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**SPECIALIST MATHS 3 & 4
TRIAL EXAMINATION 2
SOLUTIONS
2016**

Section A – Multiple-choice answers

1.	E	6.	E	11.	C	16.	C
2.	D	7.	B	12.	D	17.	A
3.	C	8.	C	13.	E	18.	A
4.	A	9.	D	14.	B	19.	A
5.	D	10.	B	15.	E	20.	D

Section A - Multiple-choice solutions

Question 1

$$\begin{aligned}
 y &= \frac{x^2 - 3x - 4}{x^2 - x - 12} \\
 &= \frac{(x-4)(x+1)}{(x-4)(x+3)} \\
 &= \frac{x+1}{x+3} \quad \text{provided } x-4 \neq 0, \text{ so } x \neq 4 \\
 &= 1 - \frac{2}{x+3}
 \end{aligned}$$

The graph has an asymptote at $x = -3$. It has a point of discontinuity at $x = 4$.
It also has an asymptote at $y = 1$ but this is not offered in the possible answers.
The answer is E.

Question 2

For $f(x) = \arccos(3-x)$ to be defined we require

$$-1 \leq 3-x \leq 1$$

$$-4 \leq -x \leq -2$$

$$2 \leq x \leq 4$$

So $d_f = [2, 4]$.

The answer is D.

Question 3

$$\begin{aligned}
 \frac{2x^2 + 1}{(x^2 - 1)(x^2 + 4)} &= \frac{2x^2 + 1}{(x-1)(x+1)(x^2 + 4)} \\
 &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2 + 4}
 \end{aligned}$$

The answer is C.

Question 4

$$\text{period} = \frac{2\pi}{n} = a \quad (\text{from the graph})$$

$$\text{so } n = \frac{2\pi}{a}$$

So the coefficient of x must be $\frac{2\pi}{a}$.

This eliminates options B, D and E.

The graph of $y = \sec\left(\frac{2\pi x}{a}\right)$ has been translated $\frac{a}{4}$ units to the right to obtain the graph shown.

For example, the point $(0,1)$ lies on the graph of $y = \sec\left(\frac{2\pi x}{a}\right)$.

On the graph shown, the point where the function equals 1 is $\left(\frac{a}{4}, 1\right)$. So the rule for the graph

shown must be $y = \sec\left(\frac{2\pi}{a}\left(x - \frac{a}{4}\right)\right)$.

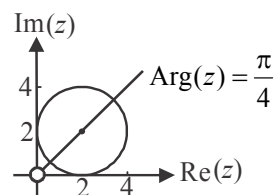
The answer is A.

Question 5

Do a quick sketch.

The graph of $|z - 2 - 2i| = 2$ or $|z - (2 + 2i)| = 2$, is the graph of a circle with centre at $z = 2 + 2i$ and radius of 2 units.

As an example, the graph of $\text{Arg}(z) = \frac{\pi}{4}$ is shown.



If the graph of $\text{Arg}(z) = \theta$ is to intersect with the circle then $0 \leq \theta \leq \frac{\pi}{2}$.

The answer is D.

Question 6

For $z_1 = 1 - \sqrt{3}i$,

$$r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$$

$$= -\frac{\pi}{3}$$

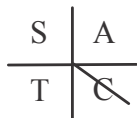
$$z_1 = 2\text{cis}\left(-\frac{\pi}{3}\right)$$

$$\text{So } (z_1)^{12} = 2^{12} \text{cis}\left(12 \times -\frac{\pi}{3}\right) \quad (\text{De Moivre})$$

$$= 2^{12} \text{cis}(-4\pi)$$

$$\text{So } z_2 = 2^{12} \text{cis}(0)$$

So $\text{Arg}(z_2) = 0$ and option A is incorrect.



$$\begin{aligned} \text{For option B, } z_2 &= 2^{12}(\cos(0) + i\sin(0)) \\ &= 2^{12}(1 + 0) \\ &= 2^{12} \end{aligned}$$

$$\text{Im}(z_2) + \text{Re}(z_2) = 0 + 2^{12} > 0$$

Option B is incorrect.

$$|z_2| = 2^{12} \text{ so option C is incorrect.}$$

$$\text{Re}(z_2) = 2^{12} \text{ so option D is incorrect.}$$

The answer is E.

Question 7

$$\begin{aligned}
& \int_1^2 (x-3)(2x+1)^5 dx \\
&= \int_3^5 \left(\frac{u-7}{2} \right) u^5 \times \frac{1}{2} \frac{du}{dx} dx \\
&= \frac{1}{4} \int_3^5 (u-7)u^5 du \\
&= \frac{1}{4} \int_3^5 (u^6 - 7u^5) du
\end{aligned}$$

The answer is B.

Question 8

For $x=0$, $\frac{dy}{dx}=0$, so eliminate options A and E.

The slopes $\left(\frac{dy}{dx}\right)$ are influenced by x -values only so eliminate option D (option E has already been eliminated).

We are left with $\frac{dy}{dx}=xe^x$ and $\frac{dy}{dx}=xe^{-x}$.

From the direction field, as $x \rightarrow \infty$, $\frac{dy}{dx} \rightarrow 0$.

For $\frac{dy}{dx}=xe^x$, as $x \rightarrow \infty$, $\frac{dy}{dx} \rightarrow \infty$.

