

**SPECIALIST MATHS 3 & 4
TRIAL EXAMINATION 1
SOLUTIONS
2016**

Question 1 (4 marks)

a.
$$\begin{aligned}\frac{\bar{z}_1}{1+i} &= \frac{-2i}{1+i} \times \frac{1-i}{1-i} \\ &= \frac{-2i-2}{1+1} \\ &= -1-i\end{aligned}$$

(1 mark)

b. Let $P(z) = z^3 - 3z^2 + 4z - 12$.

Substitute $z_1 = 2i$ into $P(z)$.

$$\begin{aligned}P(2i) &= (2i)^3 - 3(2i)^2 + 4(2i) - 12 \\ &= -8i + 12 + 8i - 12 \\ &= 0 \text{ as required}\end{aligned}$$

(1 mark)

- c. Since the coefficients of the terms in the equation are real we know that $-2i$ is also a solution (conjugate root theorem). **(1 mark)**

So $z - 2i$ and $z + 2i$ are factors.

So $(z - 2i)(z + 2i) = z^2 + 4$ is also a factor.

Method 1 – using the constant term

$$P(z) = z^3 - 3z^2 + 4z - 12$$

$$P(3) = 27 - 27 + 12 - 12 = 0$$

So $z = 3$ is the third solution.

The solutions are $z = \pm 2i, 3$.

(1 mark)

Method 2 – using long division

$$\begin{array}{r} z-3 \\ z^2+4 \overline{) z^3-3z^2+4z-12} \\ \underline{z^3 + 4z} \\ -3z^2 -12 \\ \underline{-3z^2 -12} \\ 0 \end{array}$$

$$\text{So } z^3 - 3z^2 + 4z - 12 = 0$$

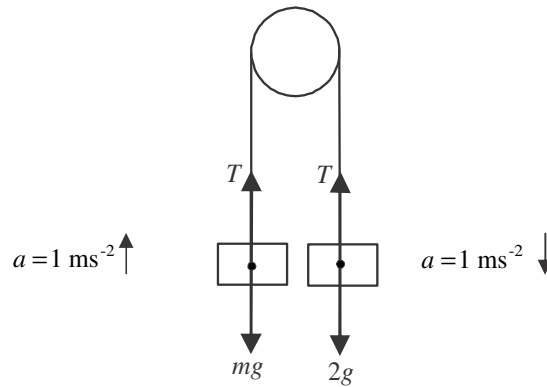
$$\text{becomes } (z^2 + 4)(z - 3) = 0$$

The solutions are $z = \pm 2i, 3$.

(1 mark)

Question 2 (4 marks)

- a. Mark in the forces. The tension force in the string is T .



Around the m kg particle

$$T - mg = m \times 1$$

$$T = m + mg$$

Around the 2 kg particle

$$2g - T = 2 \times 1$$

$$T = 2g - 2$$

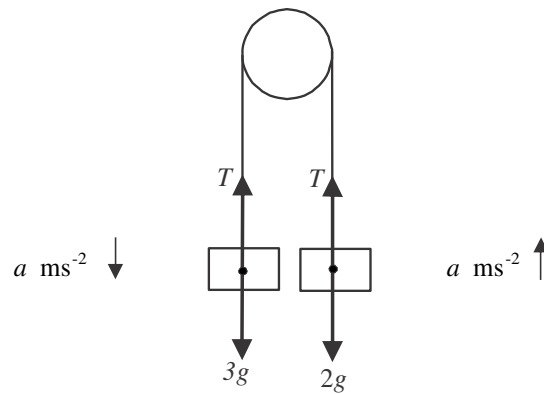
(1 mark)

So $m(1 + g) = 2g - 2$

$$m = \frac{2g - 2}{1 + g}$$

(1 mark)

- b. Mark in the forces.



