

Year 2016

VCE

Specialist Mathematics

Trial Examination 1

Solutions



KILBAHA MULTIMEDIA PUBLISHING
PO BOX 2227
KEW VIC 3101
AUSTRALIA

TEL: (03) 9018 5376
FAX: (03) 9817 4334
kilbaha@gmail.com
<http://kilbaha.com.au>

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Question 1

Let $\underline{a} = 4\hat{i} - 2\hat{j} - 2\hat{k}$, $\underline{b} = 3\hat{i} - 4\hat{j} + 5\hat{k}$

$$|\underline{b}| = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2} \quad , \quad \hat{\underline{b}} = \frac{1}{5\sqrt{2}}(3\hat{i} - 4\hat{j} + 5\hat{k})$$

$$\underline{a} \cdot \underline{b} = 12 + 8 - 10 = 10$$

A1

resolving $\underline{a}$ parallel to $\underline{b}$

$$(\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}} = \left(\frac{\underline{a} \cdot \underline{b}}{|\underline{b}|} \right) \hat{\underline{b}} = \frac{10}{5\sqrt{2}} \left(\frac{1}{5\sqrt{2}} (3\hat{i} - 4\hat{j} + 5\hat{k}) \right)$$

$$(\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}} = \frac{1}{5}(3\hat{i} - 4\hat{j} + 5\hat{k})$$

A1

resolving $\underline{a}$ perpendicular to $\underline{b}$

$$\begin{aligned} \underline{a} - (\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}} &= (4\hat{i} - 2\hat{j} - 2\hat{k}) - \frac{1}{5}(3\hat{i} - 4\hat{j} + 5\hat{k}) \\ &= \frac{1}{5}(5(4\hat{i} - 2\hat{j} - 2\hat{k}) - (3\hat{i} - 4\hat{j} + 5\hat{k})) \\ &= \frac{1}{5}(17\hat{i} - 6\hat{j} - 15\hat{k}) \end{aligned}$$

A1

Question 2

a. $z = \sqrt{5} \operatorname{cis}\left(-\frac{\pi}{2}\right) = -\sqrt{5}i$ by the conjugate root theorem, $\bar{z} = \sqrt{5}i$ is also a root

$$(z - \sqrt{5}i)(z + \sqrt{5}i) = (z^2 - 5i^2)$$

$$= (z^2 + 5) \text{ is the quadratic factor}$$

A1

b. $f(z) = z^4 - 6z^3 + 17z^2 - 30z + 60$

$$= (z^2 + 5)(z^2 - 6z + 12) = 0$$

$$= (z - \sqrt{5}i)(z + \sqrt{5}i)(z^2 - 6z + 9 + 3) = 0$$

M1

$$= (z - \sqrt{5}i)(z + \sqrt{5}i)((z-3)^2 - 3i^2) = 0$$

$$= (z - \sqrt{5}i)(z + \sqrt{5}i)(z - 3 - \sqrt{3}i)(z - 3 + \sqrt{3}i) = 0$$

$$\text{all the roots are } z = \pm\sqrt{5}i, 3 \pm \sqrt{3}i$$

A1

Question 3

$$\frac{dy}{dx} = \frac{24y^2}{9+4x^2} \text{ and } y\left(\frac{3}{2}\right) = 1$$

$$\text{using variables separable } \int \frac{1}{y^2} dy = \int \frac{24}{9+4x^2} dx, \text{ integrating} \quad \text{M1}$$

$$-\frac{1}{y} = 4 \tan^{-1}\left(\frac{2x}{3}\right) + c \text{ using } x = \frac{3}{2} \text{ when } y = 1 \quad \text{A1}$$

$$-1 = 4 \tan^{-1}(1) + c \Rightarrow -1 = \pi + c \Rightarrow c = -1 - \pi$$

$$-\frac{1}{y} = 4 \tan^{-1}\left(\frac{2x}{3}\right) - 1 - \pi$$

$$y = \frac{1}{\pi + 1 - 4 \tan^{-1}\left(\frac{2x}{3}\right)} \quad \text{A1}$$

Question 4

a. $y = x^2 - 16$

$$x^2 = y + 16$$

$$V_y = \pi \int_a^b x^2 dy$$

$$V = \pi \int_0^{20} (y + 16) dy \quad \text{M1}$$

$$= \pi \left[\frac{1}{2} y^2 + 16y \right]_0^{20} = \pi \left[\left(\frac{1}{2} \times 400 + 16 \times 20 \right) - 0 \right]$$

