



Units 3 and 4 Specialist Maths: Exam 1

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Any questions?

Check the Engage website for updated solutions, then email practiceexams@ee.org.au.

Section A – Multiple-choice questions

Question 1

- a. For $f(x)$ to be a probability density function, the total area under the curve has to be 1.

$$\int_{-a}^a \frac{3}{16} x^2 dx = 1$$

$$\left[\frac{1}{16} x^3 \right]_{-a}^a = 1$$

$$\frac{1}{16} a^3 + \frac{1}{16} a^3 = 1 \quad [1]$$

$$a^3 = 8$$

$$a = 2 \quad [1]$$

- b. Mean $E(\bar{X}) = \mu = E(X) = 0$ (the probability density function is symmetrical) $[1/2]$

$$\text{Variance } \text{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{E(X^2) - E^2(X)}{100} = \frac{\int_{-2}^2 x^2 f(x) dx}{100} = \frac{\int_{-2}^2 \frac{3}{16} x^4 dx}{100} = \left[\frac{3}{80} x^5 \right]_{-2}^2 = \frac{3}{40} \times 2^5 = \frac{3 \times 32}{40} = \frac{12}{5} \quad [1]$$

Hence, $\bar{X} \sim N\left(0, \frac{12}{5}\right)$ (due to central limit theorem) $[1/2]$

Question 2

Let W be the weight of a given individual mice.

$W \sim N(\mu, 196)$ where μ is the true mean of the population

$\bar{W} = \frac{1}{n} \sum W_i \sim N\left(\mu, \frac{256}{n}\right)$ since it is a sum of identical independent normal random variables

Thus,

$$Z = \frac{\bar{W} - \mu}{\left(\frac{14}{\sqrt{n}}\right)} \sim N(0,1)$$

$$\Pr(-1.96 < Z < 1.96) = 0.95$$

$$\Rightarrow Z \in (-1.96, 1.96) \text{ with 95\% confidence} \quad \left[\frac{1}{2} \right]$$

$$\Rightarrow \frac{\bar{W} - \mu}{\left(\frac{14}{\sqrt{n}}\right)} \in (-1.96, 1.96)$$

$$\Rightarrow \mu \in \left(\bar{H} - 1.96 \left(\frac{14}{\sqrt{n}} \right), \bar{H} + 1.96 \left(\frac{14}{\sqrt{n}} \right) \right)$$

Substituting the given numbers:

$$\mu \in \left(197 - 1.96 \left(\frac{14}{10} \right), 197 + 1.96 \left(\frac{14}{10} \right) \right)$$

$$\mu \in (194, 200) \text{ with 95\% confidence} \quad [1/2]$$

Since the prediction, 176cm is an element of the 95% confidence interval, the prediction cannot be rejected at the 0.05 significance level. $[1]$

Explanation of significance: if the sampling were to be repeated multiple times, 95% of the times, the confidence interval would contain the **predicted** population mean of 200 g. $[1]$

Question 3

$$\frac{d}{dx}(x^2 y + \log_e y + x) = \frac{d}{dx}(2)$$

$$2xy + x^2 \frac{dy}{dx} + \frac{1}{y} \times \frac{dy}{dx} + 1 = 0$$

good attempt at implicit differentiation $[1]$

$$\frac{dy}{dx} \left(x^2 + \frac{1}{y} \right) = -1 - 2xy$$

$$\therefore \frac{dy}{dx} = \frac{-1-2xy}{x^2 + \frac{1}{y}}$$

correct answer [1]

Question 4

$$\begin{aligned} P(z) &= z^2 + 5z + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 8 \\ &= \left(z + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{8 * 4}{4} \\ &= \left(z + \frac{5}{2}\right)^2 + \frac{7}{4} \\ &= \left(z + \frac{5}{2}\right)^2 - \left(\frac{\sqrt{7}}{2}i\right)^2 \\ &= \left(z + \frac{5}{2} + \frac{\sqrt{7}}{2}i\right)\left(z + \frac{5}{2} - \frac{\sqrt{7}}{2}i\right) \\ &= 0 \end{aligned}$$

successful completion of the square [1]

(other methods also ok, eg. quadratic formula)

So, by the null factor theorem,

$$\begin{aligned} z + \frac{5}{2} + \frac{\sqrt{7}}{2}i &= 0 \text{ or } z + \frac{5}{2} - \frac{\sqrt{7}}{2}i = 0 \\ \therefore z &= -\frac{5}{2} \pm \frac{\sqrt{7}}{2}i \end{aligned}$$

both answers correct [2]

