Neap:

Trial Examination 2015

VCE Specialist Mathematics Units 3&4

Written Examination 2

Suggested Solutions

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SECTION 1

Question 1 **C**

Forming $\frac{(x+2)^2}{9} - \frac{y^2}{16} = 0$ and solving for y gives $y = \pm\frac{4}{3}(x+2)$.

Alternatively, note that for $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, the asymptotes have equations $y = \pm\frac{b}{a}(x-h)+k$.

Question 2 **B**

Vertical asymptotes occur for values of x such that the denominator is zero.

If $x = -2$ and $x = 1$ are vertical asymptotes, then we require $(x+2)$ and $(x-1)$ in the denominator.

The horizontal asymptote has equation $y = 3$.

So as $x \to \pm\infty$, y should approach 3.

Question 3 **D**

In general, if the graph of $y = f(x)$ is dilated by a factor of 3 parallel to the y-axis and by a factor of $\frac{1}{2}$ parallel to the x-axis, then $y = 3f(2x)$.

So $y = 3\sec(2x)$.

Question 4 **C**

Using $\cos(2\theta) = 2\cos^2(\theta) - 1$ with $\theta = \frac{x}{2}$, we obtain $\cos(x) = 2\cos^2\left(\frac{x}{2}\right) - 1$.

Solving $\frac{1}{3} = 2\cos^2\left(\frac{x}{2}\right) - 1$ for $\cos\left(\frac{x}{2}\right)$ gives $\cos\left(\frac{x}{2}\right) = \pm\frac{\sqrt{6}}{3}$.

As x is an acute angle, $\cos\left(\frac{x}{2}\right)$ is positive, and so $\cos\left(\frac{x}{2}\right) = \frac{\sqrt{6}}{3}$.

Question 5 **B**

For $y = \tan^{-1}(x)$, the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For $y = 2\tan^{-1}(x)$, the range is $(-\pi, \pi)$.

So for $y = 2\tan^{-1}(x) + \pi$ and hence $y = 2\tan^{-1}(x-1) + \pi$, the range is $(0, 2\pi)$.

Question 6 **B**

$$\begin{aligned} z &= x^3 + y^3 i \\ \bar{z} &= x^3 - y^3 i \\ z\bar{z} &= (x^3 + y^3 i)(x^3 - y^3 i) \\ &= x^6 + y^6 \end{aligned}$$

$\text{Im}(z\bar{z}) = 0$ and so **B** is incorrect.

Note that $z - \bar{z} = 2y^3 i$ and so $\text{Im}(z - \bar{z}) = 2y^3$.

Question 7 A

Method 1:

$|z| = |z + 5|$

This set of points is the perpendicular bisector of the line joining (0, 0) and (–5, 0). This corresponds to $\text{Re}(z) = -\frac{5}{2}$.

Method 2:

Let $z = x + yi$.

Solving $|z| = |z + 5|$ for x gives $x = -\frac{5}{2}$.

So $\text{Re}(z) = -\frac{5}{2}$.

Question 8 A

$i^3 = -i$

So $u = -iz$.

Hence multiplying by $-i$ corresponds to rotating z through $\frac{\pi}{2}$ in a clockwise direction about the origin.

Question 9 D

If $P(-2 + i) = 0$ then $P(-2 - i) = 0$.

Solving $(z - (-2 + i))(z - (-2 - i))(z - w) = z^3 + z^2 - 7z - 15$ for w gives $w = 3$.

Question 10 E

The direction field indicates the gradient of the solution curve at various values of x and y.

The direction field is steep and positive for values of x close to zero, and less steep and positive as x increases.

This matches up with the behaviour of the graph of $y = \log_e(x)$.

Question 11 A

There is a repeated linear factor in the denominator, that is, $x^2 - 6x + 9 = (x - 3)^2$.

So the partial fraction form is $\frac{A}{(x-3)} + \frac{B}{(x-3)^2}$.

Question 12 E

$$\frac{dx}{dt} = 8$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$\frac{d}{dx}(\log_e(\cos(2x))) = -2\tan(2x)$$

$$\text{At } x = \frac{\pi}{6}, \frac{dy}{dx} = -2\tan\left(\frac{\pi}{3}\right)$$

$$= -2\sqrt{3}$$

$$\frac{dy}{dt} = (-2\sqrt{3})(8)$$

$$= -16\sqrt{3} \text{ (m/s)}$$

Question 13 C

Let V be the volume.