For $\frac{dy}{dx}=xe^{-x}$, as $x \rightarrow \infty$, $\frac{dy}{dx} \rightarrow 0$.

The answer is C.

Question 9

$$\frac{dy}{dx} = x^{\frac{3}{2}} + 2x, \quad h = 0.1, \quad x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n) \quad (\text{formula sheet})$$

$$x_0 = 1 \quad y_0 = 2$$

$$\begin{aligned}
x_1 &= 1.1 & y_1 &= 2 + 0.1(1^{\frac{3}{2}} + 2) \\
& & &= 2 + 0.1 \times 3 \\
& & &= 2.3
\end{aligned}$$

$$\begin{aligned}
x_2 &= 1.2 & y_2 &= 2.3 + 0.1(1.1^{\frac{3}{2}} + 2.2) \\
& & &= 2.63536... \\
& & &= 2.635 \quad (\text{to 3 decimal places})
\end{aligned}$$

The answer is D.

Question 10

At $t=2$, $t=3$ and $t=4$ the particle changes direction. Between $t=0$ and $t=2$, the distance travelled by the particle in the same direction, is greater than the distance travelled between $t=2$ and $t=3$ or between $t=3$ and $t=4$ or between $t=4$ and $t=6$. Also the distance travelled between $t=3$ and $t=4$ is less than the distance travelled between $t=2$ and $t=3$ so the particle is **not** furthest from its initial position in the $t=3$ to $t=4$ period.

The particle is furthest from its initial position during the time interval $(1.5, 2.5)$.

The answer is B.

$$\text{Let } u = 2x+1 \quad x=2, u=5$$

$$\text{so } \frac{du}{dx} = 2 \quad x=1, u=3$$

Also, since $u = 2x+1$

$$u-1 = 2x$$

$$x = \frac{u-1}{2}$$

$$x-3 = \frac{u-1}{2} - 3$$

$$x-3 = \frac{u-7}{2}$$

Question 11

The vector resolute $\underline{a}$ in the direction of $\underline{b}$ is given by $(\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}}$.

$$\begin{aligned}\text{So } (\underline{a} \cdot \hat{\underline{b}})\hat{\underline{b}} &= \left((2\hat{i} + \hat{j} + \hat{k}) \cdot \frac{1}{\sqrt{2}}(\hat{i} - \hat{k}) \right) \frac{1}{\sqrt{2}}(\hat{i} - \hat{k}) \\ &= \frac{1}{2}(2 + 0 - 1)(\hat{i} - \hat{k}) \\ &= \frac{1}{2}(\hat{i} - \hat{k})\end{aligned}$$

The answer is C.

Question 12

Start by finding the Cartesian equation of the path.

$$x = 2\cos(t) \qquad y = \sin(t)$$

$$\frac{x^2}{4} = \cos^2(t) \qquad y^2 = \sin^2(t)$$

$$\frac{x^2}{4} + y^2 = \cos^2(t) + \sin^2(t)$$

$$\frac{x^2}{4} + y^2 = 1$$

The path follows an ellipse with a semi-major axis of 2 and a semi-minor axis of 1.

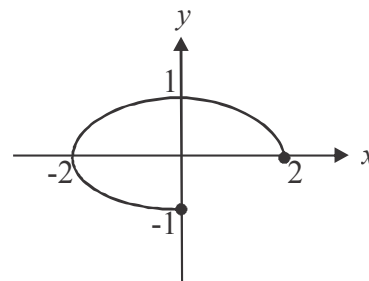
When $t = 0$, $\underline{r}(t) = 2\hat{i} + 0\hat{j}$.

The starting point is $(2, 0)$.

When $t = \frac{3\pi}{2}$, $\underline{r}(t) = 0\hat{i} - \hat{j}$.

The finishing point is $(0, -1)$.

The graph is shown to the right.



The answer is D.

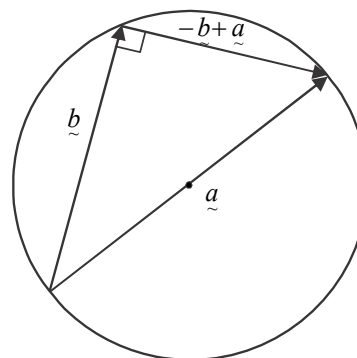
Question 13

The vector $-\underline{b} + \underline{a}$, together with vectors $\underline{a}$ and $\underline{b}$ form a right angled triangle since $\underline{a}$ spans the diameter of the circle.

$$\text{So } (-\underline{b} + \underline{a}) \cdot \underline{b} = 0$$

$$\text{or } (\underline{a} - \underline{b}) \cdot \underline{b} = 0$$

The answer is E.

