Year 2016

VCE

Specialist Mathematics

Trial Examination 2

Solutions

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SECTION 1

ANSWERS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
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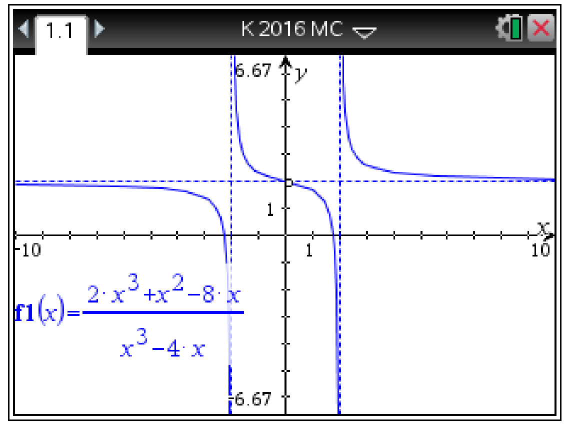
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E

SECTION A

Question 1 **Answer E**



$$f(x)=\frac{2x^3+x^2-8x}{x^3-4x}=\frac{x(2x^2+x-8)}{x(x^2-4)}$$

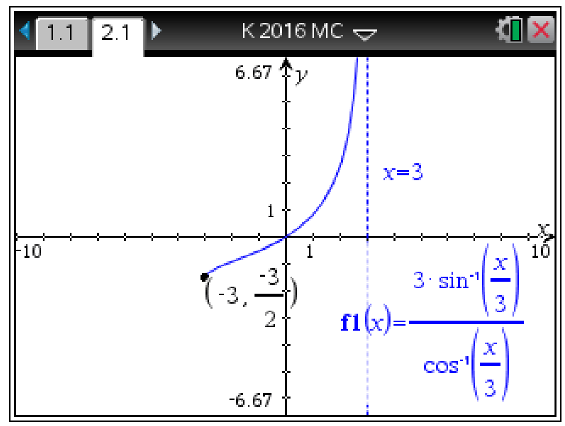
$$=2+\frac{x}{(x-2)(x+2)}$$

vertical asymptotes at $x=-2$ and $x=2$

and a horizontal asymptote at $y=2$,

and a point of discontinuity at $x=0$

Question 2 **Answer B**



The domain of $f(x)=\dfrac{a\sin^{-1}\left(\frac{x}{a}\right)}{\cos^{-1}\left(\frac{x}{a}\right)}$ is $[-a,a)$

$f(-a)=-\dfrac{a}{2}$, $x=a$ is a vertical asymptote,

the range is $\left[-\dfrac{a}{2},\infty\right)$

Question 3 **Answer A**

$|z-a|=|z+ai|$, let $z=x+yi$

$|(x-a)+yi|=|x+(y+a)i|$

$\sqrt{(x-a)^2+y^2}=\sqrt{x^2+(y+a)^2}$

$x^2-2xa+a^2+y^2=x^2+y^2+2ya+a^2$

$a(y+x)=0$ since $a\neq 0$, and $y=\text{Im}(z)$ and $x=\text{Re}(z)$

$\text{Re}(z)+\text{Im}(z)=0$

Alternatively the set of points equidistant from $(a,0)$ and $(0,-a)$ is the line $y=-x$.

Note that **E.** does not include the origin and is therefore incorrect.

Question 4 **Answer C**

$$\frac{x^2}{x^4-a^4}=\frac{x^2}{(x^2-a^2)(x^2+a^2)}=\frac{x^2}{(x-a)(x+a)(x^2+a^2)}$$, since we have the non-linear factor,

the partial fractions are given by $\dfrac{A}{x-a}+\dfrac{B}{x+a}+\dfrac{Cx+D}{x^2+a^2}$

Question 5 **Answer A**

$V=\frac{1}{3}\pi r^2 h$ but $h=2r \Rightarrow r=\frac{h}{2}$ $V=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h=\frac{\pi h^3}{12} \Rightarrow \frac{dV}{dh}=\frac{\pi h^2}{4}$

$\frac{dV}{dt}=\text{inflow}-\text{outflow}=Q-c\sqrt{h}$. By the chain rule $\frac{dt}{dh}=\frac{dt}{dV}\frac{dV}{dh}=\frac{\pi h^2}{4\left(Q-c\sqrt{h}\right)}$

$t=\int_{h_0}^{0}\frac{\pi h^2}{4\left(Q-c\sqrt{h}\right)}dh=\int_{0}^{h_0}\frac{\pi h^2}{4\left(c\sqrt{h}-Q\right)}dh$ by properties of definite integrals

Question 6 **Answer D**

$\underset{\sim}{r}(t)=\cos(t)\underset{\sim}{i}+\cos(3t)\underset{\sim}{j}$

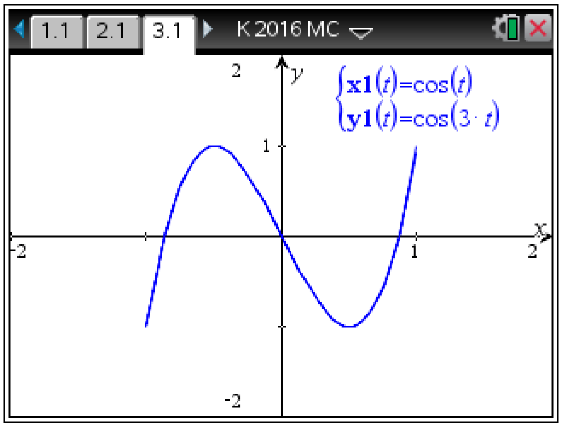
The parametric equations are

$x=\cos(t)$ and $y=\cos(3t)$.

$y=\cos(3t)=4\cos^3(t)-3\cos(t)$

so that $y=4x^3-3x$, since $t\geq 0$ $x\in[-1,1]$

The particle moves on part of a cubic.



Question 7 **Answer B**

when $x=-1$, the gradient m is infinite,
when $y=1$, $m=0$,

only $m=\frac{dy}{dx}=\frac{y-1}{x+1}$ satisfies these conditions.

Question 8 **Answer C**

Initially no x is present, $x(0)=0$, after a time of t, equal parts of x combine, leaving $\left(a-\frac{x}{2}\right)$ and $\left(b-\frac{x}{2}\right)$ of a and b respectively, since $k>0$ and initial the reaction rate is fastest, and slowing down as time goes on, then $\frac{dx}{dt}=k\left(a-\frac{x}{2}\right)\left(b-\frac{x}{2}\right)$, $x(0)=0$

Question 9 **Answer B**

$\int_0^1\frac{x}{\sqrt{b-ax}}dx$ Let $u=b-ax$, $\frac{du}{dx}=-a$ $\Rightarrow dx=\frac{-1}{a}du$ and $x=\frac{1}{a}(b-u)$

terminals, when $x=0$ $u=b$ and when $x=1$ $u=b-a$, then

$$\int_0^1\frac{x}{\sqrt{b-ax}}dx=\int_b^{b-a}\frac{\frac{1}{a}(b-u)}{\sqrt{u}}\times-\frac{1}{a}du=-\frac{1}{a^2}\int_b^{b-a}\frac{b-u}{\sqrt{u}}du=\frac{1}{a^2}\int_{b-a}^{b}\frac{b-u}{\sqrt{u}}du$$

by properties of definite integrals

Question 10 **Answer D**

$\underset{\sim}{v}(t) = 3\cos(2t)\underset{\sim}{i} + \sin(2t)\underset{\sim}{j}$