Using $V = \pi\int_a^b y^2 dx$ we obtain $V = \int_0^{\pi} \pi\left(1 - \sin\left(\frac{x}{2}\right)\right)^2 dx$.

Question 14 E

$u = x + 1 \Rightarrow x = u - 1$ and $du = dx$

$$\int (u-2)\sqrt{u}\ du = \int\left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}}$$

$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{4}{3}(x+1)^{\frac{3}{2}}$$

Question 15 A

$\overrightarrow{OA} = \underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\overrightarrow{OB} = -3\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (-3\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}) - (\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}) \\ &= -4\underset{\sim}{i} + 3\underset{\sim}{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AP} &= \frac{1}{3}\overrightarrow{AB} \\ &= \frac{1}{3}(-4\underset{\sim}{i} + 3\underset{\sim}{j})\end{aligned}$$

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP} \\ &= (\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}) + \frac{1}{3}(-4\underset{\sim}{i} + 3\underset{\sim}{j}) \\ &= \frac{1}{3}(-\underset{\sim}{i} - 9\underset{\sim}{j} + 9\underset{\sim}{k})\end{aligned}$$

Question 16 E

A: $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c}$

B: $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{c}$

C: clearly dependent

D: $\underset{\sim}{b} - \underset{\sim}{a} = \underset{\sim}{c}$

Question 17 C

The angle between two vectors is measured 'tail-to-tail'.

So the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 60° (not 120°).

$$\begin{aligned}\underset{\sim}{a} \cdot \underset{\sim}{b} &= |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta) \\ &= 45\cos(60°) \\ &= \frac{45}{2}\end{aligned}$$

Question 18 C

$\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{AC} = \underset{\sim}{b} - \underset{\sim}{a}$

If the diagonals are at right angles then $\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$.

So $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = 0$.

Question 19 D

Resolving forces:

From $ma = F - \mu N$ we obtain $10 \times 0.25 = F - 0.1 \times 10g$.

So $F = 2.5 + g$.

Question 20 D

$F - 2000g = 0 \Rightarrow F = 2000g$

After the ballast is dropped, F still acts, but the downwards force is now $1800g$ N.

Hence the net force is now $200g$ N upwards.

So $a = \dfrac{200g}{1800}$ m/s^2 upwards.

This simplifies to $a = \dfrac{g}{9}$ m/s^2 upwards.

Question 21 B

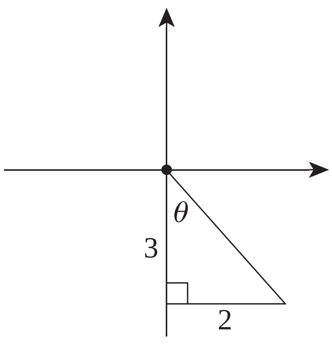
$$A = \frac{1}{2} \times 20 \times 2 + \frac{1}{2} \times 20 \times 2$$
$$= 40$$

Question 22 B

The momentum, $\underset{\sim}{p}$ kg m/s, of the particle is given by $\underset{\sim}{p} = m\underset{\sim}{v}$.

$$\underset{\sim}{p} = 2(2\underset{\sim}{i} - 3\underset{\sim}{j})$$
$$= 4\underset{\sim}{i} - 6\underset{\sim}{j}$$

$$|\underset{\sim}{p}| = \sqrt{4^2 + (-6)^2}$$
$$= \sqrt{52}$$
$$= 2\sqrt{13}$$



$\tan(\theta) = \dfrac{2}{3} \Rightarrow \theta = 33.7°$

Thus the particle has momentum $2\sqrt{13}$ kg m/s in the direction S 33.7° E.