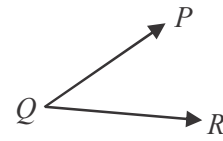


Question 14

The angle PQR is the angle between $\vec{QP}$ and $\vec{QR}$.

$$\begin{aligned}\vec{QP} &= \vec{QO} + \vec{OP} \\ &= -2\hat{j} + \hat{k} + \hat{i} + \hat{k} \\ &= \hat{i} - 2\hat{j} + 2\hat{k}\end{aligned}$$

$$\begin{aligned}\vec{QR} &= \vec{QO} + \vec{OR} \\ &= -2\hat{j} + \hat{k} + \hat{i} - \hat{j} - 2\hat{k} \\ &= \hat{i} - 3\hat{j} - \hat{k}\end{aligned}$$



$$\vec{QP} \cdot \vec{QR} = 1 + 6 - 2 = 5$$

$$\text{Also, } \vec{QP} \cdot \vec{QR} = |\vec{QP}| |\vec{QR}| \cos(\angle PQR)$$

$$5 = \sqrt{1+4+4} \sqrt{1+9+1} \cos(\angle PQR)$$

$$5 = 3\sqrt{11} \cos(\angle PQR)$$

$$\angle PQR = \cos^{-1}\left(\frac{5}{3\sqrt{11}}\right)$$

$$= 59.83321\dots^\circ$$

The closest answer is 59.8° .

The answer is B.

Question 15

$$a = \sqrt{v+4} \quad t=0, v=-3$$

Since the initial conditions are in terms of the variables t and v , we use $a = \frac{dv}{dt}$ (from the formula sheet).

$$\frac{dv}{dt} = \sqrt{v+4}$$

$$\frac{dt}{dv} = \frac{1}{\sqrt{v+4}}$$

$$t = \int \frac{1}{\sqrt{v+4}} dv$$

$$t = 2\sqrt{v+4} + c$$

When $t=0, v=-3$

$$0 = 2\sqrt{1} + c$$

$$c = -2$$

$$t = 2\sqrt{v+4} - 2$$

$$\frac{t+2}{2} = \sqrt{v+4}$$

$$v = \frac{(t+2)^2}{4} - 4$$

$$v = \frac{t^2 + 4t - 12}{4}$$

The answer is E.

Question 16

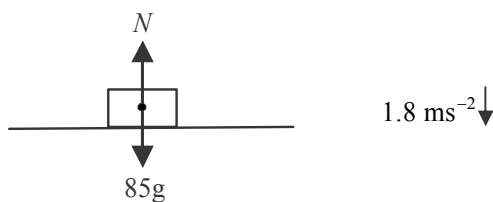
Let N be the reaction force.

$$85g - N = 1.8 \times 85$$

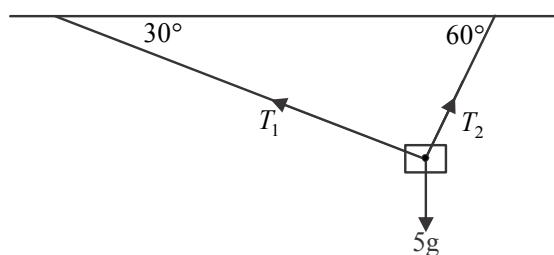
$$N = 85g - 153$$

$$= 680$$

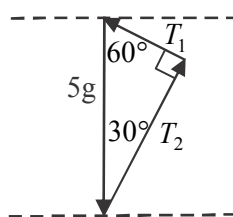
The answer is C.

**Question 17**

Draw in the forces.

Method 1

Add the force vectors head to tail.



A right-angled triangle is formed.

$$\frac{T_1}{T_2} = \tan(30^\circ)$$

$$T_1 = \frac{1}{\sqrt{3}} T_2$$

So

$$T_2 : T_1$$

becomes $\sqrt{3}T_1 : T_1$ (using the result from above)

so

$$\sqrt{3} : 1$$

and

$$1 : \frac{1}{\sqrt{3}}$$

The ratio of T_2 to T_1 is $1 : \frac{1}{\sqrt{3}}$.

The answer is A.

Method 2

Resolve the forces horizontally.

$$T_1 \cos(30^\circ) = T_2 \cos(60^\circ)$$

$$T_1 \frac{\sqrt{3}}{2} = \frac{T_2}{2}$$

$$T_2 = \sqrt{3} T_1$$

$$\text{So } T_2 : T_1$$

$$\text{becomes } \sqrt{3}T_1 : T_1$$

(using the result from above)

$$\text{so } \sqrt{3} : 1$$

$$1 : \frac{1}{\sqrt{3}}$$

The ratio of T_2 to T_1 is $1 : \frac{1}{\sqrt{3}}$.

The answer is A.

Question 18

An approximate 95% confidence interval is $\left(\bar{x} - 1.96 \frac{s}{\sqrt{n}}, \bar{x} + 1.96 \frac{s}{\sqrt{n}} \right)$ (formula sheet).

In this case, $\bar{x} = 748.6$, $s = 2.7$ and $n = 40$.

$$\begin{aligned} \text{So the interval is given by } & \left(748.6 - 1.96 \frac{2.7}{\sqrt{40}}, 748.6 + 1.96 \frac{2.7}{\sqrt{40}} \right) \\ & = (747.76..., 749.43...) \\ & = (747.8, 749.4) \quad (\text{correct to one decimal place}) \end{aligned}$$

The answer is A.