Around the 3 kg particle

$$3g - T = 3a$$

$$T = 3g - 3a$$

Around the 2 kg particle

$$T - 2g = 2a$$

$$T = 2g + 2a$$

(1 mark)

So $3g - 3a = 2g + 2a$

$$5a = g$$

$$a = \frac{g}{5}$$

(1 mark)

Question 3 (3 marks)

$$f(x) = \arcsin(3x)$$

Method 1

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}} \\ &= \frac{1}{\sqrt{\frac{1-9x^2}{9}}} \\ &= \frac{3}{\sqrt{1-9x^2}} \\ &= 3(1-9x^2)^{-\frac{1}{2}} \end{aligned}$$

Method 2

$$\text{Let } y = \arcsin(u) \quad \text{where } u = 3x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \quad (\text{chain rule}) \\ &= \frac{1}{\sqrt{1-u^2}} \times 3 \\ &= \frac{3}{\sqrt{1-9x^2}} \\ &= 3(1-9x^2)^{-\frac{1}{2}} \end{aligned}$$

(1 mark)

$$\begin{aligned} f''(x) &= 3 \times -\frac{1}{2}(1-9x^2)^{-\frac{3}{2}} \times -18x \quad (\text{chain rule}) \\ &= \frac{27x}{\sqrt{(1-9x^2)^3}} \end{aligned}$$

(1 mark)

$$\begin{aligned} f''\left(\frac{1}{6}\right) &= \frac{27}{6} \div \sqrt{\left(\frac{3}{4}\right)^3} \\ &= \frac{9}{2} \div \sqrt{\frac{27}{64}} \\ &= \frac{9}{2} \times \frac{8}{3\sqrt{3}} \\ &= \frac{12}{\sqrt{3}} \\ &= 4\sqrt{3} \end{aligned}$$

(1 mark)**Question 4** (3 marks)

$$xy^2 + y \log_e(x) - 2y - 3 = 0$$

$$y^2 + 2xy \frac{dy}{dx} + \frac{dy}{dx} \log_e(x) + \frac{y}{x} - 2 \frac{dy}{dx} = 0$$

(1 mark)

$$\begin{aligned} (2xy + \log_e(x) - 2) \frac{dy}{dx} &= -y^2 - \frac{y}{x} \\ &= \frac{-xy^2 - y}{x} \end{aligned}$$

$$\frac{dy}{dx} = \frac{-xy^2 - y}{x(2xy + \log_e(x) - 2)}$$

At (1, 3),

$$\frac{dy}{dx} = \frac{-9-3}{6+\log_e(1)-2}$$

(1 mark)

$$= \frac{-12}{4}$$

$$= -3$$

(1 mark)

Question 5 (6 marks)

- a. The approximate 95% confidence interval for μ is $\left(\bar{x} - 1.96 \frac{s}{\sqrt{100}}, \bar{x} + 1.96 \frac{s}{\sqrt{100}}\right)$.

$$\text{So } \bar{x} - 1.96 \frac{s}{10} = 15.02 \quad (1)$$

$$\text{and } \bar{x} + 1.96 \frac{s}{10} = 16.98 \quad (2)$$

(1 mark)

Solve these two equations simultaneously.

$$(1) + (2) \text{ gives } 2\bar{x} = 32$$

$$\bar{x} = 16$$

$$\text{In (1), } 16 - 0.196s = 15.02$$

$$-0.196s = -0.98$$

$$s = \frac{0.98}{0.196}$$

$$s = \frac{980}{196}$$

$$s = 5$$

(1 mark)

- b. i. $H_0: \mu = 5.2$
 $H_1: \mu > 5.2$

(1 mark)

ii. $E(\bar{X}) = \mu = 5.2$ $sd(\bar{X}) = \frac{1.5}{\sqrt{25}}$
 $= \frac{1.5}{5}$
 $= 0.3$

$$p \text{ value} = \Pr(\bar{X} \geq 5.8 | \mu = 5.2)$$

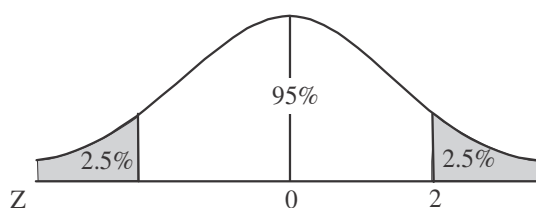
(1 mark)

$$= \Pr\left(Z \geq \frac{5.8 - 5.2}{0.3}\right)$$

$$= \Pr\left(Z \geq \frac{0.6}{0.3}\right)$$

$$= \Pr(Z \geq 2)$$

$$= 0.025$$

**(1 mark)**

- iii. From part ii., $p \text{ value} = 0.025$.
 Since $p \text{ value} < 0.05$ there is good evidence to reject the null hypothesis at the 5% level.

