

**2016 VCAA Specialist Mathematics
Sample (v2 April) Exam 2 Solutions** © 2016 itute.com

CAS should be used whenever possible to speed up the solution process.

SECTION A

1	2	3	4	5	6	7	8	9	10
B	B	D	D	D	B	B	B	E	C

11	12	13	14	15	16	17	18	19	20
C	E	B	C	A	D	D	E	A	C

Q1 $x^2 - 6x + y^2 + 4y = b$, $(x-3)^2 + (y+2)^2 = b+9+4$
 $\therefore a=3$, $b+13=5^2$, $b=12$ **B**

Q2 Domain: $-1 \leq 4x-1 \leq 1$, $0 \leq 4x \leq 2$, $0 \leq x \leq \frac{1}{2}$
 Range: $-\frac{\pi}{2} \leq \sin^{-1}(4x-1) \leq \frac{\pi}{2}$, $-\frac{3\pi}{2} \leq 3\sin^{-1}(4x-1) \leq \frac{3\pi}{2}$
 $-\frac{3\pi}{2} + \frac{\pi}{2} \leq 3\sin^{-1}(4x-1) + \frac{\pi}{2} \leq \frac{3\pi}{2} + \frac{\pi}{2}$,
 $-\pi \leq 3\sin^{-1}(4x-1) + \frac{\pi}{2} \leq 2\pi$ **B**

Q3 $f(x) = \frac{(x-3)(x-1)}{(x-3)(x+2)}$, $f(x) = \frac{x-1}{x+2}$ and $x \neq 3$ and -2 **D**

Q4 **D**

Q5 $(z+\bar{z})^2 - (z-\bar{z})^2 = 16$, $4x^2 + 4y^2 = 16$, $x^2 + y^2 = 2^2$ **D**

Q6 $z = -3i$ is also a root (conjugate root theorem) **B**

Q7 **B**

Q8 $3 \times \int_0^{\pi} \sin^3 x \, dx = 3 \times \int_0^{\pi} (1 - \cos^2 x) \sin x \, dx$
 $= -3 \times \int_1^{-1} (1 - u^2) du = 3 \int_{-1}^1 (1 - u^2) du$ **B**

Q9 $x_0 = 0$, $y_0 = 2$, $\frac{dy}{dx} = 2$
 $x_1 = 0.10$, $y_1 \approx 2 + 0.10 \times 2 = 2.20$ **E**

Q10 $x = \frac{1}{2} \sin y$, when $x = \frac{1}{2}$, $y = \frac{\pi}{2}$

$V = \int_0^{\frac{\pi}{2}} \pi x^2 \, dy = \int_0^{\frac{\pi}{2}} \pi \left(\frac{1}{2} \sin y \right)^2 \, dy$
 $= \frac{\pi}{8} \int_0^{\frac{\pi}{2}} (1 - \cos(2y)) \, dy$ **C**



Q11 $\cos \theta = \frac{(3\tilde{i} + 6\tilde{j} - 2\tilde{k})(2\tilde{i} - 2\tilde{j} + \tilde{k})}{|3\tilde{i} + 6\tilde{j} - 2\tilde{k}| |2\tilde{i} - 2\tilde{j} + \tilde{k}|} = \frac{-8}{7 \times 3}$

$\therefore \theta = 112.4$ **C**

Q12 $\hat{b} = \frac{1}{3}(2\tilde{i} - \tilde{j} - 2\tilde{k})$, $\tilde{a} \cdot \hat{b} = \frac{1}{3}(6+2) = \frac{8}{3}$ **E**

Q13 $\tilde{r}(t) = -\tilde{i} - \frac{6}{2\sqrt{t}} \tilde{j}$, $\dot{\tilde{r}}(9) = -\tilde{i} - \tilde{j}$ **B**

Q14 The diagonals of a rhombus are perpendicular. **C**

Q15 $a = \frac{1}{2} \frac{dv^2}{dx} = \frac{1}{2} \frac{d}{dx} (3x^2 - x^3 + 16) = 3x - \frac{3x^2}{2}$