Question 19

Because of the Central Limit Theorem, we can assume that the distribution of the sample mean $\bar{X}$ is approximately normal with

$$\begin{aligned} E(\bar{X}) &= \mu \\ &= 1554 \end{aligned}$$

$$\begin{aligned} \text{and } \text{sd}(\bar{X}) &= \frac{\sigma}{\sqrt{n}} \\ &= \frac{162}{\sqrt{50}} \\ &= 22.9102... \end{aligned}$$

$$\begin{aligned} \Pr(\bar{X} < 1500) &= \Pr\left(Z < \frac{1500 - 1554}{22.9102...}\right) \\ &= \Pr(Z < -2.35702...) \\ &= 0.009211... \end{aligned}$$

The closest answer is 0.0092.

The answer is A.

Question 20

If H_0 is rejected, then this **could** mean that

- a correct decision has been made (if H_0 is not true)
- a type 1 error has been made (if H_0 is true)
- an incorrect decision has been made (if it is a type 1 error)
- H_1 is true (if H_0 is not true).

It **could not** mean that a type 2 error has been made because that occurs when H_0 is not rejected.

The answer is D.

SECTION B

Question 1 (11 marks)

- a. A point of inflection occurs if $f''(x) = 0$ **and** the sign of the function f'' changes on opposite sides of the point (i.e. there is a change in concavity).

At $x = 1$,
$$f''(1) = \frac{-8(1-1)}{\pi(1^2 - 2 + 2)} = 0 \quad (1 \text{ mark})$$

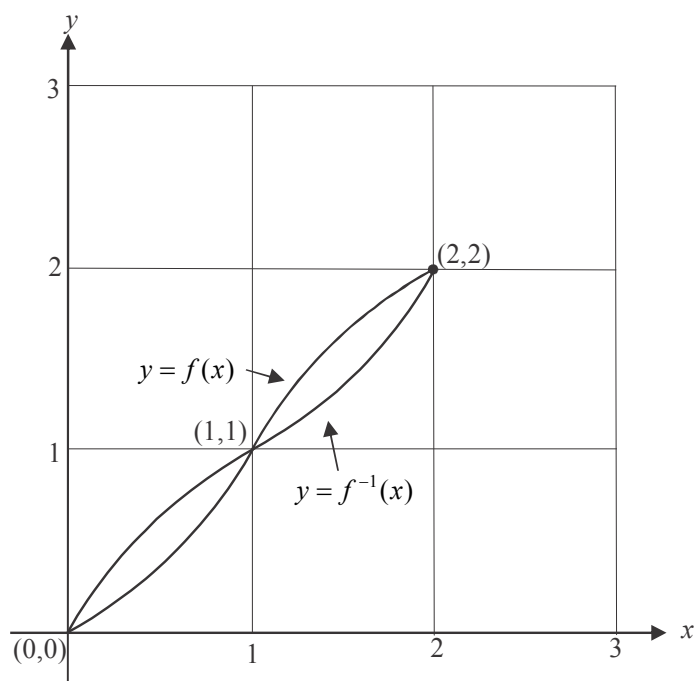
At $x = 0$, (for example) $f''(0) = \frac{2}{\pi}$

At $x = 2$, (for example) $f''(2) = -\frac{2}{\pi}$

So $f''(x) = 0$ and the sign of f'' changes on opposite sides of the point of inflection.

(1 mark)

b.



(1 mark) correct shapes

(1 mark) – correct endpoints and point of inflection

c.
$$f(x) = 1 + \frac{4}{\pi} \arctan(x-1)$$

Let $y = 1 + \frac{4}{\pi} \arctan(x-1)$

Swap x and y for inverse.

$$x = 1 + \frac{4}{\pi} \arctan(y-1) \quad (1 \text{ mark})$$

$$\frac{\pi}{4}(x-1) = \arctan(y-1)$$

$$y-1 = \tan\left(\frac{\pi}{4}(x-1)\right)$$

$$y = 1 + \tan\left(\frac{\pi}{4}(x-1)\right)$$

$$f^{-1}(x) = 1 + \tan\left(\frac{\pi}{4}(x-1)\right)$$

So $a = 1$, $b = \pi$ and $c = 4$.

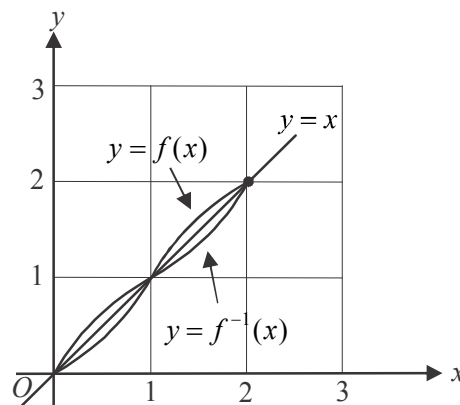
(1 mark)

- d. Looking at the graph, we could draw the line $y = x$ through the points of intersection of f and f^{-1} .