$\underset{\sim}{a}(t) = -6\sin(2t)\underset{\sim}{i} + 2\cos(2t)\underset{\sim}{j}$

$|\underset{\sim}{a}(t)| = \sqrt{(-6\sin(2t))^2 + (2\cos(2t))^2} = \sqrt{36\sin^2(t) + 4\cos^2(2t)}$

$= \sqrt{36\sin^2(2t) + 4(1-\sin^2(2t))^2} = \sqrt{32\sin^2(2t) + 4}$

when $\sin(2t) = 1$ $\quad |\underset{\sim}{a}(t)|_{\max} = 6$, $m = 3$ $\quad F_{\max} = m|\underset{\sim}{a}(t)|_{\max} = 18$ newtons

Question 11 **Answer E**

$|\underset{\sim}{u}| = 3$ and $|\underset{\sim}{v}| = 4$ and $\underset{\sim}{u}.\underset{\sim}{v} = 1$

$|\underset{\sim}{u} + \underset{\sim}{v}|^2 = (\underset{\sim}{u} + \underset{\sim}{v}).(\underset{\sim}{u} + \underset{\sim}{v}) = \underset{\sim}{u}.\underset{\sim}{u} + \underset{\sim}{v}.\underset{\sim}{u} + \underset{\sim}{u}.\underset{\sim}{v} + \underset{\sim}{v}.\underset{\sim}{v}$

$|\underset{\sim}{u} + \underset{\sim}{v}|^2 = |\underset{\sim}{u}|^2 + 2\underset{\sim}{u}.\underset{\sim}{v} + |\underset{\sim}{v}|^2 = 9 + 2 + 16 = 27 = 9 \times 3$

$|\underset{\sim}{u} + \underset{\sim}{v}| = 3\sqrt{3}$

Question 12 **Answer B**

$\frac{dy}{dx} = y\sec^2(x)$

$\int \frac{1}{y}\,dy = \int \sec^2(x)\,dx$

$\log_e(y) = \tan(x) + c$ when $x = 0$ $y = 2$

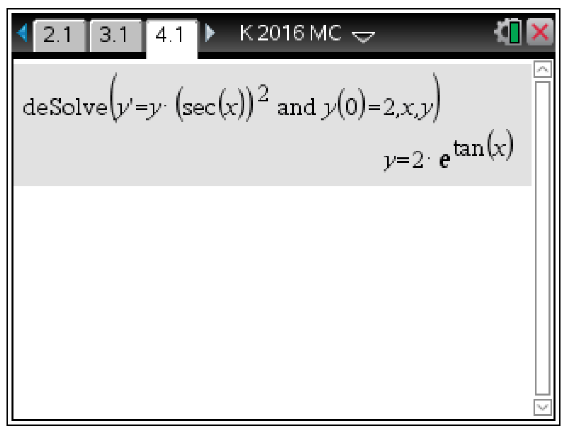
$\log_e(2) = \tan(0) + c \Rightarrow c = \log_e(2)$

$\log_e(y) = \tan(x) + \log_e(2)$

$\log_e(y) - \log_e(2) = \tan(x)$

$\log_e\left(\frac{y}{2}\right) = \tan(x)$

$\frac{y}{2} = e^{\tan(x)} \Rightarrow y = 2e^{\tan(x)}$



Question 13 **Answer C**

$\frac{dy}{dx} = bxy$ where $b \in R \setminus \{0\}$ and $y = 2$ when $x = 1$.

$\frac{dy}{dx} = f(x, y) = bxy \quad y_0 = 2 \quad x_0 = 1 \quad h = \frac{1}{2}$, using Euler's Method

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 2 + \frac{1}{2} \times b \times 1 \times 2 = 2 + b \text{ and } x_1 = \frac{3}{2}$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 2 + b + \frac{1}{2} \times b \times \frac{3}{2} \times (2 + b) = 2 + b + \frac{3b}{4}(2 + b)$$

$$= 2 + b + \frac{3b}{2} + \frac{3b^2}{4} = 2 + \frac{5b}{2} + \frac{3b^2}{4}$$

Question 14 **Answer E**

$$y = \cos\left(\sqrt{x}\right) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-\sin\left(\sqrt{x}\right)}{2\sqrt{x}}$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{-\sin\left(\sqrt{x}\right)}{2\sqrt{x}}\right)^2}\, dx$$

$$s = \int_a^b \sqrt{1 + \frac{\sin^2\left(\sqrt{x}\right)}{4x}}\, dx = \int_a^b \sqrt{\frac{4x + \sin^2\left(\sqrt{x}\right)}{4x}}\, dx = \frac{1}{2}\int_a^b \sqrt{\frac{4x + \sin^2\left(\sqrt{x}\right)}{x}}\, dx$$

Question 15 **Answer C**

$$f(x) = \sqrt{x^4 + 16} \quad \Rightarrow \quad g(x) = \int_0^x \sqrt{u^4 + 16}\, du + c$$

now $g(1) = 3$

$$g(1) = 3 = \int_0^1 \sqrt{u^4 + 16}\, du + c$$

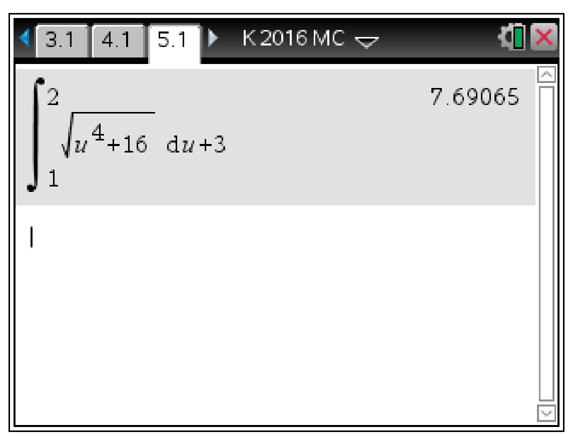
$$\Rightarrow \quad c = 3 - \int_0^1 \sqrt{u^4 + 16}\, du$$

$$g(x) = \int_0^x \sqrt{u^4 + 16}\, du + 3 - \int_0^1 \sqrt{u^4 + 16}\, du$$

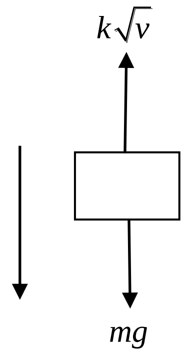
$$g(x) = \int_0^x \sqrt{u^4 + 16}\, du + \int_1^0 \sqrt{u^4 + 16}\, du + 3$$

