

SPECIALIST MATHEMATICS

Written examination 2



(TSSM's 2015 trial exam updated for the current study design)

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: A

Explanation:

$$x = 5 - 6 \sec(2t), y = 3 + 5 \tan(2t) \Rightarrow \frac{(x-5)^2}{36} - \frac{(y-3)^2}{25} = 1$$

$$\Rightarrow \text{the equation of the asymptotes: } \frac{x-5}{6} \pm \frac{y-3}{5} = 0$$

$$\Rightarrow y = \frac{5}{6}x - \frac{7}{6} \text{ or } y = -\frac{5}{6}x + \frac{43}{6}$$

Question 2

Answer: E

Explanation

From the graph we can see the semi-major axis $a = 7 - 2 = 5$
and the semi-minor axis $b = 3$

Therefore the equation of this ellipse is $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{9} = 1$

Question 3**Answer: B***Explanation:*

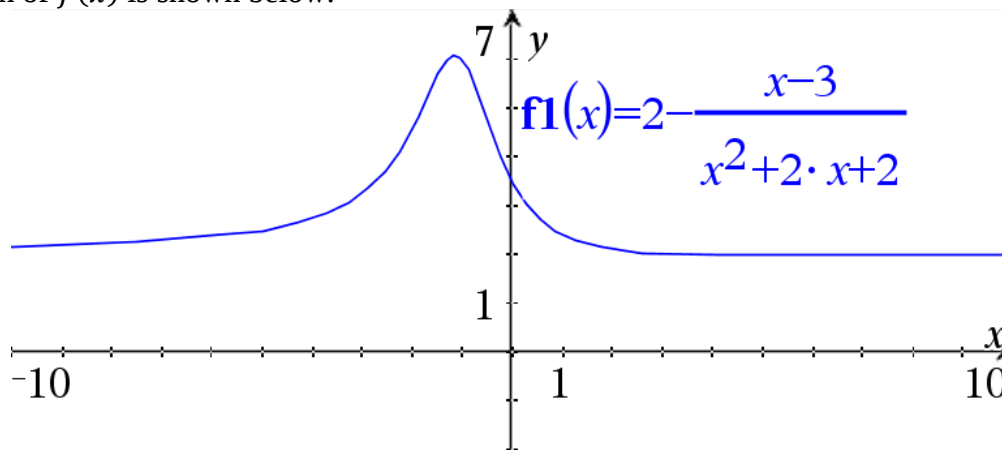
$$f(x) = \frac{2x^2 + 3x + 7}{x^2 + 2x + 2} = \frac{2(x^2 + 2x + 2) - x + 3}{x^2 + 2x + 2} = 2 - \frac{x - 3}{x^2 + 2x + 2}$$

$\Rightarrow y = 2$ is the horizontal asymptote

For $x^2 + 2x + 2 = (x + 1)^2 + 1$, the discriminant $\Delta = 2^2 - 4 \times 1 \times 2 = -4 < 0$. Hence $x^2 + 2x + 2 \neq 0$ and $(-1, 1)$ is the minimal point of $(x + 1)^2 + 1$.

Therefore there are no vertical asymptotes. $f(x)$ has a maximal point at $x = -1$.

The graph of $f(x)$ is shown below.

**Question 4****Answer: E***Explanation:*

$$\arctan(2 - 3x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow 4 \arctan(2 - 3x) + 5 \in \left(-\frac{\pi}{2} \times 4 + 5, \frac{\pi}{2} \times 4 + 5\right) = (5 - 2\pi, 5 + 2\pi)$$

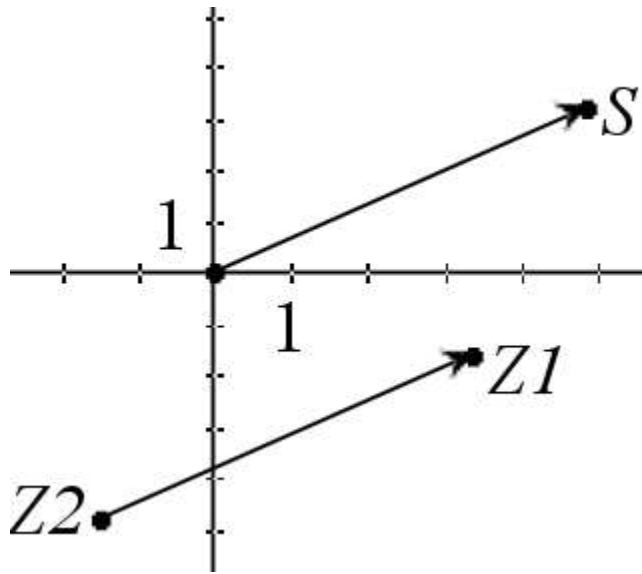
Question 5

Answer: D

Explanation:

Any complex number can be regarded as a position vector.