Question 5

$$\text{area} = \left| \int_{-2}^{-1} \frac{7x+1}{x^2+2x-8} dx \right|$$

correct integral for area [1]

$$\frac{7x+1}{x^2+2x-8} = \frac{7x+1}{(x+4)(x-2)} \equiv \frac{A}{x+4} + \frac{B}{x-2}$$

$$7x+1 \equiv A(x-2) + B(x+4)$$

$$\text{Let } x = -4$$

$$7 \times -4 + 1 = A(-4-2) + B(-4+4)$$

$$\therefore A = \frac{9}{2}$$

$$\text{Let } x = 2$$

$$7 \times 2 + 1 = A(2-2) + B(2+4)$$

$$\therefore B = \frac{5}{2}$$

$$\Rightarrow \int_{-2}^{-1} \frac{7x+1}{x^2+2x-8} dx = \int_{-2}^{-1} \frac{9}{2(x+4)} + \frac{5}{2(x-2)} dx$$

correct splitting of fraction [1]

$$= \left[\frac{9}{2} \log_e |x+4| + \frac{5}{2} \log_e |x-2| \right]_{-2}^{-1}$$

correct integration [1]

$$= \left[\frac{9}{2} \log_e |-1+4| + \frac{5}{2} \log_e |-1-2| \right] - \left[\frac{9}{2} \log_e |-2+4| + \frac{5}{2} \log_e |-2-2| \right]$$

$$= \frac{9}{2} \log_e (3) + \frac{5}{2} \log_e (3) - \frac{9}{2} \log_e (2) - \frac{5}{2} \log_e (4)$$

$$= 7 \log_e (3) - \frac{9}{2} \log_e (2) - \frac{5}{2} \log_e (4)$$

answer [1]

Question 6a

$$x = \sec^2(t) \text{ and } y^2 = \tan^2(t)$$

Using the trigonometric identity $1 + \tan^2(x) = \sec^2(x)$: [1]

$$1 + y^2 = x$$

$$y^2 = x - 1$$

$$\therefore y = \pm \sqrt{x-1} \text{ but } y > 0 \text{ in the provided domain,}$$

$$\text{so } y = \sqrt{x-1}$$

answer [1]

Question 6b

$$x = \sec^2(t) \text{ for } t \in \left\{ t: 0 \leq t \leq \frac{\pi}{4} \right\}$$

$$\sec^2(0) = \left(\frac{1}{\cos(0)} \right)^2 = 1$$

$$\sec^2\left(\frac{\pi}{4}\right) = \left(\frac{1}{\cos\left(\frac{\pi}{4}\right)} \right)^2 = 2$$

$$\text{Hence } 1 \leq x \leq 2.$$

$$y = \tan(t) \text{ for } t \in \left\{ t: 0 \leq t \leq \frac{\pi}{4} \right\}$$

$$\tan(0) = 0$$

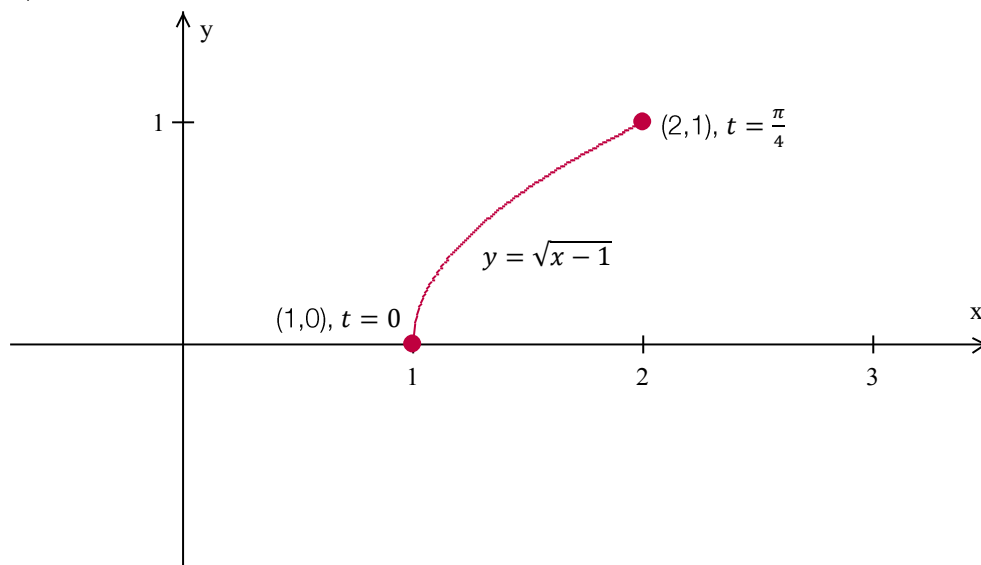
$$\tan\left(\frac{\pi}{4}\right) = 1$$

$$\text{Hence } 0 \leq y \leq 1.$$

$$\text{dom} = [1,2] \text{ and } \text{ran} = [0,1]$$

answers [2]

Question 6c



graph shape [1], intercept with coordinate and t value [1], closed endpoints with coordinates and t values [1]