$$g(x) = \int_1^x \sqrt{u^4 + 16}\, du + 3$$

$$g(2) = \int_1^2 \sqrt{u^4 + 16}\, du + 3 \approx 7.69$$



Question 16 **Answer D**



$m\ddot{x} = mg - k\sqrt{v}$, $v(0) = 0$

$mv\dfrac{dv}{dx} = mg - k\sqrt{v}$

$\dfrac{dv}{dx} = \dfrac{g}{v} - \dfrac{k\sqrt{v}}{mv}$

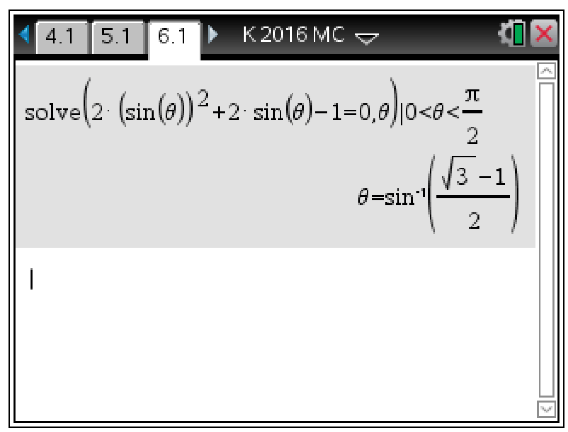
$\dfrac{dv}{dx} = \dfrac{g}{v} - \dfrac{k}{m\sqrt{v}}$ Albert is correct.

When $\ddot{x} = 0$, the terminal velocity is $\left(\dfrac{mg}{k}\right)^2$, Colin is correct.

$\dfrac{dv}{dx} = \dfrac{mg - k\sqrt{v}}{mv} \Rightarrow \dfrac{dx}{dv} = \dfrac{mv}{mg - k\sqrt{v}}$

$x = \displaystyle\int \frac{mv}{mg - k\sqrt{v}}\, dv + c$, so Ben is incorrect.

Question 17 **Answer E**



resolving horizontally

(1) $2F\cos(\theta) + F\sin(2\theta) - P = 0$

resolving vertically

(2) $2F\sin(\theta) - F\cos(2\theta) = 0$

(2) $\Rightarrow F\left(2\sin(\theta) - \cos(2\theta)\right) = 0$

$2\sin(\theta) - \cos(2\theta) = 0$

$2\sin(\theta) - \left(1 - 2\sin^2(\theta)\right) = 0$

$2\sin^2(\theta) + 2\sin(\theta) - 1 = 0$

$\sin(\theta) = \dfrac{\sqrt{3} - 1}{2}$

since $0 < \sin(\theta) < 1$ and $0 < \theta < \dfrac{\pi}{2}$

$\Rightarrow \theta = \sin^{-1}\left(\dfrac{\sqrt{3} - 1}{2}\right)$

Question 18 **Answer A**

Decision Rule	Reality: H_0 true	Reality: H_0 false
Accept H_0	CORRECT	Type 2 ERROR
Reject H_0	Type 1 ERROR	CORRECT

A type 1 error occurs when H_0 is rejected when H_0 is true.

A type 2 error occurs when H_0 is accepted when H_0 is false.

Question 19 **Answer D**

The null hypothesis is what is assumed $H_0: \mu = 20$

The alternative hypothesis is what we are trying to show $H_1: \mu < 20$

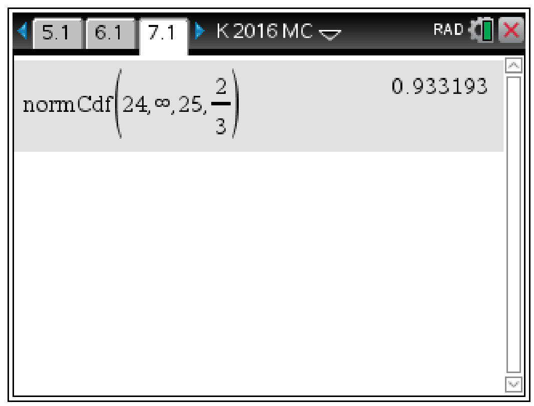
Question 20 **Answer A**

X is the heights of the trees

$X \stackrel{d}{=} N\left(\mu = 25, \sigma^2 = 4^2\right)$, $\sigma_{\bar{X}} = \dfrac{\sigma}{\sqrt{n}}$

$\bar{X} \stackrel{d}{=} N\left(\mu_{\bar{X}} = 25, \sigma_{\bar{X}}{}^2 = \dfrac{4^2}{36}\right) \quad \Rightarrow \quad \sigma_{\bar{X}} = \dfrac{2}{3}$

$\Pr\left(\bar{X} > 24\right) = 0.933$



END OF SECTION A SUGGESTED ANSWERS

SECTION B

Question 1

a.i. $x = 4\sin^2(t)$ $\qquad y = 4\tan(t)\sin^2(t) = \dfrac{4\sin^3(t)}{\cos(t)}$

$\dot{x} = \dfrac{dx}{dt} = 8\cos(t)\sin(t)$ $\qquad$ using the quotient rule M1

$$\dot{y} = \frac{dy}{dt} = \frac{12\sin^2(t)\cos^2(t) + 4\sin^4(t)}{\cos^2(t)}$$

$$= \frac{4\sin^2(t)\left(3\cos^2(t) + \sin^2(t)\right)}{\cos^2(t)}$$

$$= \frac{4\sin^2(t)\left(3\cos^2(t) + 1 - \cos^2(t)\right)}{\cos^2(t)}$$

$$= \frac{4\sin^2(t)\left(2\cos^2(t) + 1\right)}{\cos^2(t)}$$ A1

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{4\sin^2(t)\left(2\cos^2(t)+1\right)}{\cos^2(t)} \times \frac{1}{8\cos(t)\sin(t)}$$

$$= \frac{\sin(t)\left(2\cos^2(t)+1\right)}{2\cos^3(t)}$$ M1

ii. gradient is 2, $\dfrac{dy}{dx} = \dfrac{\sin(t)\left(2\cos^2(t)+1\right)}{2\cos^3(t)} = 2$ solving with $t \in [0, \pi]$

solution by CAS is $t = \dfrac{\pi}{4}$ A1

$x\left(\dfrac{\pi}{4}\right) = 4\sin^2\left(\dfrac{\pi}{4}\right) = 2$, $y\left(\dfrac{\pi}{4}\right) = 4\tan\left(\dfrac{\pi}{4}\right)\sin^2\left(\dfrac{\pi}{4}\right) = 2$

coordinate is $(2, 2)$ A1

b.i. $\underset{\sim}{r}(t)=4\sin^2(t)\underset{\sim}{i}+4\tan(t)\sin^2(t)\underset{\sim}{j}=x(t)\underset{\sim}{i}+y(t)\underset{\sim}{j}$

$$\dot{\underset{\sim}{r}}(t)=8\cos(t)\sin(t)\underset{\sim}{i}+\left(\frac{4\sin^2(t)\left(2\cos^2(t)+1\right)}{\cos^2(t)}\right)\underset{\sim}{j}=\dot{x}(t)\underset{\sim}{i}+\dot{y}(t)\underset{\sim}{j}$$

$$\dot{\underset{\sim}{r}}(t)=4\sin(2t)\underset{\sim}{i}+4\tan^2(t)\left(2\cos^2(t)+1\right)\underset{\sim}{j}$$ M1