Because of the symmetry of the graphs of f and f^{-1} around the line $y = x$, we can write

$$A = 4 \int_0^1 (f^{-1}(x) - x) dx$$

So $g(x) = x$ **(1 mark)**



- e. Method 1 – “hence”

$$A = 4 \int_0^1 (f^{-1}(x) - x) dx$$

(1 mark)

$$= 0.234915\dots$$

$$= 0.2 \text{ (correct to one decimal place)}$$

(1 mark)

Method 2 – “otherwise”

$$A = 2 \int_0^1 (f^{-1}(x) - f(x)) dx$$

(1 mark)

$$= 0.234915\dots$$

$$= 0.2 \text{ (correct to one decimal place)}$$

(1 mark)

f.
$$\text{volume} = \pi \int_0^1 x^2 dy$$

$$f(x) = 1 + \frac{4}{\pi} \arctan(x-1)$$

$$\text{so let } y = 1 + \frac{4}{\pi} \arctan(x-1).$$

From part c. when we were finding f^{-1} , we know that $f^{-1}(x) = 1 + \tan\left(\frac{\pi}{4}(x-1)\right)$

So rearranging the rule for f , without swapping the x and y , we get

$$x = 1 + \tan\left(\frac{\pi}{4}(y-1)\right).$$

$$\text{volume} = \pi \int_0^1 \left(1 + \tan\left(\frac{\pi}{4}(y-1)\right)\right)^2 dy$$

(1 mark)

$$= 1.2274\dots$$

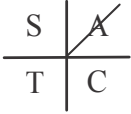
$$= 1.2 \text{ cubic units (correct to one decimal place)}$$

(1 mark)

Question 2 (10 marks)

a. i. $u = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$$r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$$

$$\theta = \tan^{-1}\left(\frac{1}{2} \div \frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$


So $u = \text{cis}\left(\frac{\pi}{6}\right)$ (1 mark) – correct modulus
(1 mark) – correct argument

ii. Since $u = \text{cis}\left(\frac{\pi}{6}\right)$ is one root of the equation $z^3 - i = 0$, then we are effectively finding the cubed roots of i so the 3 roots will be spaced $\frac{2\pi}{3}$ apart.

The second root is $\text{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)$

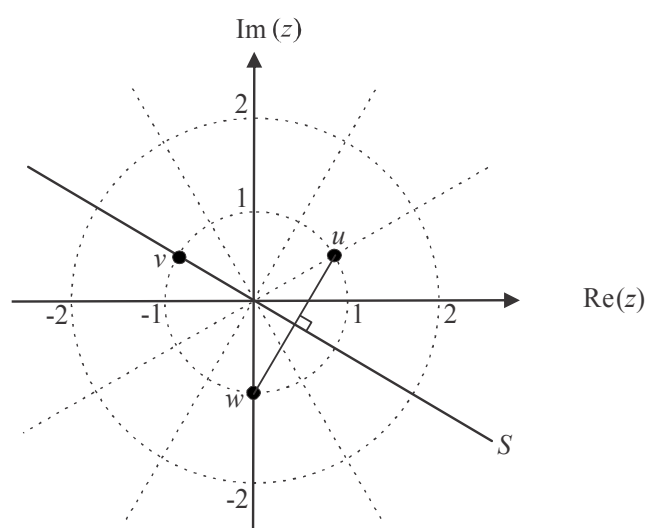
$$= \text{cis}\left(\frac{5\pi}{6}\right)$$

and the third root is $\text{cis}\left(\frac{5\pi}{6} + \frac{2\pi}{3}\right)$

$$= \text{cis}\left(\frac{9\pi}{6}\right)$$

$$= \text{cis}\left(\frac{3\pi}{2}\right)$$

So $u = \text{cis}\left(\frac{\pi}{6}\right)$, $v = \text{cis}\left(\frac{5\pi}{6}\right)$ and $w = \text{cis}\left(\frac{3\pi}{2}\right)$ because $\text{Im}(w) = -1$.



(1 mark) – correctly showing u and v
(1 mark) correctly showing w
(1 mark)

iii. See graph above.

iv. From part ii., $w = \text{cis}\left(\frac{3\pi}{2}\right) = -i$

$$|z - u| = |z - w|$$

$$\left|x + iy - \frac{\sqrt{3}}{2} - \frac{1}{2}i\right| = |x + iy + i|$$

$$\sqrt{\left(x - \frac{\sqrt{3}}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2} = \sqrt{x^2 + (y + 1)^2}$$

$$x^2 - \sqrt{3}x + \frac{3}{4} + y^2 - y + \frac{1}{4} = x^2 + y^2 + 2y + 1$$

$$-\sqrt{3}x - 3y = 0$$

$$y = -\frac{x}{\sqrt{3}}$$

(1 mark)

From part ii., $v = \text{cis}\left(\frac{5\pi}{6}\right)$

$$= \cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right)$$

$$= -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Show that $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ satisfies $y = -\frac{x}{\sqrt{3}}$.

$$LS = y$$

$$RS = -\frac{x}{\sqrt{3}}$$

$$= \frac{1}{2}$$

$$= +\frac{\sqrt{3}}{2} \div \sqrt{3}$$

$$= \frac{1}{2}$$

$$= RS$$

Have shown.