Question 7

$$\operatorname{cosec}\left(\frac{5\pi}{12}\right) = \frac{1}{\sin\left(\frac{5\pi}{12}\right)}$$

use of reciprocal circular identities [1]

$$\frac{5\pi}{12} = \frac{4\pi}{12} + \frac{\pi}{12} = \frac{4\pi}{12} + \left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{3} + \frac{\pi}{3} - \frac{\pi}{4} = \frac{2\pi}{3} - \frac{\pi}{4}$$

$$\sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y)$$

$$\therefore \sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{2\pi}{3}\right) \cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{2\pi}{3}\right) \sin\left(\frac{\pi}{4}\right)$$

correct selection of double angle formula [1]

$$= \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} + \frac{1}{2} \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\therefore \operatorname{cosec}\left(\frac{5\pi}{12}\right) = \frac{4}{\sqrt{6} + \sqrt{2}}$$

$$= \frac{4(\sqrt{6} - \sqrt{2})}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})}$$

$$= \frac{4(\sqrt{6} - \sqrt{2})}{6 - 2}$$

$$= \frac{4(\sqrt{6} - \sqrt{2})}{4} = \sqrt{6} - \sqrt{2}, \text{ as required.}$$

working leading to correct answer [1]

Question 8

$$\frac{d}{dx} \left(\tan^{-1} \left(\frac{x}{5} \right) \right) = \frac{5}{25 + x^2}$$

$$\text{Let } u = \tan^{-1} \left(\frac{x}{5} \right), \frac{du}{dx} = \frac{5}{25 + x^2}$$

correct substitution [1]

$$\int \frac{\tan^{-1} \frac{x}{5}}{25 + x^2} dx = \frac{1}{5} \int \tan^{-1} \frac{x}{5} * \frac{5}{25 + x^2} dx$$

$$= \frac{1}{5} \int u \frac{du}{dx} dx$$

$$= \frac{1}{5} \int u du$$

successful working [1]

$$\frac{1}{5} \times \frac{1}{2} u^2 + c \text{ where } c \in \mathbb{R}$$

$$\therefore \int \frac{\tan^{-1} \frac{x}{5}}{25 + x^2} dx = \frac{1}{10} \left(\tan^{-1} \frac{x}{5} \right)^2 + c$$

answer [1]

Question 9

From formula sheet:

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1 = 1 - 2 \sin^2(x)$$

$$\Rightarrow \cos^2(x) = \frac{\cos(2x) + 1}{2} \text{ and } \sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\therefore \int_0^{\pi} \cos^2(x) \sin^2(x) dx = \int_0^{\pi} \left(\frac{\cos(2x) + 1}{2} \right) \left(\frac{1 - \cos(2x)}{2} \right) dx$$

$$= \frac{1}{4} \int_0^{\pi} 1 - \cos^2(2x) dx$$

correct use of double-angle formula [1]

$$\cos(4x) = 2 \cos^2(2x) - 1 \Rightarrow \cos^2(2x) = \frac{1 + \cos(4x)}{2}$$

$$\therefore \frac{1}{4} \int_0^{\pi} 1 - \cos^2(2x) dx = \frac{1}{4} \int_0^{\pi} 1 - \frac{1 + \cos(4x)}{2} dx$$

second correct use of double-angle formula [1]

$$= \frac{1}{8} \int_0^{\pi} 1 + \cos(4x) dx$$

$$= \frac{1}{8} \left[x + \frac{1}{4} \sin(4x) \right]_0^{\pi}$$

correct integral found [1]

$$= \frac{1}{8} \left[\left(\pi + \frac{1}{4} \sin(4\pi) \right) - \left(0 + \frac{1}{4} \sin(0) \right) \right]$$

$$= \frac{1}{8} (\pi + 0 - 0)$$

$$= \frac{\pi}{8}$$

answer [1]

Question 10a

$$x = t^3 \Rightarrow t = x^{\frac{1}{3}}$$

$$y = \log_e(t) = \log_e(x^{\frac{1}{3}})$$

answer [1]

Question 10b

$$\mathbf{v}(t) = \frac{d}{dt}(\mathbf{r}(t))$$

$$= 3t^2 \mathbf{i} + \frac{1}{t} \mathbf{j}$$

derivative [1]

$$\text{speed} = |\mathbf{v}(t)|$$

$$= \sqrt{(3t^2)^2 + \left(\frac{1}{t}\right)^2}$$

[1]

$$= \sqrt{9t^4 + \frac{1}{t^2}}$$

answer [1]

Question 10c

Speed is a minimum when $\frac{d}{dx}(|\mathbf{v}(t)|) = 0$

stating derivative of speed should be zero for speed to be at a minimum [1]

$$\frac{d}{dx} \left(\sqrt{9t^4 + \frac{1}{t^2}} \right) = \frac{1}{2} (9t^4 + t^{-2})^{-\frac{1}{2}} (36t^3 - 2t^{-3}) = 0 \quad \text{evaluation of derivative [1]}$$

$\left(9t^4 + \frac{1}{t^2}\right)^{-\frac{1}{2}} \neq 0$ due to the negative power

$36t^3 - 2t^{-3} = 0$ by the null factor theorem.

$$36t^3 = \frac{2}{t^3}$$

$$t^6 = \frac{1}{18}$$

$$\therefore t = \left(\frac{1}{18}\right)^{\frac{1}{6}}$$

answer [1]