$$\dot{\underset{\sim}{r}}\left(\frac{\pi}{4}\right)=4\sin\left(\frac{\pi}{2}\right)\underset{\sim}{i}+4\tan^2\left(\frac{\pi}{4}\right)\left(2\cos^2\left(\frac{\pi}{4}\right)+1\right)\underset{\sim}{j}$$

$$\dot{\underset{\sim}{r}}\left(\frac{\pi}{4}\right)=4\underset{\sim}{i}+8\underset{\sim}{j}$$

$$\left|\dot{\underset{\sim}{r}}\left(\frac{\pi}{4}\right)\right|=\sqrt{16+64}=4\sqrt{5}$$ A1

ii. $x=4\sin^2(t)$

$$RHS=\frac{x^3}{4-x}$$

$$=\frac{64\sin^6(t)}{4-4\sin^2(t)}$$

$$=\frac{64\sin^6(t)}{4\left(1-\sin^2(t)\right)}$$ M1

$$=\frac{16\sin^2(t)\sin^4(t)}{\cos^2(t)}$$

$$=16\tan^2(t)\sin^4(t)=y^2=LHS$$ A1

Define $x(t)=4\cdot(\sin(t))^2$	*Done*
Define $y(t)=4\cdot\tan(t)\cdot(\sin(t))^2$	*Done*
$\frac{d}{dt}(x(t))$	$8\cdot\sin(t)\cdot\cos(t)$
$\frac{d}{dt}(y(t))$	$4\cdot(\tan(t))^2\cdot(2\cdot(\cos(t))^2+1)$
$\dfrac{\frac{d}{dt}(y(t))}{\frac{d}{dt}(x(t))}$	$\dfrac{\sin(t)\cdot(2\cdot(\cos(t))^2+1)}{2\cdot(\cos(t))^3}$
$\text{solve}\left(\dfrac{\sin(t)\cdot(2\cdot(\cos(t))^2+1)}{2\cdot(\cos(t))^3}=2,t\right)\mid 0\le t\le\pi$	$t=0.785398$
$\frac{\pi}{4}$	0.785398
$x\left(\frac{\pi}{4}\right)$	2
$y\left(\frac{\pi}{4}\right)$	2

Define $r(t)=[x(t)\ \ y(t)]$	*Done*
$\text{norm}\left(\frac{d}{dt}(r(t))\right)\mid t=\frac{\pi}{4}$	$4\cdot\sqrt{5}$

Question 2

a. $f(x)=\sqrt{\dfrac{x^3}{4-x}}$ for $x\in[0,4)$

$f'(x)=\dfrac{\sqrt{x}(6-x)}{(4-x)^{\frac{3}{2}}}$ $a=6$, $n=\dfrac{3}{2}$ A1

b. $f''(x)=\dfrac{12}{\sqrt{x}(4-x)^{\frac{5}{2}}}$ $b=12$, $m=\dfrac{5}{2}$ A1

c. For stationary points $f'(x)=0$

$x=6$ but the maximal domain of the function is $x\in[0,4)$

$x=0$ but the gradient function is not defined at the end-points,

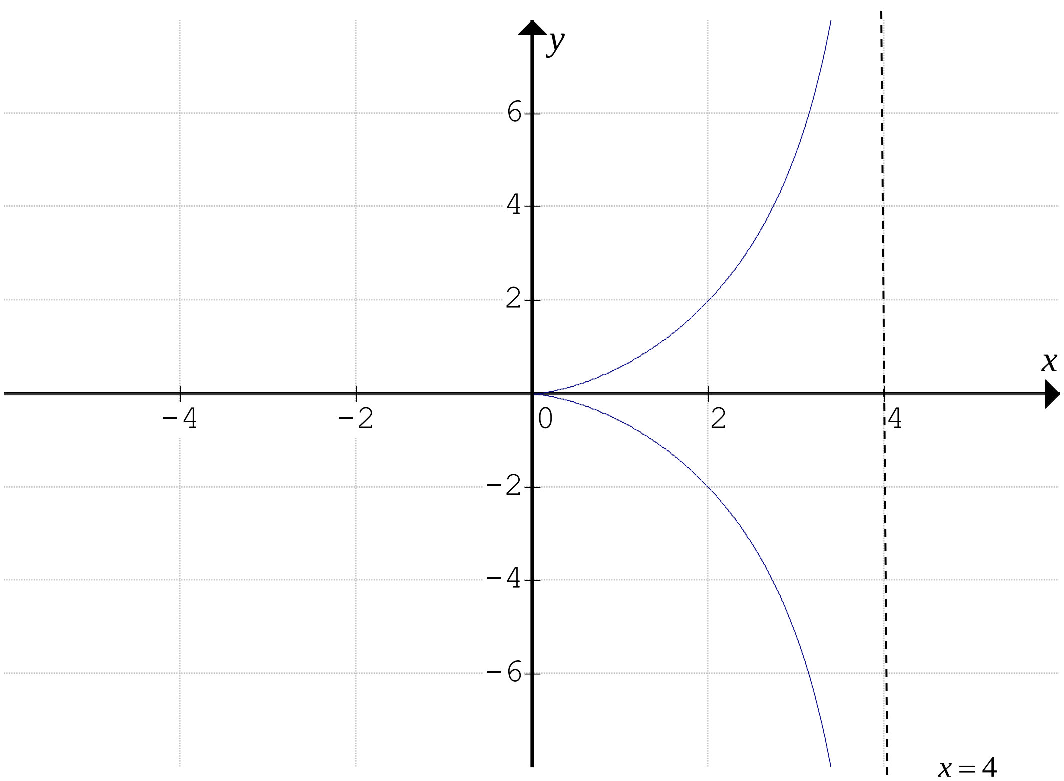
$f'(x)$ is defined for $x\in(0,4)$, so there are no stationary points. A1

$f''(x)\neq 0$ so there are no points of inflexion. A1

d. $y^2=\dfrac{x^3}{4-x}$ $\Rightarrow$ $y=\pm\sqrt{\dfrac{x^3}{4-x}}$ reflection in the x-axis

$x=4$ is a vertical asymptote. Correct behaviour at the origin, A1

correct shape and the graphs must pass through $(2,2)$ and $(2,-2)$, from **Q1.a.ii** A1



e.i. $RHS = \dfrac{64}{4-x} - x^2 - 4x - 16 = \dfrac{64}{4-x} - \left(x^2 + 4x + 16\right)$

$= \dfrac{64 - \left(x^2 + 4x + 16\right)\left(4 - x\right)}{4-x}$

$= \dfrac{64 - \left(4x^2 + 16x + 64\right) + \left(x^3 + 4x^2 + 16x\right)}{4-x}$

