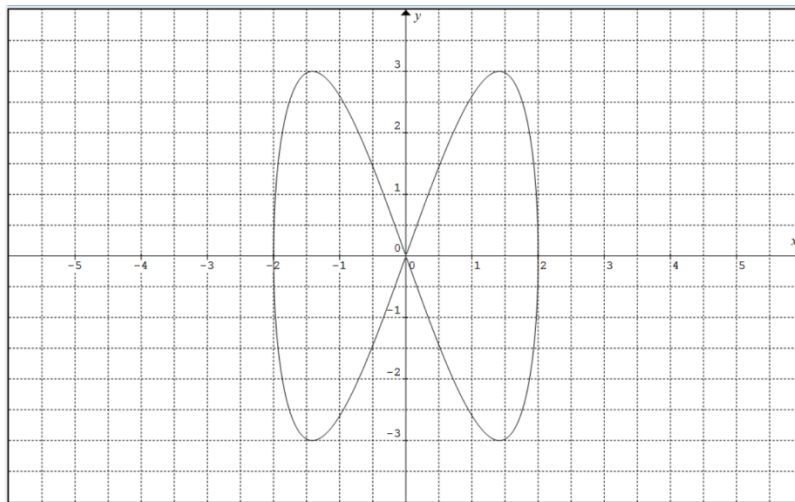
$$\begin{aligned} y^2 &= 36\sin^2(t)\cos^2(t) \\ &= \frac{9}{4} \times (4 - 4\cos^2(t)) \times 4\cos^2(t) \\ &= \frac{9}{4} \times (4 - x^2)x^2 \end{aligned}$$
 [M1]

$$k = 9/4$$
 [A1]

d. $2y \frac{dy}{dx} = \frac{9}{4}(8x - 4x^3)$ [M1]

$$\frac{dy}{dx} = \frac{9x(2 - x^2)}{2y}$$
 [A1]

e.



[A1]

f. i.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$L = \int_0^{2\pi} \sqrt{((-2\sin(t))^2 + (6\cos(2t))^2)} dt$$

[A1]

$$= \int_0^{2\pi} \sqrt{(4\sin^2(t) + 36\cos^2(2t))} dt$$

ii.

$$L = 26.20$$

[A1]

Question 3 (8 marks)

a. Solve

$$0 = 3.5m + c$$

$$8 = 5.5m + c$$

$$m = 4, c = -14$$

[A1]

b.

$$V = \pi \int_0^h x^2 dy$$

$$= \pi \int_0^h \frac{(y+14)^2}{16} dy \quad [\text{M1}]$$

$$V = \frac{\pi(h+14)^3}{48} - \frac{343\pi}{6} \quad [\text{A1}]$$

c. Let $h = 8$, $V = 517.32\text{cm}^3$ [A1]

d. (i) $\frac{dV}{dt} = -3\sqrt{h}$, $\frac{dV}{dh} = \frac{\pi(h+14)^2}{16}$ and $\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$ [A1]

therefore $\frac{dh}{dt} = -\frac{48\sqrt{h}}{\pi(h+14)^2}$ [A1]

(ii) Find the time taken for h to decrease from 8 to 0.

$$t = -\pi \int_8^0 \frac{(h+14)^2}{48\sqrt{h}} dh \quad [\text{M1}]$$

$$t = 104.95\text{s} \quad [\text{A1}]$$

Question 4 (12 marks)