(1 mark)

Question 6 (3 marks)

$$\int_0^{\frac{\pi}{4}} \cos^2(x) \sin(2x) dx$$

$$= \int_0^{\frac{\pi}{4}} 2 \sin(x) \cos^3(x) dx$$

since $\sin(2x) = 2 \sin(x) \cos(x)$ **(1 mark)**

$$= -2 \int_1^{\frac{1}{\sqrt{2}}} u^3 \frac{du}{dx} dx$$

Let $u = \cos(x)$ also $x = \frac{\pi}{4}$ so $u = \frac{1}{\sqrt{2}}$

$$= -2 \int_1^{\frac{1}{\sqrt{2}}} u^3 du$$

$$\frac{du}{dx} = -\sin(x)$$

and $x = 0$ so $u = 1$ **(1 mark)**

$$= -2 \left[\frac{u^4}{4} \right]_1^{\frac{1}{\sqrt{2}}}$$

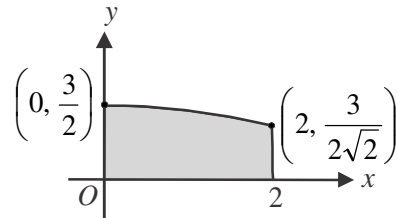
$$= -2 \left(\frac{1}{16} - \frac{1}{4} \right)$$

$$= -\frac{1}{8} + \frac{1}{2}$$

$$= \frac{3}{8}$$

(1 mark)**Question 7** (3 marks)**a.** Do a quick sketch.

$$\begin{aligned} \text{volume} &= \pi \int_0^2 y^2 dx \\ &= \pi \int_0^2 \frac{9}{x^2 + 4} dx \end{aligned}$$

**(1 mark)**

$$\begin{aligned} \text{b. volume} &= \frac{9\pi}{2} \int_0^2 \frac{2}{x^2 + 4} dx \\ &= \frac{9\pi}{2} \left[\tan^{-1} \left(\frac{x}{2} \right) \right]_0^2 \\ &= \frac{9\pi}{2} (\tan^{-1}(1) - \tan^{-1}(0)) \\ &= \frac{9\pi}{2} \left(\frac{\pi}{4} - 0 \right) \\ &= \frac{9\pi^2}{8} \text{ cubic units} \end{aligned}$$

(1 mark)

Question 8 (5 marks)

- a. We require $\underline{a} \bullet \underline{b} = 0$

$$2 + m + 8 = 0$$

$$m = -10$$

(1 mark)

- b. $\underline{a} = 2\underline{i} + \underline{j} - 2\underline{k}$

$$|\underline{a}| = \sqrt{2^2 + 1^2 + (-2)^2}$$

$$= 3$$

$$\hat{\underline{a}} = \frac{1}{3}(2\underline{i} + \underline{j} - 2\underline{k})$$

(1 mark)

$$\text{So } \underline{d} = 6\hat{\underline{a}}$$

$$= 6 \times \frac{1}{3}(2\underline{i} + \underline{j} - 2\underline{k})$$

$$= 4\underline{i} + 2\underline{j} - 4\underline{k}$$

(1 mark)

- c. If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly dependent then $\alpha\underline{a} + \beta\underline{b} = \underline{c}$ where α and $\beta \in \mathbb{R}$.

$$\text{So we require } \alpha(2\underline{i} + \underline{j} - 2\underline{k}) + \beta(\underline{i} + m\underline{j} - 4\underline{k}) = -\underline{i} + 3\underline{j}.$$

For the $\underline{i}$ components,

$$2\alpha + \beta = -1 \quad (1)$$

For the $\underline{j}$ components,

$$\alpha + m\beta = 3 \quad (2)$$

For the $\underline{k}$ components,

$$-2\alpha - 4\beta = 0$$

$$\text{So } \alpha = -2\beta$$

(1 mark)

$$\text{In (1) } -3\beta = -1$$

$$\beta = \frac{1}{3}$$

$$\text{So } \alpha = -\frac{2}{3}$$

$$\text{In (2) } -\frac{2}{3} + \frac{m}{3} = 3$$

$$m = 11$$

(1 mark)

If you have time, check your answer.