(1 mark)

b.

$$z^2 + 6i \sin(\alpha)z - 9 = 0$$

We have a quadratic equation in z .

$$z = \frac{-6i \sin(\alpha) \pm \sqrt{-36 \sin^2(\alpha) + 36}}{2}$$

$$= \frac{-6i \sin(\alpha) \pm 6\sqrt{1 - \sin^2(\alpha)}}{2}$$

$$= -3i \sin(\alpha) \pm 3\sqrt{\cos^2(\alpha)}$$

$$= -3i \sin(\alpha) \pm 3 \cos(\alpha)$$

(1 mark)

$$z = 3 \cos(\alpha) - 3i \sin(\alpha) \quad \text{or} \quad z = -3 \cos(\alpha) - 3i \sin(\alpha)$$

$$= 3(\cos(-\alpha) + i \sin(-\alpha)) \quad = -3(\cos(\alpha) + i \sin(\alpha))$$

$$\text{So } z_1 = 3\text{cis}(-\alpha) \quad \text{and} \quad z_2 = -3\text{cis}(\alpha)$$

or vice versa

(1 mark)

(1 mark)

Question 3 (11 marks)

- a. At the point of projection, we have:

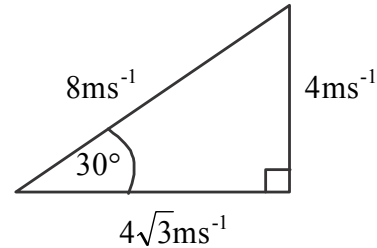
Horizontal component of velocity

$$8\cos 30^\circ = 4\sqrt{3}$$

Vertical component of velocity

$$8\sin 30^\circ = 4$$

So initial velocity = $4\sqrt{3}\mathbf{i} + 4\mathbf{j}$. **(1 mark)**



- b.
$$\ddot{\mathbf{r}}(t) = -\frac{t}{50}\mathbf{i} + \left(\frac{t}{20} - g\right)\mathbf{j}$$

$$\dot{\mathbf{r}}(t) = -\frac{t^2}{100}\mathbf{i} + \left(\frac{t^2}{40} - gt\right)\mathbf{j} + \mathbf{c}_1$$

$$\dot{\mathbf{r}}(0) = 4\sqrt{3}\mathbf{i} + 4\mathbf{j} \text{ from part a.}$$

So $\mathbf{c}_1 = 4\sqrt{3}\mathbf{i} + 4\mathbf{j}$

$$\dot{\mathbf{r}}(t) = \left(4\sqrt{3} - \frac{t^2}{100}\right)\mathbf{i} + \left(\frac{t^2}{40} - gt + 4\right)\mathbf{j} \quad \textbf{(1 mark)}$$

$$\mathbf{r}(t) = \left(4\sqrt{3}t - \frac{t^3}{300}\right)\mathbf{i} + \left(\frac{t^3}{120} - \frac{gt^2}{2} + 4t\right)\mathbf{j} + \mathbf{c}_2$$

$$\mathbf{r}(0) = 0\mathbf{i} + 1.5\mathbf{j} \quad \textbf{(1 mark)}$$

So $\mathbf{c}_2 = 0\mathbf{i} + 1.5\mathbf{j}$

$$\mathbf{r}(t) = \left(4\sqrt{3}t - \frac{t^3}{300}\right)\mathbf{i} + \left(\frac{t^3}{120} - \frac{gt^2}{2} + 4t + 1.5\right)\mathbf{j} \text{ as required} \quad \textbf{(1 mark)}$$

- c. The ball hits the ground when $\frac{t^3}{120} - \frac{gt^2}{2} + 4t + 1.5 = 0$

Use CAS to solve this equation for t .

(1 mark)

$$t = 1.097341\dots$$

Substitute into $4\sqrt{3}t - \frac{t^3}{300} = 7.59820\dots$

The ball travels 7.6 m horizontally before hitting the ground (correct to one decimal place). **(1 mark)**

- d. distance = $\int_0^{1.097341\dots} \sqrt{\left(4\sqrt{3} - \frac{t^2}{100}\right)^2 + \left(\frac{t^2}{40} - gt + 4\right)^2} dt$ (formula sheet – arc length)

(using the expression for $\dot{\mathbf{r}}(t)$ in part b.) **(1 mark)** – correct integrand

$$= 8.41767\dots$$

(1 mark) – correct terminals

distance = 8.4 metres (correct to one decimal place) **(1 mark)**

- e. speed = $|\dot{\mathbf{r}}(t)| = \sqrt{\left(4\sqrt{3} - \frac{t^2}{100}\right)^2 + \left(\frac{t^2}{40} - gt + 4\right)^2}$

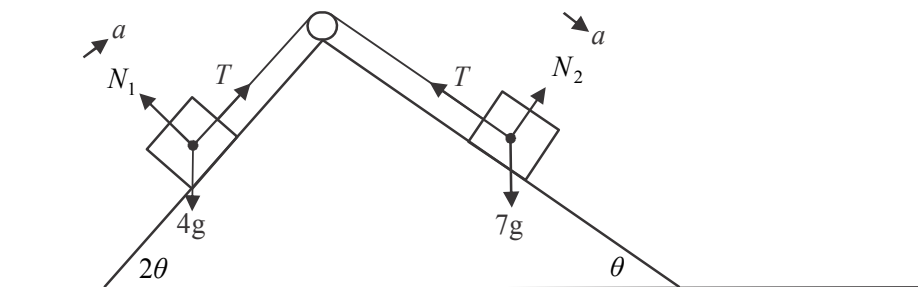
When the ball hits the ground, $t = 1.097341\dots$ from part c. **(1 mark)**

So speed = 9.64589...

= 9.6ms^{-1} (correct to one decimal place) **(1 mark)**

Question 4 (9 marks)

- a. Draw the forces.