$F = ma = 12 \left(3x - \frac{3x^2}{2} \right)$ **A**

Q16 $\frac{dv}{dt} = \frac{v}{\log_e v}$, $\frac{dt}{dv} = \frac{\log_e v}{v}$

$t = \int \frac{\log_e v}{v} dv = \int u \, du = \frac{u^2}{2} + c = \frac{1}{2} (\log_e v)^2 + c$

$v = 5$ when $t = 0$, $\therefore t = \frac{1}{2} (\log_e v)^2 - \frac{1}{2} (\log_e 5)^2$

$(\log_e v)^2 = 2t + (\log_e 5)^2$, $\log_e v = \sqrt{2t + (\log_e 5)^2}$
 $v = e^{\sqrt{2t + (\log_e 5)^2}}$ **D**

Q17 $2T \sin 60^\circ - 12g = 0$, $T = \frac{6g}{\sin 60^\circ} = 4\sqrt{3} \, g$ **D**

Q18 $\mu_z = \bar{X} - 3\bar{Y} = 10 - 3 \times 3 = 1$

$\sigma_z = \sqrt{8^2 + (-3)^2 (2^2)} = 10$ **E**

Q19 $sd(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{7}{\sqrt{20}} \approx 1.56525$, $\mu = 30$

Normal: $\Pr(\bar{X} > 32) \approx 0.1007$ **A**

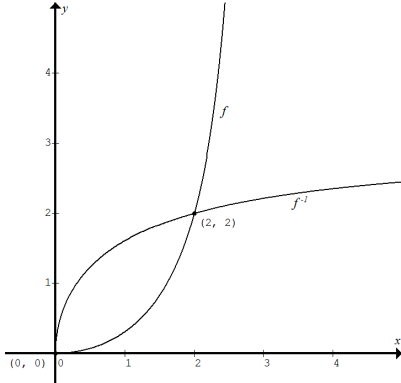
Q20 **C**



SECTION B

Q1a $f(2) = -2 + 2 \sec\left(\frac{2\pi}{6}\right) = -2 + \frac{2}{\cos \frac{\pi}{3}} = -2 + 4 = 2$

Q1b



Q1c Equation of $f: y = -2 + 2 \sec\left(\frac{\pi x}{6}\right)$

Equation of $f^{-1}: x = -2 + 2 \sec\left(\frac{\pi y}{6}\right)$

$$\frac{x+2}{2} = \sec\left(\frac{\pi y}{6}\right), \quad \frac{2}{x+2} = \cos\left(\frac{\pi y}{6}\right)$$

$$y = \frac{6}{\pi} \arccos\left(\frac{2}{x+2}\right), \therefore k = \frac{6}{\pi}$$

Q1d $A = 2 \times \int_0^2 \left(\frac{6}{\pi} \arccos\left(\frac{2}{x+2}\right) - x \right) dx \approx 1.939$

Q1ei $f(x) = -2 + 2 \left(\cos\left(\frac{\pi x}{6}\right) \right)^{-1}$,

$$f'(x) = -2 \left(\cos\left(\frac{\pi x}{6}\right) \right)^{-2} \left(-\sin\left(\frac{\pi x}{6}\right) \right) \left(\frac{\pi}{6} \right) = \frac{\pi}{3} \frac{\sin\left(\frac{\pi x}{6}\right)}{\cos^2\left(\frac{\pi x}{6}\right)}$$

Arc length $= \int_0^2 \sqrt{1 + (f'(x))^2} dx$

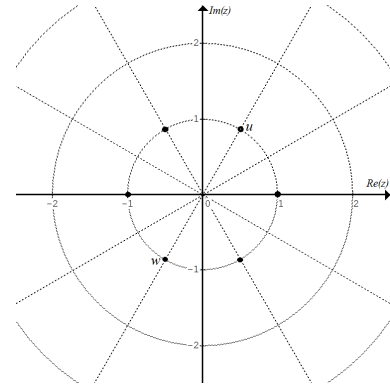
$$= \int_0^2 \sqrt{1 + \frac{\pi^2 \sin^2\left(\frac{\pi x}{6}\right)}{9 \cos^4\left(\frac{\pi x}{6}\right)}} dx$$