Hence $z = z_1 - z_2 = \overrightarrow{z_2 z_1} = \overrightarrow{OS}$.



Question 6

Answer: E

Explanation:

$|z - z_1| = |z - z_2|$ represents the locus of points with equal distance to z_1 and z_2 , which is the perpendicular bisector between z_1 and z_2 .

Question 7

Answer: B

Explanation:

$$z = -5\sqrt{3} - 5i = 10\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 10\text{cis}\left(-\frac{5\pi}{6}\right)$$

$$\Rightarrow z^{10} = 10^{10}\left(\cos\left(-\frac{5\pi}{6} \times 10\right) + \sin\left(-\frac{5\pi}{6} \times 10\right)i\right) = 10^{10}\text{cis}\left(-\frac{\pi}{3}\right)$$

Question 8**Answer: C***Explanation:*

Geometric interpretation of multiplication of complex numbers: $z = 4cis(72^\circ)z_1$ can be obtained from z_1 by a rotation of 72° around the origin in anti-clockwise, followed by a dilation of factor 4 from the origin.

Question 9**Answer: A***Explanation:*

For any quadratic equation $az^2 + bz + c, a \neq 0$, the two solutions are $u, v = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Hence $u + v = \frac{-b}{a}, u \times v = \frac{c}{a}$

$$u + v = \frac{3i}{5} = \frac{6i}{10}, u \times v = \frac{7}{10} \Rightarrow a = 10, b = -6i, c = 7.$$

Question 10**Answer: B***Explanation:*

$$\int_{\log_e(\frac{\pi}{6})}^{\log_e(\frac{\pi}{2})} \frac{e^x}{1 + e^{2x}} dx = \int_{\log_e(\frac{\pi}{6})}^{\log_e(\frac{\pi}{2})} \frac{1}{1 + (e^x)^2} de^x$$

Let $u = e^x$. Then $u = \frac{\pi}{2}$ when $x = \log_e(\frac{\pi}{2})$ and $u = \frac{\pi}{6}$ when $x = \log_e(\frac{\pi}{6})$

$$\text{Therefore } \int_{\log_e(\frac{\pi}{6})}^{\log_e(\frac{\pi}{2})} \frac{e^x}{1 + (e^x)^2} de^x = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{1 + u^2} du$$

Question 11**Answer: E***Explanation:*

According the Fundamental Theorem of Calculus, $G(5) - G(1) = \int_1^5 g(x) dx = 8$.

Therefore $G(5) = 8 + G(1) = 8 + 2 = 10$

Question 12**Answer: E***Explanation:*

A useful mathematical model for setting up differential equations of dynamic systems

$$\frac{dx}{dt} = R_{in} \times C_{in} - R_{out} \times C_{out}$$

where R_{in} and R_{out} are the flowing in and flowing out rate; C_{in} and C_{out} are the concentrations of the solutions which are flowing in and flowing out respectively.

Therefore

$$\frac{dx}{dt} = 2 \times 12 - 1.5 \times \frac{x}{85 + (2 - 1.5)t} = 24 - \frac{3x}{170 + t}$$

Question 13**Answer: C***Explanation:*

Use a List and Spread Sheets in CAS to solve

	A x	B y	C dy
=			
1	2	6	1.1275
2	2.2	6.2255	1.14678
3	2.4	6.45486	1.16756

Question 14**Answer: A***Explanation:*

Look at the slope field in CAS for each of the differential equations.

Question 15**Answer: D***Explanation:*

The component coefficient matrix of the vectors is $\begin{bmatrix} 1 & -1 & 1 \\ 2 & a & 1 \\ 5 & 5 & a \end{bmatrix}$.