Around the 4kg particle (parallel to its plane)

$$T - 4g \sin(2\theta) = 4a$$

(1 mark)

- b. Around the 7kg particle (parallel to its plane)

$$7g \sin(\theta) - T = 7a$$

(1 mark)

$$\text{So } T = 7g \sin(\theta) - 7a$$

$$\text{From part a., } T = 4a + 4g \sin(2\theta)$$

$$\text{So } 7g \sin(\theta) - 7a = 4a + 4g \sin(2\theta)$$

(1 mark)

$$-11a = 8g \sin(\theta) \cos(\theta) - 7g \sin(\theta)$$

$$a = \frac{g \sin(\theta)(8 \cos(\theta) - 7)}{-11}$$

$$a = \frac{g \sin(\theta)}{11} (7 - 8 \cos(\theta))$$

as required

(1 mark)

- c. When the system is in equilibrium, $a = 0$.

$$\text{So solve } 0 = \frac{g \sin(\theta)}{11} (7 - 8 \cos(\theta)) \text{ for } \theta$$

(1 mark)

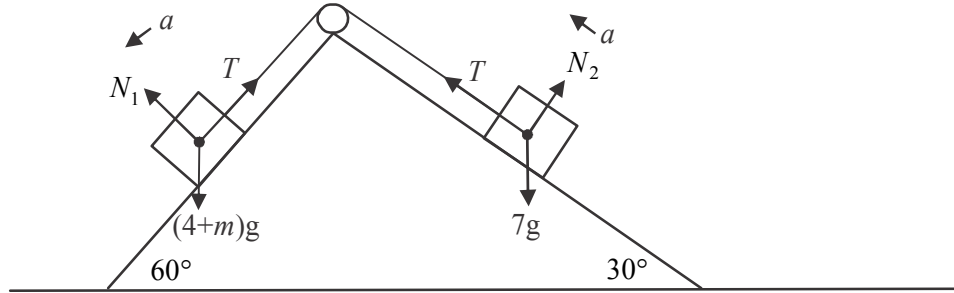
$$\sin(\theta) = 0 \quad \text{or} \quad \cos(\theta) = \frac{7}{8}$$

$$\theta = 0^\circ, 180^\circ, \dots \quad \theta = 28.9550 \dots^\circ$$

$$\text{but } 0^\circ < \theta < 45^\circ \quad \text{so} \quad \theta = 29.0^\circ \text{ (correct to one decimal place)}$$

(1 mark)

d.



Around the $(4 + m)$ kg particle
(parallel to its plane)

$$(4 + m)g \sin(60^\circ) - T = (4 + m)a$$

$$T = \frac{\sqrt{3}g}{2}(4 + m) - (4 + m)a$$

Around the 7 kg particle
(parallel to its plane)

$$T - 7g \sin(30^\circ) = 7a$$

$$T = \frac{7g}{2} + 7a$$

Combining these, we have

$$\frac{\sqrt{3}g}{2}(4 + m) - (4 + m)a = \frac{7g}{2} + 7a$$

$$\frac{\sqrt{3}g}{2}(4 + m) - \frac{7g}{2} = 7a + (4 + m)a$$

$$\frac{4\sqrt{3}g + \sqrt{3}mg - 7g}{2} = (11 + m)a$$

(1 mark)

$$a = \frac{g(4\sqrt{3} + \sqrt{3}m - 7)}{2(m + 11)}$$

Since $a = \frac{g(11\sqrt{3} - 14)}{50}$, which is given in the question,

$$\text{we have } \frac{g(11\sqrt{3} - 14)}{50} = \frac{g(4\sqrt{3} + \sqrt{3}m - 7)}{2(m + 11)}$$

(1 mark)

Method 1 – use CAS to solve

$$m = \frac{3}{2}$$

(1 mark)

Method 2 – by hand

$$(11\sqrt{3} - 14)(m + 11) = 25(4\sqrt{3} + \sqrt{3}m - 7)$$

$$11\sqrt{3}m + 121\sqrt{3} - 14m - 154 = 100\sqrt{3} + 25\sqrt{3}m - 175$$

$$11\sqrt{3}m - 14m - 25\sqrt{3}m = 100\sqrt{3} - 175 - 121\sqrt{3} + 154$$

$$-14m - 14\sqrt{3}m = -21\sqrt{3} - 21$$

$$-14m(1 + \sqrt{3}) = -21(1 + \sqrt{3})$$

$$m = \frac{-21(1 + \sqrt{3})}{-14(1 + \sqrt{3})}$$

$$m = \frac{3}{2}$$

(1 mark)

Question 5 (10 marks)