$= \dfrac{x^3}{4-x} = LHS$ alternatively use long division M1

ii. $V = \pi \int_a^b y^2 dx$

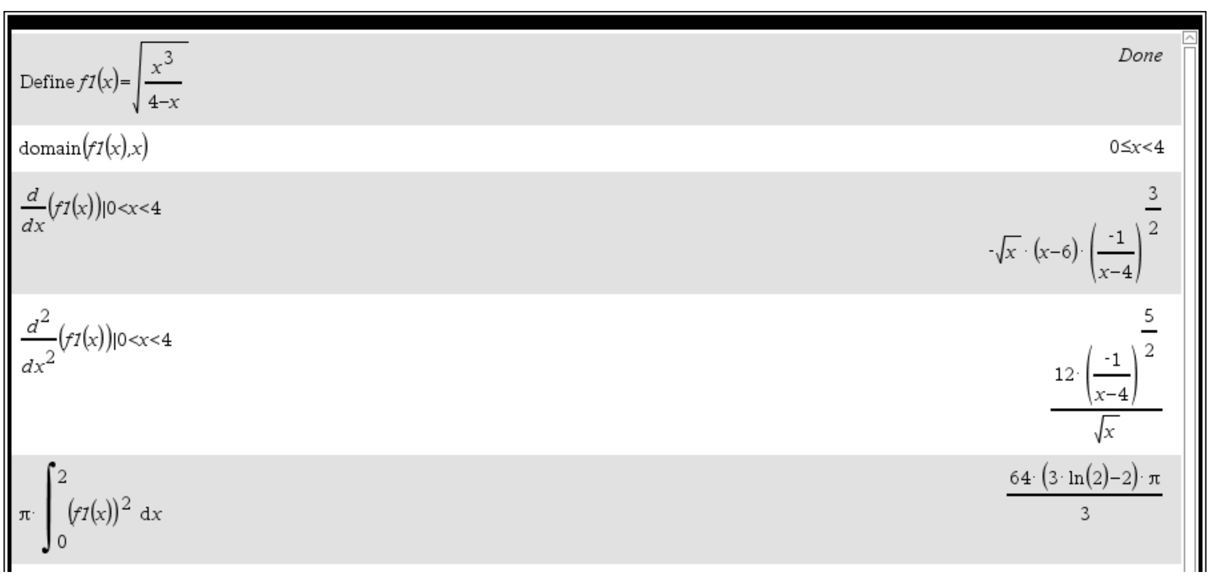
$V = \pi \int_0^2 \dfrac{x^3}{4-x} dx$

$= \pi \int_0^2 \left(\dfrac{64}{4-x} - x^2 - 4x - 16 \right) dx$

$= \pi \left[-64 \log_e\left(4-x\right) - \dfrac{1}{3}x^3 - 2x^2 - 16x \right]_0^2$ A1

$= \pi \left(-64 \log_e\left(2\right) - \dfrac{8}{3} - 8 - 32 + 64 \log_e\left(4\right) \right)$

$= \pi \left(64 \log_e\left(2\right) - \dfrac{128}{3} \right) \Rightarrow c = 64 \,,\; p = 128 \,,\; q = 3$ A1



Question 3

a. $\overrightarrow{OA} = -4\underset{\sim}{i} + 4\underset{\sim}{j}$, $\overrightarrow{OB} = -\left(3+\sqrt{3}\right)\underset{\sim}{i} + \left(1+\sqrt{3}\right)\underset{\sim}{j}$, $\overrightarrow{OC} = -\left(1+\sqrt{3}\right)\underset{\sim}{i} + \left(3+\sqrt{3}\right)\underset{\sim}{j}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(1-\sqrt{3}\right)\underset{\sim}{i} + \left(\sqrt{3}-3\right)\underset{\sim}{j}$ A1

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(3-\sqrt{3}\right)\underset{\sim}{i} + \left(\sqrt{3}-1\right)\underset{\sim}{j}$ A1

b. $\left|\overrightarrow{AB}\right| = \sqrt{\left(1-\sqrt{3}\right)^2 + \left(\sqrt{3}-3\right)^2} = \sqrt{1-2\sqrt{3}+3+3-6\sqrt{3}+9} = \sqrt{16-8\sqrt{3}}$

$\left|\overrightarrow{AB}\right| = 2\sqrt{4-2\sqrt{3}}$ A1

$\left|\overrightarrow{AC}\right| = \sqrt{\left(3-\sqrt{3}\right)^2 + \left(\sqrt{3}-1\right)^2} = \sqrt{9-6\sqrt{3}+3+3-2\sqrt{3}+1} = \sqrt{16-8\sqrt{3}}$

$\left|\overrightarrow{AC}\right| = 2\sqrt{4-2\sqrt{3}}$ A1

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 2\underset{\sim}{i} + 2\underset{\sim}{j}$

$\left|\overrightarrow{BC}\right| = 2\sqrt{2}$

since $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AC}\right| \neq \left|\overrightarrow{BC}\right| \Rightarrow ABC$ is an isosceles triangle A1

c. $\cos(\theta) = \dfrac{\overrightarrow{AB}.\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$

$\cos(\theta) = \dfrac{3-3\sqrt{3}-\sqrt{3}+3+3-\sqrt{3}-3\sqrt{3}+3}{4\left(4-2\sqrt{3}\right)}$ M1

$= \dfrac{4\left(3-2\sqrt{3}\right)}{4\left(4-2\sqrt{3}\right)} = \dfrac{3-2\sqrt{3}}{2\left(2-\sqrt{3}\right)} \times \dfrac{2+\sqrt{3}}{2+\sqrt{3}} = \dfrac{6-4\sqrt{3}+3\sqrt{3}-6}{2(4-3)} = -\dfrac{\sqrt{3}}{2}$ M1

$\theta = \cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ A1

$\theta = \dfrac{5\pi}{6}$ $\left(\text{or } 150^0\right)$

d. $\text{Area} = \dfrac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin(\theta)$

$= \dfrac{1}{2}\left(16-8\sqrt{3}\right)\sin\left(\dfrac{5\pi}{6}\right)$

$= 2\left(2-\sqrt{3}\right)$ A1

$a:=[-4 \quad 4]$	$[-4 \quad 4]$
$b:=[-(3+\sqrt{3}) \quad 1+\sqrt{3}]$	$[-(\sqrt{3}+3) \quad \sqrt{3}+1]$
$c:=[-(1+\sqrt{3}) \quad 3+\sqrt{3}]$	$[-(\sqrt{3}+1) \quad \sqrt{3}+3]$
$ab:=b-a$	$[1-\sqrt{3} \quad \sqrt{3}-3]$
$ac:=c-a$	$[3-\sqrt{3} \quad \sqrt{3}-1]$
$bc:=c-b$	$[2 \quad 2]$
$\text{norm}(ab)$	$2\cdot\sqrt{3}-2$
$\text{norm}(ac)$	$2\cdot\sqrt{3}-2$
$\text{norm}(bc)$	$2\cdot\sqrt{2}$
$\cos^{-1}\left(\frac{\text{dotP}(ab,ac)}{\text{norm}(ab)\cdot\text{norm}(ac)}\right)$	$\frac{5\cdot\pi}{6}$
$\frac{1}{2}\cdot\text{norm}(ab)\cdot\text{norm}(ac)\cdot\sin\left(\frac{5\cdot\pi}{6}\right)$	$-2\cdot(\sqrt{3}-2)$

Question 4

a. The complex number $b=-(3+\sqrt{3})+(1+\sqrt{3})i$ is in the second quadrant.