Let $\det\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & a & 1 \\ 5 & 5 & a \end{bmatrix}\right) = 0$. Solve for a by CAS, shown below

$$\text{solve}\left(\det\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & a & 1 \\ 5 & 5 & a \end{bmatrix}\right)=0, a\right) \quad a=0 \text{ or } a=3$$

Question 16**Answer: C***Explanation:*

$$\cos(\theta) = \frac{\tilde{a} \cdot \tilde{b}}{|\tilde{a}| |\tilde{b}|} = \frac{3+2}{\sqrt{13}\sqrt{13}} = \frac{5}{13} \Rightarrow \tan(\theta) = \frac{12}{5} \Rightarrow \tan(2\theta) = \frac{2 \tan(\theta)}{1 - \tan^2(\theta)} = -\frac{120}{119}$$

Question 17**Answer: A***Explanation:*

$$\tilde{r}(t) = \int \left(\frac{e^t + e^{-t}}{2} \tilde{i} + \frac{e^t - e^{-t}}{2} \tilde{j} + 2t \tilde{k} \right) dt = \frac{e^t - e^{-t}}{2} \tilde{i} + \frac{e^t + e^{-t}}{2} \tilde{j} + t^2 \tilde{k} + \tilde{c}$$

$$\tilde{r}(0) = 2\tilde{j} + 2\tilde{k} = \tilde{j} + \tilde{c} \Rightarrow \tilde{c} = \tilde{j} + 2\tilde{k}$$

$$\Rightarrow \tilde{r}(t) = \frac{e^t - e^{-t}}{2} \tilde{i} + \frac{e^t + e^{-t} + 2}{2} \tilde{j} + (t^2 + 2) \tilde{k} = \frac{e^t - e^{-t}}{2} \tilde{i} + \frac{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^2}{2} \tilde{j} + (t^2 + 2) \tilde{k}$$

Question 18

Answer: E

Explanation:

The resultant force of any number of forces acting upon the body is the sum of all the forces.

Question 19

Answer: C

Explanation:

The velocity $\vec{v}(t) = -24 \cos(3t) \vec{i} - 45 \sin(3t) \vec{j}$, $t \geq 0$.

The speed $|\vec{v}(t)| = \sqrt{24^2 \cos^2(3t) + 45^2 \sin^2(3t)} = \sqrt{24^2 + 1449 \sin^2(3t)}$

Therefore the minimum speed is 24 m/s

Question 20

Answer: B

Explanation:

$$v^2 - u^2 = 2as \Rightarrow v = \sqrt{u^2 + 2as} = \sqrt{(-2)^2 + 2 \times 9.8 \times 10} = 10\sqrt{2}$$

Question 21

Answer: B

Explanation:

$$a = v \frac{dv}{dx} = ve^{\frac{x^2}{100}} \Rightarrow \frac{dv}{dx} = e^{\frac{x^2}{100}} \Rightarrow v(5) - v(1) = \int_1^5 e^{\frac{x^2}{100}} dx$$

$$v(5) = v(1) + \int_1^5 e^{\frac{x^2}{100}} dx \approx 7.45$$

Question 22

Answer: D

Explanation:

$$\text{Displacement} = \int_0^6 v(t) dt = \int_0^6 (t^3 - 4t^2 - 4t + 16) dt = 60$$

SECTION 2: Extended Response questions

Question 1 (10 marks)

a.

$$x = 2 + 3 \tan(t), \quad y = 4 \sec(t) \Rightarrow \tan(t) = \frac{x-2}{3}, \sec(t) = \frac{y}{4} \Rightarrow \frac{y^2}{16} - \frac{(x-2)^2}{9} = 1$$

1 mark

b. The equation of the asymptotes are

$$\frac{y}{4} \pm \frac{x-2}{3} = 0 \Rightarrow y = \pm \frac{4}{3}(x-2)$$

1 mark

c. Differentiate both sides of $\frac{y^2}{16} - \frac{(x-2)^2}{9} = 1$, we get

$$\frac{2yy'}{16} - \frac{2(x-2)}{9} = 0 \Rightarrow y' = \frac{16(x-2)}{9y}$$

$$\Rightarrow y' = \frac{16(6-2)}{9 \times \frac{20}{3}} = \frac{16}{15} \text{ when } x = 6, y = \frac{20}{3}.$$

The equation of the tangent at $(6, \frac{20}{3})$ is

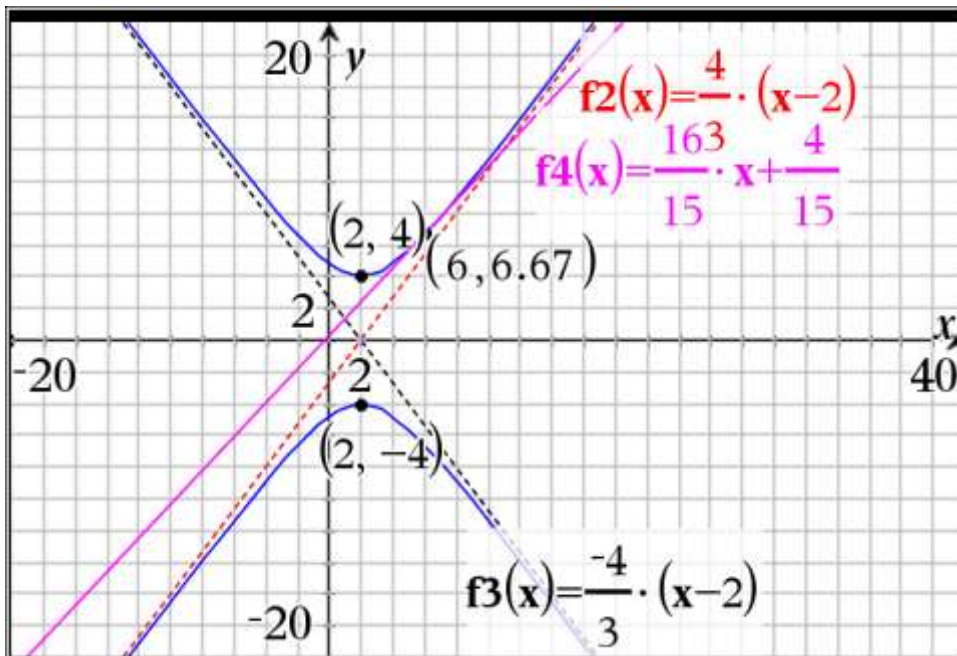
$$y - \frac{20}{3} = \frac{16}{15}(x - 6).$$

Or

$$y = \frac{16}{15}x + \frac{4}{15}$$

1 mark

d.



4 marks

e. $V = \pi \int_0^4 \left(16 + \frac{16(x-2)^2}{9}\right) dx$

1 mark

f. By CAS

$$V = \pi \int_0^4 \left(16 + \frac{16(x-2)^2}{9}\right) dx \approx 230.85$$

1 mark

Question 2

a. $z = 2\sqrt{2} + 2\sqrt{2}i = 4\text{cis}\left(\frac{\pi}{4}\right) \Rightarrow z^4 = 4^4\text{cis}(\pi) = -256$

1 mark

b. $z^4 = 4^4\text{cis}(\pi) \Rightarrow z = 4\text{cis}\left(\frac{\pi}{4}\right), 4\text{cis}\left(\frac{\pi}{4} + \frac{\pi}{2}\right), 4\text{cis}\left(\frac{\pi}{4} + \frac{2\pi}{2}\right), 4\text{cis}\left(\frac{\pi}{4} + \frac{3\pi}{2}\right)$

Therefore the other solutions are

$$z = 4\text{cis}\left(\frac{3\pi}{4}\right), \quad 4\text{cis}\left(-\frac{3\pi}{4}\right), \quad 4\text{cis}\left(-\frac{\pi}{4}\right)$$

1 mark

c. The shape of the locus represented by $|z - 2\sqrt{2} - 2\sqrt{2}i| = |z + 2\sqrt{2} + 2\sqrt{2}i|$ is the perpendicular bisector of the line segment connecting $A(2\sqrt{2}, 2\sqrt{2})$ and $B(-2\sqrt{2}, -2\sqrt{2})$.

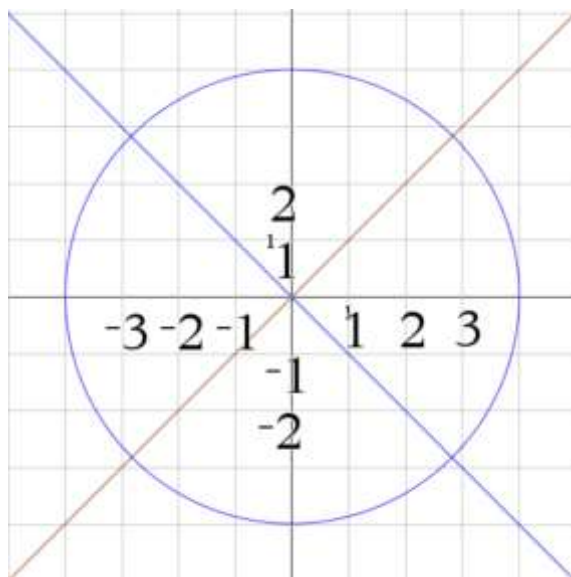
The gradient of AB is 1 $\Rightarrow$ the gradient of the bisector is -1 .

The midpoint of AB is $(0, 0)$

Therefore the equation of the bisector is $y = -x$

d.

The shape of the locus represented by $|z| = |2\sqrt{2} + 2\sqrt{2}i|$ is a circle with centre $(0, 0)$ and radius $r = |2\sqrt{2} + 2\sqrt{2}i| = 4$.



1 mark

e. i. From the CAS

$$\sqrt{\left(2\sqrt{2} - \frac{8\sqrt{42}}{7}\right)^2 + (2\sqrt{2})^2} + \sqrt{\left(2\sqrt{2} + \frac{8\sqrt{42}}{7}\right)^2 + (2\sqrt{2})^2}$$

16

Hence LHS=RHS.

ii. From the equation given in ei, the semi-major axis of the ellipse is 8 and the centre is at (0,0).

Let (0, b) be a vertex of the ellipse on the y-axis. Substitute $z = bi$ into the equation then

then we have $2\sqrt{\left(\frac{8\sqrt{42}}{7}\right)^2 + b^2} = 16$. Solve for b by CAS

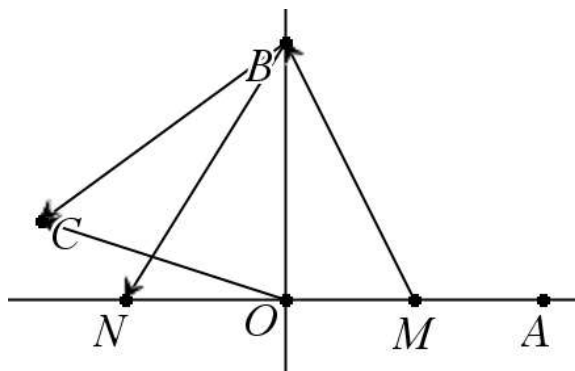
$$\text{solve}\left(b^2 + \left(\frac{8\sqrt{42}}{7}\right)^2 = 8, b \mid b > 0\right) \quad b = \frac{8\sqrt{7}}{7}$$

Therefore $b = \frac{8\sqrt{7}}{7}$.

Hence the equation of the ellipse is

$$\frac{x^2}{64} + \frac{7y^2}{64} = 1$$

Question 3



a. $|\overrightarrow{MB}| = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2}$

1 mark

b. $\overrightarrow{ON} = -|\overrightarrow{MB}|i + \frac{1}{2}i = \frac{-\sqrt{5}+1}{2}i$

2 marks

c. $|\overrightarrow{BN}| = \sqrt{1^2 + \left(\frac{-\sqrt{5}+1}{2}\right)^2} = \frac{\sqrt{10-2\sqrt{5}}}{2}$

1 mark

d. Let θ be the angle between $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

Then $\cos(\theta) = \frac{\overrightarrow{OB} \cdot \overrightarrow{OC}}{|\overrightarrow{OB}| |\overrightarrow{OC}|} = \frac{\frac{\sqrt{5}-1}{4}}{1 \times 1} = \frac{\sqrt{5}-1}{4}.$

While $\cos\left(2 \times \frac{\pi}{5}\right) = 2\cos^2\left(\frac{\pi}{5}\right) - 1 = 2 \times \frac{5+2\sqrt{5}+1}{16} - 1 = \frac{\sqrt{5}-1}{4}$

Therefore $\theta = \frac{2\pi}{5}.$

3 marks

e. $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\underset{\sim}{j} + \frac{-\sqrt{10+2\sqrt{5}}}{4}\underset{\sim}{i} + \frac{\sqrt{5}-1}{4}\underset{\sim}{j} = \frac{\sqrt{10+2\sqrt{5}}}{4}\underset{\sim}{i} + \frac{\sqrt{5}-5}{4}\underset{\sim}{j}$

$$|\overrightarrow{BC}| = \sqrt{\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 + \left(\frac{\sqrt{5}-5}{4}\right)^2} = \frac{\sqrt{10-2\sqrt{5}}}{2}$$

$$\therefore |\overrightarrow{BC}| = |\overrightarrow{BN}|$$

3 marks

f. The side length can be found using the cosine rule:

$$l = \sqrt{1^2 + 1^2 - 2 \times 1 \times 1 \times \cos\left(\frac{2\pi}{5}\right)} = \frac{\sqrt{10-2\sqrt{5}}}{2}$$

2 marks

Question 4

a. $\frac{dh}{dr} = \frac{dh/dt}{dr/dt} = \frac{h}{1+h} \times \frac{r+1}{r+2}$

2 marks

b. $v = \pi r^2 h$

$$\begin{aligned} \frac{dv}{dt} &= 2\pi r \times \frac{dr}{dt} \times h + \pi r^2 \times \frac{dh}{dt} \\ &= 2\pi \times 5 \times \frac{7}{6} \times 8 + \pi \times 5^2 \times \frac{8}{9} \\ &= \frac{1040\pi}{9} \text{ cm}^3/\text{s} \end{aligned}$$

3 marks

c. $\frac{dx}{dt} = 8 \times 40 - 8 \times \frac{x}{400} = \frac{16000-x}{50}$

2 marks

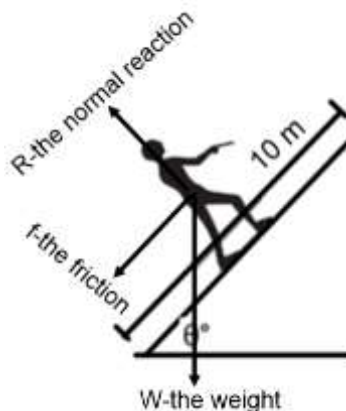
d. $t = \int_0^{200} \frac{50}{16000-x} dx \approx 0.63 \text{ minutes}$

2 marks

Question 5

- a. The forces acting on Nick and his board are shown in the diagram below.

2 marks



b. $mg \sin(\theta) + \mu mg \cos(\theta) = -ma$

$$\Rightarrow a = -(g \sin(\theta) + \mu g \cos(\theta))$$

$$\Rightarrow a = -9.8(\sin(45^\circ) + 0.01 \cos(45^\circ)) = -7 \text{ m/s}^2$$

2 marks

c. $v^2 - u^2 = 2as \Rightarrow v = \sqrt{u^2 + 2as} = \sqrt{15^2 - 2 \times 7 \times 10} = 9.22 \text{ m/s}$

2 marks

- d. Let t be the time that Nick comes back to the same level after leaving the first skate ramp.
Then

$$s = ut - \frac{1}{2}gt^2 \Rightarrow 0 = 9.22 \sin(45^\circ)t - \frac{1}{2} \times 9.8 \times t^2$$

Solve for t , $t = 0$ (reject) or $t \approx 1.33 \text{ s}$ (accept).

In the 1.33 seconds the horizontal distance travelled by Nick is

$$d = 1.33 \times 9.22 \times \cos(45^\circ) \approx 8.67 \text{ m}$$

That is more than 8m. Therefore Nick can land on the other side safely.

3 marks

- e. Let t be the time after leaving the first ramp when Nick lands on the second ramp and let h be the vertical distance below the top of the ramps when Nick lands on the second ramp.
Then

$$x = 9.22 \cos(45^\circ) \cdot t$$

$$-h = 9.22 \sin(45^\circ)t - \frac{1}{2} \times 9.8 \times t^2$$

$$\frac{h}{x - 8} = \tan(30^\circ)$$

Solve them simultaneously

$$\begin{cases} x = 4.25 \text{ m} \\ h = -2.17 \text{ m (reject)} \\ t = 0.65 \text{ s} \end{cases} \quad \text{OR} \quad \begin{cases} x = 9.44 \text{ m} \\ h = 0.83 \text{ m (accept)} \\ t = 1.45 \text{ s} \end{cases}$$

3 marks

- f. Let θ be the minimal angle, in degrees, between the first skate ramp and the ground such that Nick can safely land on the second skate ramp t seconds after leaving the first ramp. Then

$$\begin{aligned} 9.22 \cos(\theta) t &= 8 \\ 9.22 \sin(\theta) t - \frac{1}{2} \times 9.8 \times t^2 &= 0 \end{aligned}$$

Solve them simultaneously,

$$t = 1.04 \text{ s}, \quad \theta = 33.6^\circ$$

Therefore the required minimal angle is 33.6° .

3 marks