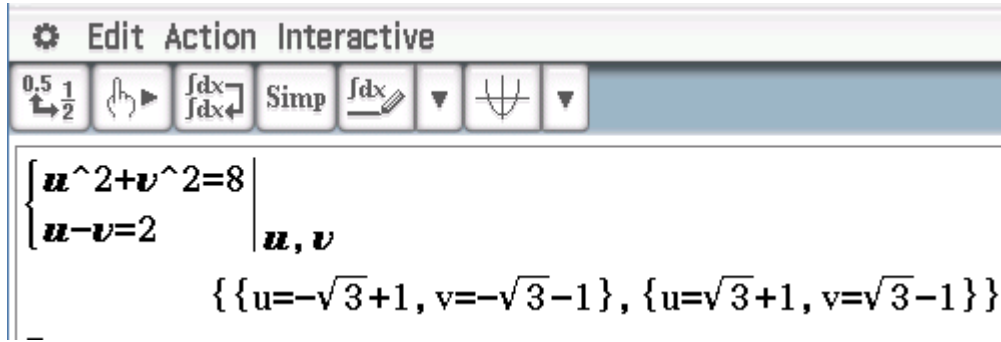
a. $\overrightarrow{\text{OA}} = 2\hat{i} - 2\hat{j}$ [A1]

$$\overrightarrow{\text{OB}} = u\hat{i} + v\hat{j} \quad [\text{A1}]$$

b. $|\overrightarrow{\text{OA}}| = |\overrightarrow{\text{OB}}|$ therefore $\sqrt{u^2 + v^2} = 2\sqrt{2}$ so $u^2 + v^2 = 8$ [A1]

$$\overrightarrow{\text{OA}} \cdot \overrightarrow{\text{OB}} = \sqrt{8}\sqrt{8} \cos(60^\circ) = 4 \quad \text{and} \quad \overrightarrow{\text{OA}} \cdot \overrightarrow{\text{OB}} = 2u - 2v \quad \text{therefore} \quad u - v = 2 \quad [\text{A1}]$$

- c. Solving $u^2 + v^2 = 8$ and $u - v = 2$ gives



TI-Nspire interface showing the solution to the system of equations $u^2 + v^2 = 8$ and $u - v = 2$. The calculator displays the equations in a list, then the variables u, v , and finally the solution set: $\{\{u=-\sqrt{3}+1, v=-\sqrt{3}-1\}, \{u=\sqrt{3}+1, v=\sqrt{3}-1\}\}$.

but $u > 0$ and $v > 0$ therefore the solution is $u = 1 + \sqrt{3}$ and $v = -1 + \sqrt{3}$ [A1]

d. $\overrightarrow{OC} = \overrightarrow{AB} = (\sqrt{3} - 1)\underline{i} + (\sqrt{3} + 1)\underline{j}$

Therefore C is the point $((\sqrt{3} - 1), (\sqrt{3} + 1))$ [A1]

e. $\overrightarrow{OB} = (1 + \sqrt{3})\underline{i} + (-1 + \sqrt{3})\underline{j}$ and $\overrightarrow{AC} = (-3 + \sqrt{3})\underline{i} + (3 + \sqrt{3})\underline{j}$ [A1]

$\overrightarrow{OB} \cdot \overrightarrow{AC} = (1 + \sqrt{3})(-3 + \sqrt{3}) + (-1 + \sqrt{3})(3 + \sqrt{3}) = 0$ [A1]

f.
$$\begin{aligned} \text{Area} &= \frac{1}{2} |\overrightarrow{OA}| |\overrightarrow{OP}| \sin(60^\circ) \\ &= \frac{1}{2} \sqrt{8} \times \frac{1}{3} \sqrt{8} \times \frac{\sqrt{3}}{2} = \frac{2\sqrt{3}}{3} \end{aligned}$$
 [A1]

g. $\overrightarrow{BX} = (\overrightarrow{BP} \cdot \widehat{BA}) \widehat{BA}$ [M1]

$\overrightarrow{BP} = -\frac{2}{3}(\sqrt{3} + 1)\underline{i} - \frac{2}{3}(\sqrt{3} - 1)\underline{j}$ and $\widehat{BA} = \frac{1}{\sqrt{8}}((1 - \sqrt{3})\underline{i} - (1 + \sqrt{3})\underline{j})$

$\widehat{BX} = \frac{1}{3}((1 - \sqrt{3})\underline{i} - (1 + \sqrt{3})\underline{j})$ [A1]