$$\text{Arg}(b)=\pi-\tan^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\right)$$

$$=\pi-\tan^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\times\frac{3-\sqrt{3}}{3-\sqrt{3}}\right)=\pi-\tan^{-1}\left(\frac{3+3\sqrt{3}-\sqrt{3}-3}{9-3}\right)$$ M1

$$=\pi-\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

$$=\pi-\frac{\pi}{6}$$ A1

$$=\frac{5\pi}{6}$$

$$|b|=\sqrt{(3+\sqrt{3})^2+(1+\sqrt{3})^2}=\sqrt{9+6\sqrt{3}+3+1+2\sqrt{3}+3}=\sqrt{16+8\sqrt{3}}=2(\sqrt{3}+1)$$

$$b=2(\sqrt{3}+1)\text{cis}\left(\frac{5\pi}{6}\right)$$ A1

b. $S=\{z:|z-a|=2(\sqrt{3}-1)\}$, let $z=x+yi$

$$|(x+4)+(y-4)i|=2(\sqrt{3}-1)$$

$$(x+4)^2+(y-4)^2=\left(2(\sqrt{3}-1)\right)^2$$ M1

S is a circle with centre $(-4,4)$, radius $2(\sqrt{3}-1)$ A1

c. $T = \{ z : \operatorname{Arg}(z) = \frac{5\pi}{6} \}$, let $z = x + yi$

T is the ray from the origin not included, making an angle of $\frac{5\pi}{6}$ with the positive real axes A1

$\tan^{-1}\left(\frac{y}{x}\right) = \frac{5\pi}{6}$ for $x < 0$ and $y > 0$

$\frac{y}{x} = \tan\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

$y = -\frac{\sqrt{3}\,x}{3}$ for $x < 0$ and $y > 0$ A1

d. $b = -\left(3+\sqrt{3}\right) + \left(1+\sqrt{3}\right)i$ and $a = -4 + 4i$

Now $b - a = \left(1-\sqrt{3}\right) + \left(\sqrt{3}-3\right)i$ and $|b-a| = 2\left(\sqrt{3}-1\right)$ from **Q3** or using CAS

so b lies on the circle S $|z-a| = 2\left(\sqrt{3}-1\right)$ A1

since $b = 2\left(\sqrt{3}+1\right)\operatorname{cis}\left(\frac{5\pi}{6}\right)$ and $\operatorname{Arg}(b) = \frac{5\pi}{6}$ so b lies on the ray T

so $b \in S \cap T$. The ray T is a tangent to the circle S, touching at b. A1

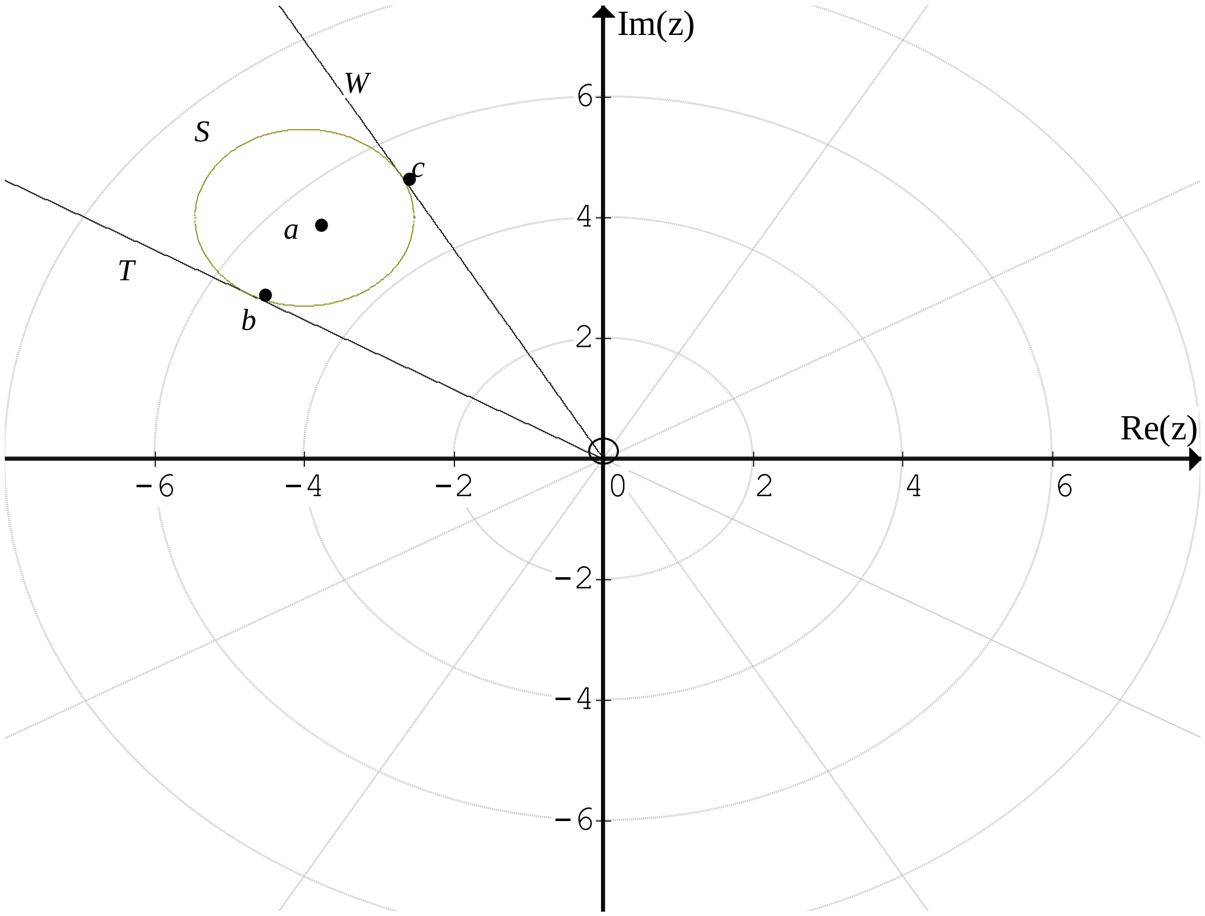
e. $c = -\left(1+\sqrt{3}\right) + \left(3+\sqrt{3}\right)i = 2\left(\sqrt{3}+1\right)\operatorname{cis}\left(\frac{2\pi}{3}\right)$ A1

f. Now $c - a = \left(3-\sqrt{3}\right) + \left(\sqrt{3}-1\right)i$ and $|c-a| = 2\left(\sqrt{3}-1\right)$ from **Q4** or using CAS so c lies on the circle S, $|z-a| = 2\left(\sqrt{3}-1\right)$.