$$-\frac{2}{3}\underline{a} + \frac{1}{3}\underline{b} = \underline{c}$$

$$LS = -\frac{4}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{4}{3}\underline{k} + \frac{1}{3}\underline{i} + \frac{11}{3}\underline{j} - \frac{4}{3}\underline{k}$$

$$= -\underline{i} + 3\underline{j}$$

$$= RS$$

Question 9 (5 marks)

$$\frac{dy}{dx} = \frac{x\sqrt{x^2-1}}{e^{2y}}$$

$$\int e^{2y} dy = \int x\sqrt{x^2-1} dx \quad (\text{separation of variables}) \quad (1 \text{ mark})$$

$$\frac{e^{2y}}{2} + c_1 = \int \frac{1}{2} \frac{du}{dx} u^{\frac{1}{2}} dx \quad \left| \quad \begin{array}{l} u = x^2 - 1 \\ \frac{du}{dx} = 2x \end{array} \right.$$

$$= \frac{1}{2} \int u^{\frac{1}{2}} du$$

$$= \frac{1}{2} u^{\frac{3}{2}} \times \frac{2}{3} + c_2$$

$$\text{So } \frac{e^{2y}}{2} = \frac{1}{3} (x^2 - 1)^{\frac{3}{2}} + c \quad \text{where } c = c_2 - c_1$$

(1 mark) – left side

(1 mark) – right side

When $x=1$, $y=0$.

$$\text{So } \frac{e^0}{2} = \frac{1}{3} \times 0 + c$$

$$c = \frac{1}{2}$$

(1 mark)

$$\text{So } \frac{e^{2y}}{2} = \frac{1}{3} (x^2 - 1)^{\frac{3}{2}} + \frac{1}{2}$$

$$e^{2y} = \frac{2}{3} (x^2 - 1)^{\frac{3}{2}} + 1$$

$$2y = \log_e \left(\frac{2}{3} (x^2 - 1)^{\frac{3}{2}} + 1 \right)$$

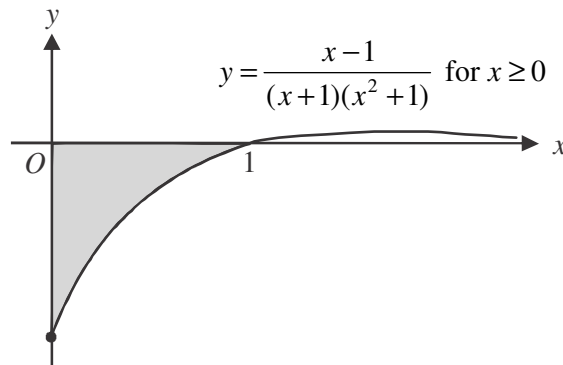
$$y = \log_e \sqrt{\frac{2}{3} (x^2 - 1)^{\frac{3}{2}} + 1}$$

So $a=2$, $b=3$ and $c=1$.

(1 mark)

Question 10 (4 marks)

A sketch of $y = \frac{x-1}{(x+1)(x^2+1)}$ for $x \geq 0$ is shown below.



The shaded region shown above is equal in area to the shaded region shown in the question, but is below the x -axis.

The x -intercept for both graphs occurs when $x-1=0$ i.e. at $(1,0)$.

$$\begin{aligned} \text{Let } \frac{(x-1)}{(x+1)(x^2+1)} &\equiv \frac{A}{(x+1)} + \frac{Bx+C}{x^2+1} && \text{(1 mark)} \\ &\equiv \frac{A(x^2+1) + (Bx+C)(x+1)}{(x+1)(x^2+1)} \end{aligned}$$

$$\text{True iff } x-1 = A(x^2+1) + (Bx+C)(x+1)$$

$$\text{Put } x = -1, \quad -2 = 2A, \quad A = -1$$

$$\text{Put } x = 1, \quad 0 = -2 + 2B + 2C \quad \text{_____ (1)}$$

$$\text{Put } x = 0, \quad -1 = -1 + C, \quad C = 0$$

$$\text{In (1) } \quad \text{so } B = 1 \quad \text{(1 mark)}$$

$$\text{So } \frac{x-1}{(x+1)(x^2+1)} = \frac{-1}{x+1} + \frac{x}{x^2+1}$$

$$\text{area required} = -\int_0^1 \left(\frac{-1}{x+1} + \frac{x}{x^2+1} \right) dx \quad \text{(1 mark)}$$

$$= \left[\log_e |x+1| \right]_0^1 - \left[\frac{1}{2} \log_e (x^2+1) \right]_0^1$$

$$= \log_e (2) - \log_e (1) - \frac{1}{2} \log_e (2) + \frac{1}{2} \log_e (1)$$

$$= \log_e (2) - \log_e \sqrt{2}$$

$$= \log_e \left(\frac{2}{\sqrt{2}} \right)$$

$$= \log_e (\sqrt{2}) \text{ square units}$$

(1 mark)