- a. $E(4R + 4C) = 4E(R) + 4E(C)$ (formula sheet)
 $= 4 \times 2.2 + 4 \times 4.9$
 $= 28.4$
 So $E(W) = 28.4 \text{ kg}$ (1 mark)
- $\text{var}(4R + 4C) = 4^2 \text{ var}(R) + 4^2 \text{ var}(C)$ (formula sheet)
 $= 16 \times 0.18 + 16 \times 0.31$
 $= 7.84$
 standard deviation of $W = \sqrt{7.84} = 2.8 \text{ kg}$ (1 mark)
- b. $H_0 : \mu = 8.6$ (1 mark)
 $H_1 : \mu < 8.6$ (1 mark)
- c. $E(\bar{X}) = \mu = 8.6$, $\text{sd}(\bar{X}) = \frac{0.54}{\sqrt{36}} = 0.09$
 $p \text{ value} = \Pr(\bar{X} \leq 8.4 | \mu = 8.6)$ (1 mark)
 $= \Pr\left(Z \leq \frac{8.4 - 8.6}{0.09}\right)$
 $= \Pr(Z \leq -2.2)$
 $= 0.0131341 \dots$
 $= 0.013$ (correct to three decimal places) (1 mark)
- d. H_0 **should be** rejected at the 5% level of significance. This is because $p < 0.05$, that is, $0.013 < 0.05$. (1 mark)
- e. If H_0 is not to be rejected at the 1% level then the p value for the test must be greater than or equal to 0.01. (1 mark)
 When $0.01 = \Pr(Z \leq c)$,
 then $c = -2.326347 \dots$ (inverse normal) (1 mark)
 So the minimum value of the sample mean $\bar{x}$ occurs
 when $\frac{\bar{x} - 8.6}{0.09} = -2.326347 \dots$
 $\bar{x} = 8.390628 \dots$
 So $\bar{x} = 8.391 \text{ kg}$ (correct to three decimal places) (1 mark)

Question 6 (9 marks)

a. i. concentration of salt in tank

$$= \frac{\text{amount of salt in tank at time } t}{\text{volume of brine in tank at time } t}$$

$$= \frac{x}{500 + 40t}$$

(1 mark)

Note that every minute there is an increase in the volume of the brine of $60 - 20 = 40$ litres.

ii.
$$\frac{dx}{dt} = \frac{dx_{\text{inflow}}}{dl} \cdot \frac{dl_{\text{inflow}}}{dt} - \frac{dx_{\text{outflow}}}{dl} \cdot \frac{dl_{\text{outflow}}}{dt}$$

$$= 0 \times 60 - \frac{x}{500 + 40t} \times 20$$

$$= \frac{-x}{25 + 2t}$$

(1 mark)

So
$$\frac{dx}{dt} + \frac{x}{25 + 2t} = 0.$$

(1 mark)

b. Since
$$\frac{dx}{dt} = \frac{-x}{25 + 2t},$$

$$\int -\frac{1}{x} dx = \int \frac{1}{25 + 2t} dt \quad (\text{separation of variables})$$

$$-\log_e(x) + c_1 = \frac{1}{2} \log_e(25 + 2t) + c_2 \quad (\text{since } x > 0, t \geq 0 \text{ we don't need mod signs})$$

$$\log_e(x) - c_1 = -\frac{1}{2} \log_e(25 + 2t) - c_2$$

$$\log_e(x) = \log_e(25 + 2t)^{-\frac{1}{2}} + c \quad \text{where } c = c_1 - c_2$$

(1 mark)

When $t = 0$, $x = 100$

$$\log_e(100) = \log_e\left(\frac{1}{\sqrt{25}}\right) + c$$

$$c = \log_e(500)$$

So
$$\log_e(x) = \log_e\left(\frac{1}{\sqrt{25 + 2t}}\right) + \log_e(500)$$

$$= \log_e\left(\frac{500}{\sqrt{25 + 2t}}\right)$$

$$x = \frac{500}{\sqrt{25 + 2t}}$$

So
$$x(t) = \frac{500}{\sqrt{25 + 2t}} \text{ as required.}$$

(1 mark)

c.

$$x = \frac{500}{\sqrt{25+2t}}$$

$$= 500(25+2t)^{-\frac{1}{2}}$$

$$\frac{dx}{dt} = -250(25+2t)^{-\frac{3}{2}} \times 2$$

$$= \frac{-500}{\sqrt{(25+2t)^3}}$$

So for $\frac{dx}{dt} + \frac{x}{25+2t} = 0$,

$$LS = \frac{-500}{(25+2t)^{\frac{3}{2}}} + \frac{500}{(25+2t)^{\frac{1}{2}}} \times \frac{1}{25+2t}$$

$$= 0$$

$$= RS$$

Have verified.

(1 mark)

The initial conditions are $t = 0, x = 100$.

$$x(t) = \frac{500}{\sqrt{25+2t}}$$

When $t = 0$,

$$x = \frac{500}{\sqrt{25}}$$

$$= 100$$

Have verified.

(1 mark)

d. Method 1

No salt was added to the tank, only water was added.

At $t = 0$, $x = 100\text{g}$

At $t = 20$, $x = 62.0173\dots\text{g}$

(1 mark)

The amount of salt that flowed out in the first 20 minutes was 37.9826... or 38.0 grams (correct to one decimal place).

(1 mark)

Method 2

$$\text{outflow} = \frac{20x}{500+40t} \text{ (from part a.ii.)}$$

$$= \frac{x}{25+2t}$$

Amount of salt flowing out in first 20 minutes

$$= \int_0^{20} \frac{x}{25+2t} dt$$

(1 mark)

$$= \int_0^{20} \frac{1}{25+2t} \times \frac{500}{\sqrt{25+2t}} dt$$

$$= 37.9826\dots$$

= 38.0 grams (correct to one decimal place)

(1 mark)