$W = \{ z : \operatorname{Arg}(z) = \theta \}$ is the ray from the origin not included, making an angle of θ with the positive real axes. Since $c \in S \cap W$, so c lies on the ray W and the ray W is a tangent to the circle S, touching at c, since $\operatorname{Arg}(c) = \frac{2\pi}{3}$

$\theta = \frac{2\pi}{3}$ A1

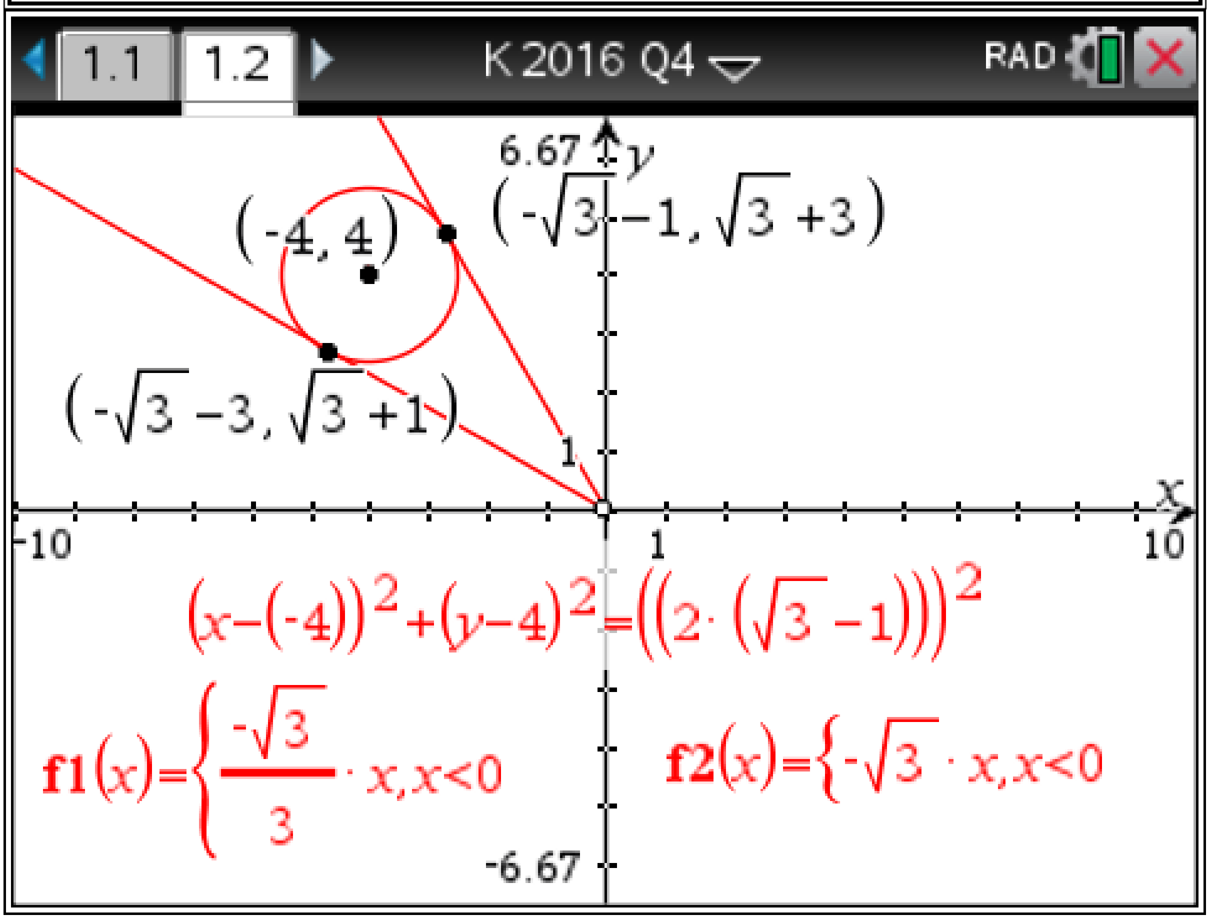
g. correct points *a*,*b*,*c*, rays and circle and open circle at origin. G3



h. $|u|_{\max}$ represents the point on the circle S furthest from the origin, this distance is the magnitude of the complex number a plus the radius of the circle S. Since $|a| = 4\sqrt{2}$ and $r = 2\left(\sqrt{3}-1\right)$

$$|u|_{\max} = |a| + r = 4\sqrt{2} + 2\left(\sqrt{3}-1\right) = 2\left(2\sqrt{2}+\sqrt{3}-1\right)$$ A1

$b:=-(3+\sqrt{3})+(1+\sqrt{3})\cdot i$	$-(\sqrt{3}+3)+(\sqrt{3}+1)\cdot i$
$\text{angle}(b)$	$\frac{5\cdot\pi}{6}$
$\lvert b\rvert$	$2\cdot\sqrt{3}+2$
$c:=-(1+\sqrt{3})+(3+\sqrt{3})\cdot i$	$-(\sqrt{3}+1)+(\sqrt{3}+3)\cdot i$
$\text{angle}(c)$	$\frac{2\cdot\pi}{3}$
$\lvert c\rvert$	$2\cdot\sqrt{3}+2$
$a:=-4+4\cdot i$	$-4+4\cdot i$
$b-a$	$1-\sqrt{3}+(\sqrt{3}-3)\cdot i$
$\lvert b-a\rvert$	$2\cdot\sqrt{3}-2$
$c-a$	$3-\sqrt{3}+(\sqrt{3}-1)\cdot i$
$\lvert c-a\rvert$	$2\cdot\sqrt{3}-2$
Define $f1(x)=\frac{-\sqrt{3}}{3}\cdot x\vert x<0$	*Done*
Define $f2(x)=-\sqrt{3}\cdot x\vert x<0$	*Done*



Question 5

a.i. $\dfrac{dN}{dt}=\dfrac{N}{4}\left(1-\dfrac{N}{500}\right)=\dfrac{N(500-N)}{2000}$ inverting $\dfrac{dt}{dN}=\dfrac{2000}{N(500-N)}$

$t=\displaystyle\int\frac{2000}{N(500-N)}dN$ A1

ii. using partial fractions

$\dfrac{2000}{N(500-N)}=\dfrac{A}{N}+\dfrac{B}{500-N}=\dfrac{A(500-N)+BN}{N(500-N)}=\dfrac{N(B-A)+500A}{N(500-N)}$ M1

$(1)\ 500A=2000 \quad (2)\ B-A=0 \quad \Rightarrow A=B=4$

$t=4\displaystyle\int\left(\frac{1}{N}+\frac{1}{(500-N)}\right)dN$ since $50\le N<500$ no need for modulus

$\dfrac{t}{4}=\log_e(N)-\log_e(500-N)+c=\log_e\left(\dfrac{N}{500-N}\right)+c$

now when $t=0\ \ N=50\quad 0=\log_e\left(\dfrac{50}{450}\right)+c\quad\Rightarrow c=-\log_e\left(\dfrac{1}{9}\right)$ M1

$\dfrac{t}{4}=\log_e\left(\dfrac{N}{500-N}\right)-\log_e\left(\dfrac{1}{9}\right)=\log_e\left(\dfrac{N}{500-N}\right)+\log_e(9)=\log_e\left(\dfrac{9N}{500-N}\right)$

$\dfrac{9N}{500-N}=e^{\frac{t}{4}}\qquad\Rightarrow\quad e^{-\frac{t}{4}}=\dfrac{500-N}{9N}$

M1

$9Ne^{-\frac{t}{4}}=500-N\qquad\Rightarrow N\left(1+9e^{-\frac{t}{4}}\right)=500$

$N=N(t)=\dfrac{500}{1+9e^{-\frac{t}{4}}}$ A1

b. as $t\to\infty\quad N\to 500$ A1

c. $\dfrac{dN}{dt}=\dfrac{1}{2000}\left(500N-N^2\right)$

$\dfrac{d^2N}{dt^2}=\dfrac{d}{dt}\left(\dfrac{dN}{dt}\right)=\dfrac{d}{dN}\left(\dfrac{dN}{dt}\right)\dfrac{dN}{dt}$

