



2016 VCAA Specialist Mathematics Sample Exam 1 (v2 April) Solutions © 2016 itute.com

Q1a Let $z = \sqrt{5} - i$,

$$(\sqrt{5} - i)^3 - (\sqrt{5} - i)(\sqrt{5} - i)^2 + 4(\sqrt{5} - i) - 4(\sqrt{5} - i) = 0$$

$\therefore \sqrt{5} - i$ is a solution.

Q1b $z^3 - (\sqrt{5} - i)z^2 + 4z - 4(\sqrt{5} - i) = (z - (\sqrt{5} - i))(z^2 + 4) = 0$

$\therefore z^2 + 4 = 0, \therefore z = \pm 2i$ are the other solutions.

Q2 $3x^2 + 2xy + y^2 = 11$ and $y > 0$ (in the first quadrant)

At $x = 1, -8 + 2y + y^2 = 0, \therefore y = 2$

Implicit differentiation: $\frac{d}{dx}(3x^2 + 2xy + y^2) = 0,$

$$6x + 2x \frac{dy}{dx} + 2y + 2y \frac{dy}{dx} = 0, \frac{dy}{dx} = -\frac{3x + y}{x + y}$$

At $(1, 2), \frac{dy}{dx} = -\frac{5}{3}, \therefore$ gradient of the normal $= \frac{3}{5}$

$\therefore$ equation of the normal: $y - 2 = \frac{3}{5}(x - 1), 3x - 5y + 7 = 0$

Q3a $\overline{X + Y} = \overline{X} + \overline{Y} = 240 + 10 = 250$ mL

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) = 8^2 + 2^2 = 68 \text{ (mL)}^2$$

Q3bi Null hypothesis: The second machine is, on average, dispensing **not** less coffee than the first.

Alternative hypothesis: The second machine is, on average, dispensing less coffee than the first.

Q3bii $a = \frac{235 - 240}{8} = -0.625, p = \Pr(Z \leq -0.625) \approx 0.266$

Since $p > 0.05$, the null hypothesis should not be rejected at the 0.05 level of significance.

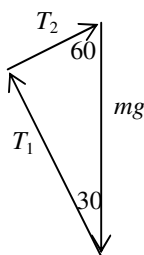
Q4a $V = \int_0^a \pi(e^{-x})^2 dx = \int_0^a \pi e^{-2x} dx$

Q4b $V(a) = \pi \left[\frac{e^{-2x}}{-2} \right]_0^a = \pi \left(\frac{1 - e^{-2a}}{2} \right)$

Q4c $\pi \left(\frac{1 - e^{-2a}}{2} \right) = \frac{5\pi}{18}, 9 - 9e^{-2a} = 5, e^{-2a} = \frac{4}{9}, e^{2a} = \frac{9}{4},$

$$e^a = \frac{3}{2}, a = \log_e \left(\frac{3}{2} \right)$$

Q5a



$$\frac{T_2}{T_1} = \tan 30, \therefore T_2 = \frac{T_1}{\sqrt{3}}$$

Q5b $\frac{T_2}{mg} = \sin 30$, let $T_2 = 98$

$$\therefore m = \frac{2 \times 98}{9.8} = 20 \text{ is the maximum value.}$$

Q6

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cos^2(2x) \sin(2x) dx = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} -\frac{1}{2} u^2 \frac{du}{dx} dx$$

$$\begin{aligned} u &= \cos(2x) \\ \frac{du}{dx} &= -2 \sin(2x) \\ -\frac{1}{2} \times \frac{du}{dx} &= \sin(2x) \end{aligned}$$

$$= \int_{-1}^0 -\frac{1}{2} u^2 du = \left[-\frac{u^3}{6} \right]_{-1}^0 = -\frac{1}{6}$$

Q7 $\frac{dy}{dx} = \frac{y}{x^2}, \int \frac{1}{y} dy = \int \frac{1}{x^2} dx, \log_e |y| = -\frac{1}{x} + c$

Given $x = 1, y = -1, \log_e |-1| = -\frac{1}{1} + c, c = 1$

$$\therefore \log_e |y| = 1 - \frac{1}{x}, |y| = e^{\left(1 - \frac{1}{x}\right)}, y = \pm e^{\left(1 - \frac{1}{x}\right)}$$

Q8a Arc length

$$= \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_0^{\pi} \sqrt{(-2 \sin(2\theta))^2 + (2 \cos(2\theta))^2} d\theta$$

Q8b Arc length $= 2 \int_0^{\pi} \sqrt{\sin^2(2\theta) + \cos^2(2\theta)} d\theta = 2 \int_0^{\pi} d\theta = 2\pi$

Q9a $\tilde{b} = \tilde{i} + 2\tilde{j} + m\tilde{k}, |\tilde{b}| = \sqrt{1^2 + 2^2 + m^2} = 2\sqrt{3}$

$$\therefore m^2 + 5 = 12, m = \pm\sqrt{7}$$

Q9b $\tilde{a} \cdot \tilde{b} = 0, 1 - 2 + 2m = 0, m = \frac{1}{2}$

Q9ci $3\tilde{c} - \tilde{a} = 2\tilde{i} + 4\tilde{j} - 5\tilde{k}$

Q9cii Since $3\tilde{c} - \tilde{a} = 2\tilde{i} + 4\tilde{j} - 5\tilde{k} \therefore 3\tilde{c} - \tilde{a} = 2\tilde{b}$ if $m = -\frac{5}{2}$

$\therefore \tilde{a}, \tilde{b}$ and $\tilde{c}$ are linearly dependent if $m = -\frac{5}{2}$

Q10a $\frac{1}{x^2} + \frac{3}{x} + \frac{2x-1}{x^2+4} = \frac{(x^2+4) + 3x(x^2+4) + x^2(2x-1)}{x^2(x^2+4)}$

$$= \frac{5x^3 + 12x + 4}{x^2(x^2+4)}$$

Q10b $\int \frac{5x^3 + 12x + 4}{x^2(x^2+4)} dx = \int \frac{1}{x^2} + \frac{3}{x} + \frac{2x-1}{x^2+4} dx$

$$= \int \frac{1}{x^2} + \frac{3}{x} + \frac{2x}{x^2+4} - \frac{1}{x^2+4} dx$$

$$= -\frac{1}{x} + 3 \log_e |x| + \log_e (x^2 + 4) - \tan^{-1} \left(\frac{x}{2} \right)$$

$$= -\frac{1}{x} + \log_e |x^3| (x^2 + 4) - \tan^{-1} \left(\frac{x}{2} \right)$$

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