Q1eii 3.067 by CAS

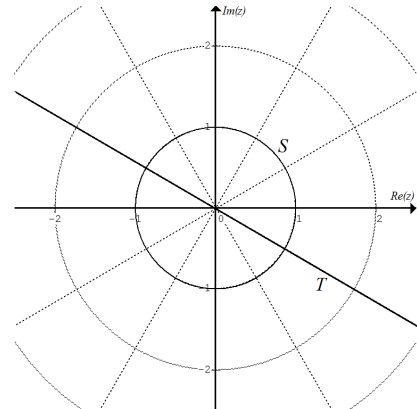
Q2ai $u = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)$

Q2aii $u^6 = \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right)^6 = \cos\left(\frac{6\pi}{3}\right) + i \sin\left(\frac{6\pi}{3}\right)$
 $= \cos(2\pi) + i \sin(2\pi) = 1$

Q2aiii



Q2bi and ii



Q2biii $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2} \right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2} \right)$

Alternatively, $S \cap T = \left\{ \frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{\sqrt{3}}{2} + \frac{1}{2}i \right\}$

Q3a $\log_e N = 6 - 3e^{-0.4t}$, $\frac{d}{dt} \log_e N = \frac{d}{dt} (6 - 3e^{-0.4t})$

$$\frac{1}{N} \frac{dN}{dt} = 1.2e^{-0.4t}, \quad \frac{1}{N} \frac{dN}{dt} = 0.4(6 - \log_e N)$$

$$\therefore \frac{1}{N} \frac{dN}{dt} + 0.4 \log_e N - 2.4 = 0$$

Q3b $t = 0$, $\log_e N = 6 - 3e^{-0.4t} = 3$, $N = e^3 \approx 20$

Q3c As $t \rightarrow \infty$, $\log_e N \rightarrow 6$, $N \rightarrow 403$ for integer N



Q3di $\frac{dN}{dt} = 0.4N(6 - \log_e N)$

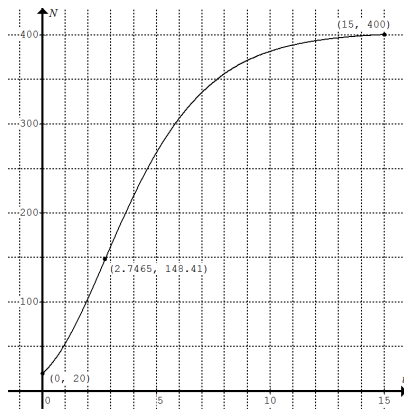
$\frac{d}{dt}\left(\frac{dN}{dt}\right) = \frac{d}{dt}(0.4N(6 - \log_e N))$, $\frac{d^2N}{dt^2} = 0.4(5 - \log_e N)\frac{dN}{dt}$
 $\frac{d^2N}{dt^2} = 0.16N(5 - \log_e N)(6 - \log_e N)$

Q3dii $\frac{d^2N}{dt^2} = 0$ and $\frac{dN}{dt} \neq 0$ and $N(t) > 0$

$\therefore 5 - \log_e N = 0$, $\therefore 5 = 6 - 3e^{-0.4t}$

$\therefore t \approx 2.7$ and $N \approx 148$

Q3e



Q4a $\tilde{r}(0) = 12 \cos 60^\circ \tilde{i} + 12 \sin 60^\circ \tilde{j} = 6 \tilde{i} + 6\sqrt{3} \tilde{j}$

Q4b $\tilde{r}(t) = \int (-0.1t \tilde{i} - (gt - 0.1t) \tilde{j}) dt$

$\tilde{r}(t) = -0.05t^2 \tilde{i} - (gt - 0.05t^2) \tilde{j} + \tilde{c}$, $\tilde{r}(0) = \tilde{c} = 6 \tilde{i} + 6\sqrt{3} \tilde{j}$