$$= 520\pi \text{ cm}^3 \quad \text{A1}$$

b. $V(h) = \pi \int_0^h (y + 16) dy$

$$\frac{dV}{dh} = \pi(h + 16), \quad \frac{dV}{dt} = 2 \quad \text{A1}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \frac{dV}{dt} = \frac{2}{\pi(h + 16)} \quad \text{M1}$$

$$\left. \frac{dh}{dt} \right|_{h=14} = \frac{2}{\pi(14 + 16)}$$

$$= \frac{1}{15\pi} \text{ cm/sec} \quad \text{A1}$$

Question 5

a.
$$\int \frac{\sin(2x)}{\cos^2(2x)} dx \quad \text{let } u = \cos(2x) \quad \frac{du}{dx} = -2\sin(2x) \quad , \quad dx = \frac{-1}{2\sin(2x)} du$$

$$\int \frac{\sin(2x)}{\cos^2(2x)} dx = \int \frac{\sin(2x)}{u^2} \times \frac{-1}{2\sin(2x)} du$$

$$= -\frac{1}{2} \int \frac{1}{u^2} du = -\frac{1}{2} \int u^{-2} du \quad \text{M1}$$

$$= \frac{1}{2} u^{-1} + c = \frac{1}{2u} + c$$

$$= \frac{1}{2\cos(2x)} + c = \frac{1}{2} \sec(2x) + c$$

b.i.
$$\dot{y}(t) = 4 \tan(2t) \sec(2t) \hat{i} + 6 \sec^2(2t) \hat{j}$$

$$\dot{z}(t) = \frac{4 \sin(2t)}{\cos(2t)} \times \frac{1}{\cos(2t)} \hat{i} + 6 \sec^2(2t) \hat{j}$$

$$= \frac{4 \sin(2t)}{\cos^2(2t)} \hat{i} + 6 \sec^2(2t) \hat{j}$$

$$\dot{z}(t) = \int \frac{4 \sin(2t)}{\cos^2(2t)} dt \hat{i} + \int 6 \sec^2(2t) dt \hat{j} \quad \text{M1}$$

$$\dot{z}(t) = 2 \sec(2t) \hat{i} + 3 \tan(2t) \hat{j} + \underline{c}$$

$$\dot{z}(0) = 3 \hat{i} + 2 \hat{j} = 2 \hat{i} + \underline{c} \Rightarrow \underline{c} = \hat{i} + 2 \hat{j}$$

$$\dot{z}(t) = (2 \sec(2t) + 1) \hat{i} + (3 \tan(2t) + 2) \hat{j} \quad \text{A1}$$

ii. The distance is $s = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2} dt$

$$a = 0, b = \frac{\pi}{5}, \quad \dot{x} = 4 \tan(2t) \sec(2t), \quad \dot{y} = 6 \sec^2(2t)$$

$$s = \int_0^{\frac{\pi}{5}} \sqrt{16 \tan^2(2t) \sec^2(2t) + 36 \sec^4(2t)} dt$$

$$s = \int_0^{\frac{\pi}{5}} \sqrt{4 \sec^2(2t) (4 \tan^2(2t) + 9 \sec^2(2t))} dt \quad \text{M1}$$

$$s = \int_0^{\frac{\pi}{5}} \sqrt{4 \sec^2(2t) (4 \tan^2(2t) + 9(1 + \tan^2(2t)))} dt$$

$$s = \int_0^{\frac{\pi}{5}} 2 \sec(2t) \sqrt{13 \tan^2(2t) + 9} dt \quad \text{A1}$$

$$p = 2, q = 13, r = 9$$

Question 6

$$y^2(4-x) = x^3 \text{ expanding } 4y^2 - xy^2 - x^3 = 0$$

using implicit differentiation and the product rule on the second term.

$$\frac{d}{dx}(4y^2) - \frac{d}{dx}(xy^2) - \frac{d}{dx}(x^3) = 0$$

$$8y \frac{dy}{dx} - y^2 - 2xy \frac{dy}{dx} - 3x^2 = 0 \quad \text{M1}$$

$$(8y - 2xy) \frac{dy}{dx} = y^2 + 3x^2$$

$$\frac{dy}{dx} = \frac{y^2 + 3x^2}{8y - 2xy} \quad \text{A1}$$

$$\text{Now if } x=2 \Rightarrow 2y^2=8, y^2=4$$

$$\text{but in the fourth quadrant } y < 0 \text{ so } y = -2, (2, -2) \quad \text{A1}$$