M1

$=\dfrac{1}{2000}(500-2N)\dfrac{dN}{dt}$

$=\dfrac{N(500-2N)(500-N)}{4{,}000{,}000}=\dfrac{N(250-N)(500-N)}{2{,}000{,}000}$ A1

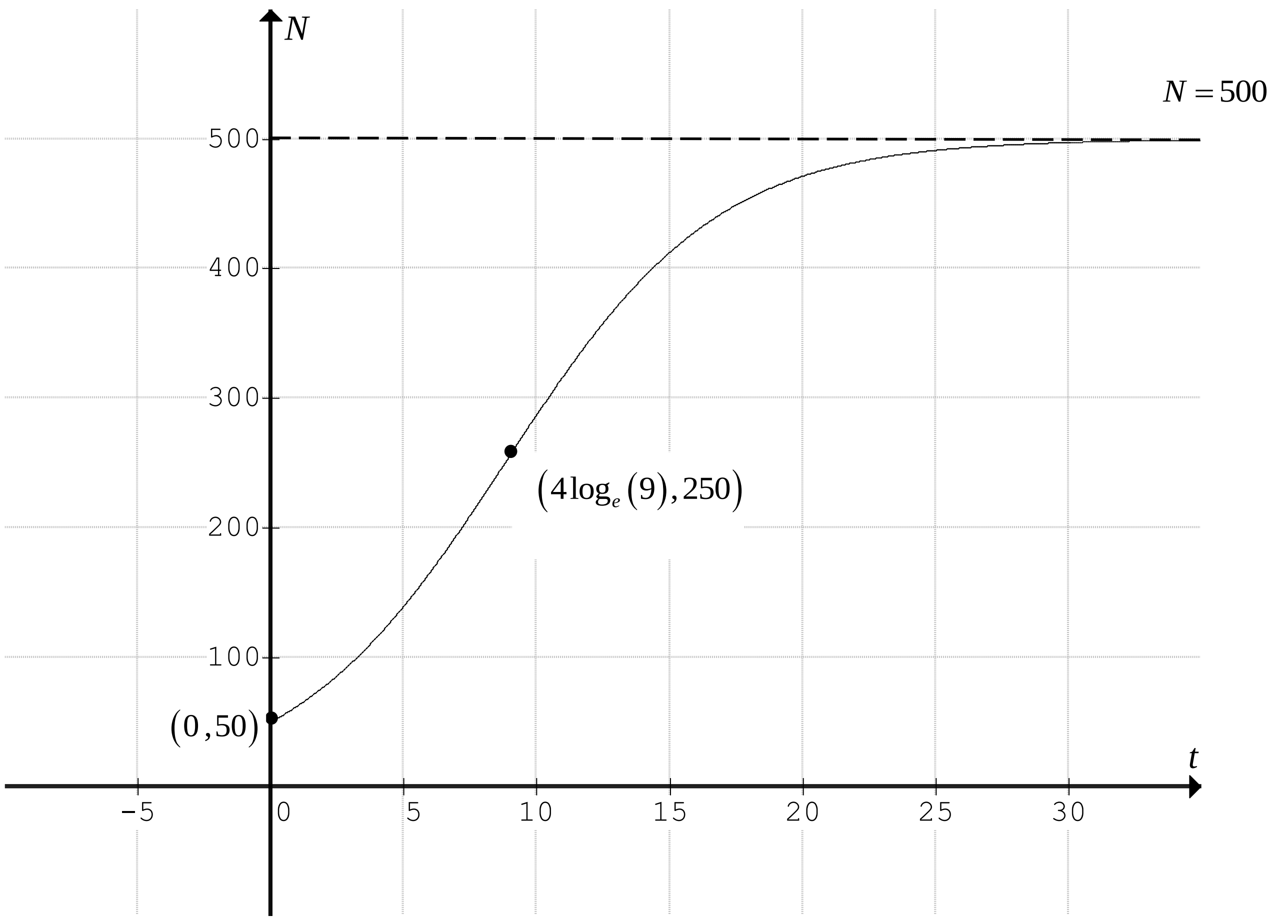
d. since $50 \le N < 500$, inflexion points $\frac{d^2N}{dt^2} = 0 \;\; \Rightarrow N = 250$

when $N = 250$ solving for $t = 4\log_e\left(\frac{9 \times 250}{500 - 250}\right) = 4\log_e(9) \approx 8.8$

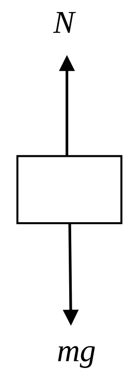
inflexion point $(4\log_e(9), 250)$ A1

correct graph, shape for $t \ge 0$, $N = 500$ is a horizontal asymptote, G1

and passing through $(0, 50)$



Question 6



a. going up $650 - mg = ma \Rightarrow (1)\ 650 = m(g+a)$ A1

going down $mg - 624 = ma \Rightarrow (2)\ 624 = m(g-a)$ A1

$(1)+(2)\ \ 1274 = 2mg$

$m = \dfrac{1274}{2\times 9.8} = 65\,\text{kg}$ substituting $a = 0.2\ \text{m/s}^2$ A1

b. Let M be the distribution of males, $M \stackrel{d}{=} N\left(85, 15^2\right)$,

let F be the distribution of females, $F \stackrel{d}{=} N\left(65, 20^2\right)$.

Let T be the total weight of 12 males and 8 females.

$T = 12M + 8F$

$E(T) = 12E(M) + 8E(F)$

$= 12\times 85 + 8\times 65$ M1

$= 1540$

$\text{Var}(T) = 12^2\,\text{Var}(M) + 8^2\,\text{Var}(F)$

$= 12^2 \times 15^2 + 8^2 \times 20^2$ M1

$= 58000$

$T \stackrel{d}{=} N(1540, 58000)$ A1

$\Pr(T > 1500) = 0.5660$

c. $\bar{x} = 1000$, $s = 250$, $n = 30$,

95% $\alpha = 0.05 \Rightarrow z_{0.025} = 1.96$

$\bar{x} - 1.96\dfrac{s}{\sqrt{n}} \le \mu \le \bar{x} + 1.96\dfrac{s}{\sqrt{n}}$ M1

$1000 - \dfrac{1.96\times 250}{\sqrt{30}} \le \mu \le 1000 + \dfrac{1.96\times 250}{\sqrt{30}}$

$910.54 \le \mu \le 1089.46$ A1

$\text{solve}(650=m\cdot(g+a) \text{ and } 624=m\cdot(g-a),\{m,a\})\vert g=9.8$	$a=0.2$ and $m=65.$
$\text{normCdf}(1500,\infty,1540,\sqrt{58000})$	0.565957
zInterval 250,1000,30,0.95: *stat.results*	"Title" "z Interval"; "CLower" 910.54; "CUpper" 1089.46; "$\bar{x}$" 1000.; "ME" 89.4597; "n" 30.; "σ" 250.

END OF SECTION B SUGGESTED ANSWERS