$\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BX} = \frac{2}{3}((\sqrt{3} + 2)\underline{i} + (\sqrt{3} - 2)\underline{j})$

X is the point $(\frac{2}{3}(\sqrt{3} + 2), \frac{2}{3}(\sqrt{3} - 2))$ [A1]

Question 5 (9 marks)

a. For a sample size of 20 $\bar{X} : N(175.5, \frac{90.57^2}{20})$

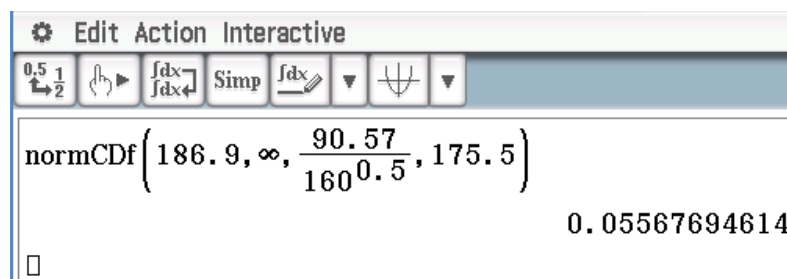
Therefore $\Pr(\bar{X} \leq 150) = 0.103992... = 0.1040$ correct to 4 decimal places [A1]

b.

i. $H_0 : \mu = 175.5$
 $H_1 : \mu > 175.5$ [A1]

ii. For a sample size of 160 $\bar{X} : N(175.5, \frac{90.57^2}{160})$

Therefore $p = \Pr(\bar{X} \geq 186.9) = 0.0557$ [A1]



normCDF(186.9, ∞, $\frac{90.57}{\sqrt{160}}$, 175.5)
 0.05567694614

iii. $p = 0.0557 > 0.05$ therefore the null hypothesis should not be rejected at the 5% level [A1]

c. $T : N(71, 4.7^2)$, for a random person $\Pr(T > 76) = 0.1437$ [A1]

Whereas, for a sample size of 4, $\bar{T} : N(71, \frac{4.7^2}{4})$, and $\Pr(\bar{T} > 74) = 0.1009$. Therefore you would be more likely to find a random person who spends more than 76 mins on Facebook [A1]

d. probability required is $\Pr(2W - Y > 0)$ [M1]

$$E(2W - Y) = 2E(W) - E(Y) = 2 \times 7.1 - 5.8 = 8.4$$

$$\text{VAR}(2W - Y) = 2^2 \text{VAR}(W) + \text{VAR}(Y) = 4 \times 4.7^2 + 3.5^2 = 100.61$$
 [A1]

therefore $2W - Y : N(8.4, 100.61)$

and the probability is $\Pr(2W - Y > 0) = 0.7988$ [A1]

Question 6 (8 marks)

a. $F = 80g - 320v$
 $= 80(g - 4v)$ newtons [A1]

b. $\frac{dv}{dt} = g - 4v$

$$t = \int \frac{1}{g - 4v} dv, \quad t = -\frac{1}{4} \ln|g - 4v| + c$$

$$v = \frac{1}{4}(Ae^{-4t} + g) \quad [M1]$$

When $t = 0$, $v = 58.8$ therefore $v = \frac{1}{4}\left(g + \frac{1127}{5}e^{-4t}\right)$ [A1]

c. $t = 300s$

$$\text{distance travelled} = \int_0^{300} v dt \quad [M1]$$

$$= 749.09m \quad [A1]$$

d. $v \frac{dv}{dx} = g - \frac{v^2}{2}$ [M1]

$$x = \int_{58.8}^v \frac{2v}{2g - v^2} dv, \quad v = \sqrt{(19.6 + 3437.84e^{-x})} \quad [A1]$$

He reaches the ground when $x = 749.09$ therefore his speed $v = 4.43\text{ms}^{-1}$ (correct to 2 decimal places) [A1]