$$m_T = \left. \frac{dy}{dx} \right|_{(2,-2)} = \frac{4+12}{-16+8} = -2$$

$$\text{gradient of the normal } m_N = \frac{1}{2} \quad \text{A1}$$

Question 7

$$H_0: \mu = 100$$

$$H_A: \mu \neq 100 \quad 2 \text{ sided test} \quad \text{A1}$$

$$\bar{x} = 103, \mu = 100, \sigma = 15, n = 36$$

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{103 - 100}{\frac{15}{\sqrt{36}}} = \frac{3}{\frac{15}{6}} = \frac{18}{15} = \frac{6}{5} = 1.2 \quad \text{A1}$$

$$\text{At the 95\% level } z = \pm 1.96, \text{ the calculated value } z = 1.2 \text{ is in the} \quad \text{M1}$$

acceptable region, there is no evidence to support the alternative

hypothesis, accept the null hypothesis, students are average IQ A1

Question 8

$$F = ma$$

$$-12x = 3a$$

$$a = \ddot{x} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right) = -4x$$

$$\frac{1}{2} v^2 = \int -4x dx = -2x^2 + c_1 \quad \text{M1}$$

$$\text{when } t = 0, v = 0, x = 4 \quad 0 = -32 + c_1 \Rightarrow c_1 = 32$$

$$v^2 = 64 - 4x^2 = 4(16 - x^2) \quad \text{A1}$$

$$v = \frac{dx}{dt} = \pm 2\sqrt{16 - x^2}$$

take the positive and separate the variables, note that if we take the negative, we obtain the same final answer.

$$2t = \int \frac{1}{\sqrt{16 - x^2}} dx$$

$$2t = \sin^{-1} \left(\frac{x}{4} \right) + c_2$$

$$\text{when } t = 0, x = 4 \quad \text{M1}$$

$$0 = \sin^{-1}(1) + c_2 \Rightarrow c_2 = -\frac{\pi}{2}$$

$$2t = \sin^{-1} \left(\frac{x}{4} \right) - \frac{\pi}{2} \quad \text{M1}$$

$$2t + \frac{\pi}{2} = \sin^{-1} \left(\frac{x}{4} \right)$$

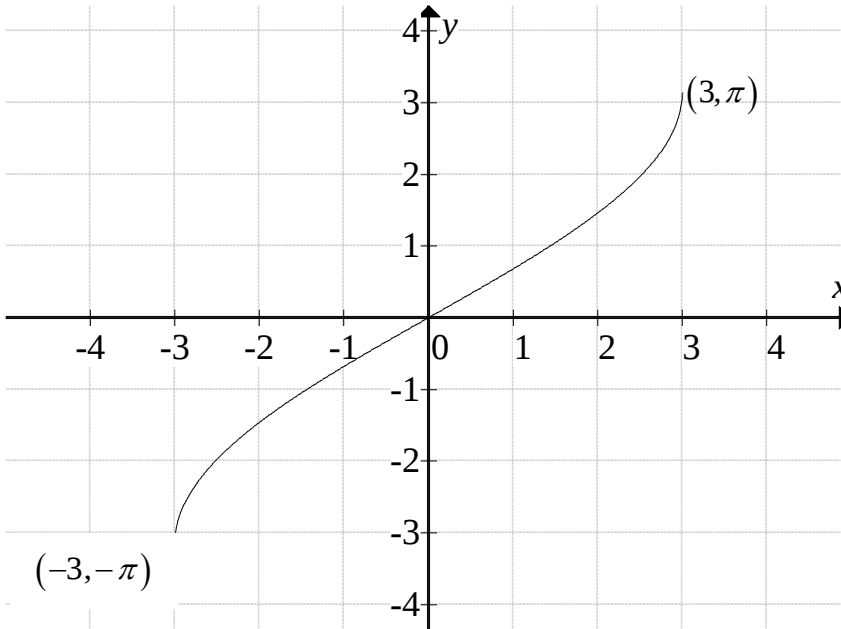
$$\frac{x}{4} = \sin \left(2t + \frac{\pi}{2} \right) = \sin(2t) \cos \left(\frac{\pi}{2} \right) + \cos(2t) \sin \left(\frac{\pi}{2} \right)$$

$$x = 4 \cos(2t) \quad \text{A1}$$

Question 9 $f(x) = 2 \arcsin\left(\frac{x}{3}\right) = 2 \sin^{-1}\left(\frac{x}{3}\right)$

- a. domain $[-3, 3]$ range $[-\pi, \pi]$, passes through the origin $(0, 0)$
endpoints $(-3, -\pi)$, $(3, \pi)$, correct graph shape

A1



b. $f: y = 2 \sin^{-1}\left(\frac{x}{3}\right)$

$$f^{-1}: x = 2 \sin^{-1}\left(\frac{y}{3}\right) \Rightarrow \frac{x}{2} = \sin^{-1}\left(\frac{y}{3}\right) \Rightarrow \frac{y}{3} = \sin\left(\frac{x}{2}\right) \Rightarrow y = 3 \sin\left(\frac{x}{2}\right)$$

to state the function, we must state its domain.

$$f^{-1}: [-\pi, \pi] \rightarrow \mathbb{R}, f^{-1}(x) = 3 \sin\left(\frac{x}{2}\right) \quad \text{A1}$$

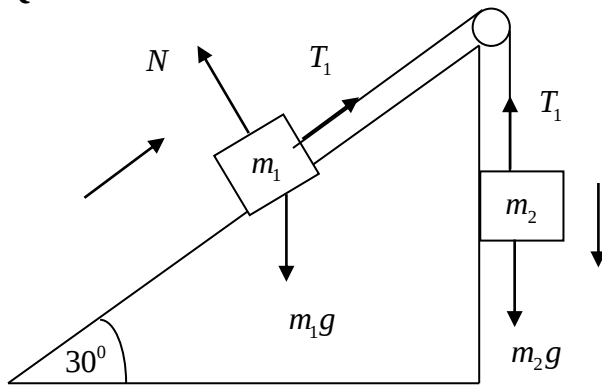
c. Let $A_1 = \int_0^3 2 \sin^{-1}\left(\frac{x}{3}\right) dx$, $A_2 = \int_0^\pi 3 \sin\left(\frac{x}{2}\right) dx$

$$A_1 + A_2 = 3\pi \quad \text{area of the rectangle}$$

$$A_2 = \int_0^\pi 3 \sin\left(\frac{x}{2}\right) dx = \left[-6 \cos\left(\frac{x}{2}\right) \right]_0^\pi = -6 \cos\left(\frac{\pi}{2}\right) + 6 \cos(0) = 6 \quad \text{A1}$$

$$A_1 = \int_0^3 2 \sin^{-1}\left(\frac{x}{3}\right) dx = 3\pi - 6 = 3(\pi - 2) \quad \text{A1}$$

Question 10



resolving up parallel to plane around the m_1 kg mass

$$(1) \quad T_1 - m_1 g \sin(30^\circ) = m_1 a \Rightarrow T_1 - \frac{m_1 g}{2} = m_1 a$$

resolving downwards around the m_2 kg mass

$$(2) \quad m_2 g - T_1 = m_2 a$$

adding to eliminate the tension in the string,

to find the acceleration a , of the system

$$(1) + (2) \quad m_2 g - \frac{m_1 g}{2} = m_1 a + m_2 a$$

$$\frac{g}{2} (2m_2 - m_1) = a (m_1 + m_2)$$

$$a = \frac{g(2m_2 - m_1)}{2(m_1 + m_2)}$$

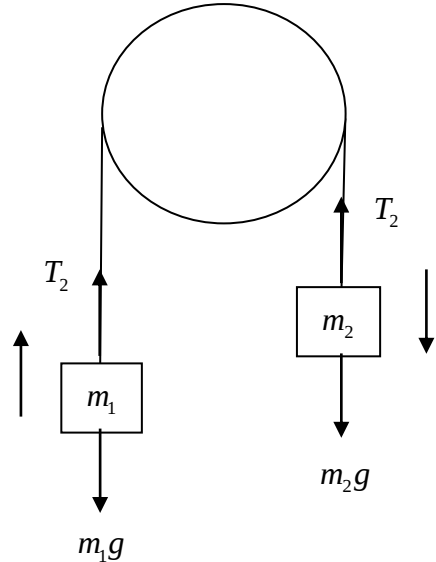
equating the accelerations

$$4(m_2 - m_1) = (2m_2 - m_1)$$

$$4m_2 - 4m_1 = 2m_2 - m_1$$

$$2m_2 = 3m_1$$

$$\frac{m_2}{m_1} = \frac{3}{2}$$



around m_2 kg mass A1

$$(3) \quad m_2 g - T_2 = \frac{m_2 a}{2}$$

around m_1 kg mass

$$(4) \quad T_2 - m_1 g = \frac{m_1 a}{2}$$

$$(3) + (4) \quad m_2 g - m_1 g = \frac{a}{2} (m_1 + m_2)$$

$$a = \frac{2g(m_2 - m_1)}{m_1 + m_2} \quad \text{A1}$$

M1

A1

END OF SUGGESTED SOLUTIONS