$\therefore \tilde{r}(t) = (6 - 0.05t^2) \tilde{i} + (6\sqrt{3} - gt + 0.05t^2) \tilde{j}$

$\tilde{r}(t) = \int ((6 - 0.05t^2) \tilde{i} + (6\sqrt{3} - gt + 0.05t^2) \tilde{j}) dt$

$\tilde{r}(t) = \left(6t - \frac{1}{60}t^3\right) \tilde{i} + \left(6\sqrt{3}t - \frac{1}{2}gt^2 + \frac{1}{60}t^3\right) \tilde{j}$, given $\tilde{r}(0) = \tilde{0}$

Q4c At $t = T$, $y = -x$, $6\sqrt{3}t - \frac{1}{2}gt^2 + \frac{1}{60}t^3 = -6t + \frac{1}{60}t^3$

$\therefore 6\sqrt{3}T + 6T - \frac{g}{2}T^2 = 0$, $6T\left(\sqrt{3} + 1 - \frac{g}{12}T\right) = 0$

Since $T > 0$, $\therefore T = \frac{12}{g}(\sqrt{3} + 1)$

Q4d $T = \frac{12}{g}(\sqrt{3} + 1) = 3.3454$, $T^2 = 11.1915$

$\tilde{r}(T) = (6 - 0.05 \times 11.1915) \tilde{i} + (6\sqrt{3} - g \times 3.3454 + 0.05 \times 11.1915) \tilde{j}$
 $= 5.4404 \tilde{i} - 21.8330 \tilde{j}$

Speed $= |\tilde{r}(T)| = \sqrt{5.4404^2 + 21.8330^2} \approx 22.5 \text{ m s}^{-1}$

Q5a Consider the two particles together: $3g - g = (3+1)a$, $a = \frac{g}{2}$

Q5b Consider the 1 kg particle: $T_1 - g = 1 \times \frac{g}{2}$, $T_1 = \frac{3g}{2}$

Q5c Consider the two particles together:

$3g \sin 30^\circ - g = (3+1)b$, $b = \frac{g}{8}$

Q5d Consider the two particles together: $3g \sin \theta^\circ - g = 0$

$\therefore \sin \theta^\circ = \frac{1}{3}$, $\theta \approx 19.5$

Q5e Consider the two particles together:

$g - 3g \sin \theta^\circ = (3+1)\frac{g}{4}\left(1 - \frac{3}{\sqrt{2}}\right)$, $1 - 3 \sin \theta^\circ = 1 - \frac{3}{\sqrt{2}}$

$\therefore \sin \theta^\circ = \frac{1}{\sqrt{2}}$, $\theta = 45$

Q6a H_0 : The mean lifetime is that claimed by the manufacturer.

($\mu = 10$)

H_1 : The mean lifetime is less than that claimed by the manufacturer.

($\mu < 10$)

Q6b $\sigma \approx s = 1$, $\text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} \approx \frac{1}{\sqrt{25}} = 0.2$, $E(\bar{X}) = \mu = 10$

p -value $= \Pr(\bar{X} \leq 9.7 | \mu = 10) = \Pr\left(Z \leq \frac{9.7 - 10}{0.2}\right) \approx 0.067$

Q6c Since p -value > 0.05 , $\therefore H_0$ should not be rejected at the 5% level of significance.

Q6d $\Pr(\bar{X} < C^* | \mu = 10) = 0.05$, $\Pr\left(Z < \frac{C^* - 10}{0.2}\right) = 0.05$

$\therefore \frac{C^* - 10}{0.2} \approx -1.64485$, $C^* \approx 9.671$

Q6ei $\Pr(\bar{X} > 9.671 | \mu = 9.5) = \Pr\left(Z > \frac{9.671 - 9.5}{0.2}\right)$

$\approx \Pr(Z > 0.855) \approx 0.196$

Q6eii Type II error:

p -value $\approx 0.196 > 0.05$, it supports that the actual mean lifetime is $\mu = 9.5$, $\therefore H_0$ is not true but it is not rejected.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors