

SPECIALIST MATHEMATICS

Written examination 1



(TSSM's 2015 trial exam updated for the current study design)

SOLUTIONS

Question 1 (6 marks)

a. $|\underline{\tilde{v}}| = \sqrt{(8 + 3\sqrt{11})^2 + 7^2} = \sqrt{212 + 48\sqrt{11}}$

Now

$$(6 + 4\sqrt{11})^2 = 36 + 48\sqrt{11} + 176 = 212 + 48\sqrt{11}$$

Therefore

$$|\underline{\tilde{v}}| = 6 + 4\sqrt{11}$$

2 marks

- b. Let θ be the angle between $\underline{\tilde{u}}$ and $\underline{\tilde{v}}$.

Then

$$\cos(\theta) = \frac{\underline{\tilde{u}} \cdot \underline{\tilde{v}}}{|\underline{\tilde{u}}||\underline{\tilde{v}}|} = \frac{16 + 6\sqrt{11} - 7}{3(6 + 4\sqrt{11})} = \frac{1}{2}$$

Therefore

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

2 marks

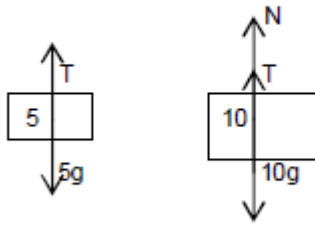
- c. The resolute vector of $\underline{\tilde{v}}$ in the direction of $\underline{\tilde{u}}$ is

$$|\underline{\tilde{v}}| \cos\left(\frac{\pi}{3}\right) \hat{\underline{\tilde{u}}} = \frac{3 + 2\sqrt{11}}{3} (2\underline{\tilde{i}} - \underline{\tilde{j}} + 2\underline{\tilde{k}})$$

2 marks

Question 2 (5 marks)

a.



$$T - 5g = 0, \quad N + T - 10g = 0$$

$$T = 5 \times 9.8 = 49N$$

4 marks

b. $N = 10 \times 9.8 - 49 = 49N$

1 mark

Question 3 (7 marks)

a. $x = 4(e^t + e^{-t}) \Rightarrow x^2 = 16(e^{2t} + e^{-2t} + 2)$
 $y = 3(e^t - e^{-t}) \Rightarrow y^2 = 9(e^{2t} + e^{-2t} - 2)$
 $\Rightarrow \frac{x^2}{64} - \frac{y^2}{36} = \frac{16(e^{2t} + e^{-2t} + 2)}{64} - \frac{9(e^{2t} + e^{-2t} - 2)}{36} = 1$

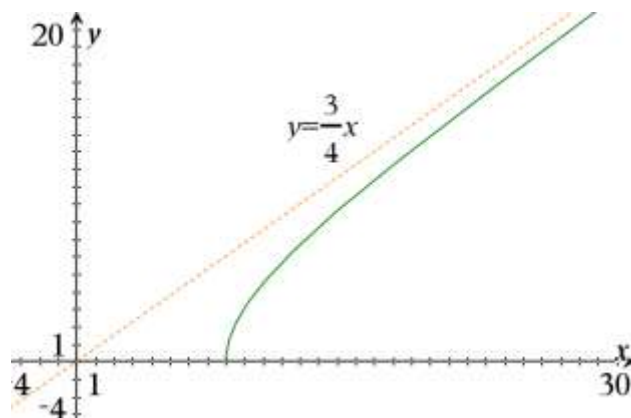
2 marks

b. Domain: $[8, \infty)$ Range: $[0, \infty)$

1 mark

c. The equation of the asymptote: $y = \frac{3}{4}x$

1 mark



Graph	1 mark
Asymptote	1 mark
x-int	1 mark

Question 4 (3 marks)

a. $P\left(-\frac{1}{2}\right) = 2 \times \frac{-1}{8} + 5 \times \frac{1}{4} + 6 \times \frac{-1}{2} + 2 = 0 \Rightarrow z = -\frac{1}{2}$ is a root of $P(z)$. 1 mark

b. According to the result of Part a, $(2z + 1)$ is a factor of $P(z)$.

$$\begin{aligned} 2z^3 + 5z^2 + 6z + 2 &= (2z + 1)(z^2 + 2z + 2) \\ &= (2z + 1)[(z + 1)^2 + 1] \\ &= (2z + 1)(z + 1 + i)(z + 1 - i) \end{aligned}$$

The other roots of $P(z)$ are

$$-1 - i = \sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{4}\right) \quad \text{and} \quad -1 + i = \sqrt{2} \operatorname{cis}\left(\frac{3\pi}{4}\right)$$

2 marks

Question 5 (4 marks)

a. Differentiate the equation of the curve, $2yy' = 4 \Rightarrow y' = \frac{2}{y}$.

The gradient of the tangent at P: $m = \frac{2}{2\sqrt{p}} = \frac{1}{\sqrt{p}}$.

1 mark

b. $\tan(\alpha) = m = \frac{1}{\sqrt{p}}, \quad \tan(\beta) = \frac{2\sqrt{p}}{p-1}$.

$$\tan(2\alpha) = \frac{2 \tan(\alpha)}{1 - \tan^2(\alpha)} = \frac{\frac{2}{\sqrt{p}}}{1 - \frac{1}{p}} = \frac{2\sqrt{p}}{p-1} = \tan(\beta)$$

Therefore $\beta = 2\alpha$ since $\alpha, \beta \in (0, \pi)$.

3 mark

Question 6 (5 marks)

a. $1 + \cos(4x) = 1 + 2 \cos^2(2x) - 1 = 2 \cos^2(2x).$ 1 mark

b. $\int \frac{\sin(2x)}{1+\cos(4x)} dx = \int \frac{\sin(2x)}{2 \cos^2(2x)} dx = -\frac{1}{4} \int \frac{1}{\cos^2(2x)} d(\cos(2x))$
 Let $u = \cos(2x)$. Then $\int \frac{\sin(2x)}{1+\cos(4x)} dx = -\frac{1}{4} \int \frac{1}{u^2} du$ 2 marks

c. When $x = \frac{\pi}{6}$, $u = \cos(2x) = \frac{1}{2}$
 $x = \frac{\pi}{8}$, $u = \cos(2x) = \frac{1}{\sqrt{2}}.$
 $\therefore \int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \frac{\sin(2x)}{1+\cos(4x)} dx = -\frac{1}{4} \int_{\frac{1}{\sqrt{2}}}^{\frac{1}{2}} \frac{1}{u^2} du = -\frac{1}{4} \left[-\frac{1}{u} \right]_{\frac{1}{\sqrt{2}}}^{\frac{1}{2}} = \frac{2-\sqrt{2}}{4}.$ 2 marks

Question 7 (4)

a. $\int \frac{16 \arctan(x)}{1+x^2} dx = 16 \int \arctan(x) d(\arctan(x)) = 8(\arctan(x))^2 + c.$ 2 marks

b. Area = $2 \int_0^1 f(x) dx = 16[(\arctan(x))^2]_0^1 = \pi^2$ 2 marks

Question 8 (6 marks)

a. $2x - x^2 = 1 - (x - 1)^2.$ 1 mark

b. $\int \frac{1}{x(x-2)} dx = \int \frac{1}{2} \left(\frac{1}{x-2} - \frac{1}{x} \right) dx = \frac{1}{2} \log_e \left| \frac{x-2}{x} \right| + c$ 2 marks

c. $\pi \int_1^{\frac{3}{2}} \left(1 + \frac{1}{\sqrt{2x-x^2}} \right)^2 dx = \pi \int_1^{\frac{3}{2}} \left(1 + \frac{2}{\sqrt{2x-x^2}} + \frac{1}{2x-x^2} \right) dx$
 $= \pi [x]_1^{\frac{3}{2}} + \pi \int_1^{\frac{3}{2}} \frac{2}{\sqrt{1-(x-1)^2}} dx - \pi \int_1^{\frac{3}{2}} \frac{1}{x(x-2)} dx$
 $= \frac{\pi}{2} + 2\pi [\arcsin(x-1)]_1^{\frac{3}{2}} - \frac{\pi}{2} \left[\log_e \left| \frac{x-2}{x} \right| \right]_1^{\frac{3}{2}}$
 $= \frac{\pi}{2} + \frac{\pi^2}{3} + \frac{\pi \log_e 3}{2}$

3 marks