SECTION 2

Question 1 (10 marks)

a. $f(x) = \log_e(9 - x^2)$

$f''(x) = \dfrac{-2(x^2 + 9)}{(x^2 - 9)^2}$ M1 A1

$(x^2 - 9)^2 > 0$ for $-3 < x < 3$, $x^2 + 9 > 0$, and so $-2(x^2 + 9) < 0$ for $-3 < x < 3$. A1

b. $A = \displaystyle\int_0^2 \log_e(9 - x^2)dx$ M1

So $A = 4.0472$ (correct to 4 decimal places). A1

c. **i.** Using $y_{n+1} = y_n + hf(x_n)$ we obtain $y_{25} = y_{24} + 0.08f(9 - x_{24}^2)$. M1

$y_{25} = y_{24} + 0.08\log_e(9 - 1.92^2)$ (or equivalent). A1

ii. $y_{25} = 3.9367 + 0.08\log_e(9 - 1.92^2)$

$= 4.0703$ (correct to 4 decimal places) A1

iii. $y \approx \displaystyle\int_0^{x_{25}} \log_e(9 - x^2)dx$ A1

$y = \displaystyle\int_0^2 \log_e(9 - x^2)dx$

$= A$ A1

Question 2 (12 marks)

a. 2 kg/L at 3 L/min, so $\dfrac{dQ}{dt_{\text{in}}} = 6$ kg/min.

$\dfrac{Q}{200 + (3 - r)t}$ kg/L at r L/min, so $\dfrac{dQ}{dt_{\text{out}}} = \dfrac{rQ}{200 + (3 - r)t}$ kg/min.

$\dfrac{dQ}{dt} = 6 - \dfrac{rQ}{200 + (3 - r)t}$ A2

When $t = 0$, $Q = 100$. A1

b. i. When $r = 3$, $\dfrac{dQ}{dt} = 6 - \dfrac{3Q}{200}$.

Attempts to solve $\dfrac{dQ}{dt} = 6 - \dfrac{3Q}{200}$ for Q with $Q(0) = 100$. M1

$Q = 400 - 300e^{-\frac{3t}{200}}$ A1

ii. As $t \to \infty$, $e^{-\frac{3t}{200}} \to 0$ and so $Q \to 400$. M1

In the long term, there will be 400 kg of salt in the tank. A1

c. i. When $r = 2$, $\dfrac{dQ}{dt} = 6 - \dfrac{2Q}{200 + (3 - 2)t}$

$$= 6 - \frac{2Q}{200 + t}$$ A1

ii. We are given $Q = 2(200 + t) + \dfrac{c}{(200 + t)^2}$.

$$\frac{dQ}{dt} = 2 - \frac{2c}{(200 + t)^3}$$ A1

$$6 - \frac{2Q}{200 + t} = 6 - \frac{2}{200 + t}\left(2(200 + t) + \frac{c}{(200 + t)^2}\right)$$

$$= 2 - \frac{2c}{(200 + t)^3}$$ A1

So the solution is verified.

When $t = 0$, $Q = 100$, and so $c = -300 \times 200^2 (= -12\ 000\ 000)$. A1

iii. $Q = 2(200 + t) - \dfrac{300 \times 200^2}{(200 + t)^2}$

When $t = 25$, $Q = 2 \times 225 - \dfrac{300 \times 200^2}{225^2}$.

So 213.0 kg of salt is in the tank (correct to 1 decimal place). A1

Question 3 (10 marks)

a. $z^n + \dfrac{1}{z^n} = \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$ A1

$= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$

$= 2\cos(n\theta)$ A1

b. $z^n - \dfrac{1}{z^n} = \cos(n\theta) + i\sin(n\theta) - \cos(-n\theta) - i\sin(-n\theta)$ A1

$= \cos(n\theta) + i\sin(n\theta) - \cos(n\theta) + i\sin(n\theta)$

$= 2i\sin(n\theta)$ A1

c. $\left(z + \dfrac{1}{z}\right)^4\left(z - \dfrac{1}{z}\right)^2 = z^6 + 2z^4 - z^2 - \dfrac{1}{z^2} + \dfrac{2}{z^4} + \dfrac{1}{z^6} - 4$ M1

$= z^6 + \dfrac{1}{z^6} + 2\left(z^4 + \dfrac{1}{z^4}\right) - \left(z^2 + \dfrac{1}{z^2}\right) - 4$ A1

d. $z^6 + \dfrac{1}{z^6} + 2\left(z^4 + \dfrac{1}{z^4}\right) - \left(z^2 + \dfrac{1}{z^2}\right) - 4 = 2\cos(6\theta) + 4\cos(4\theta) - 2\cos(2\theta) - 4$ A1

$\left(z + \dfrac{1}{z}\right)^4\left(z - \dfrac{1}{z}\right)^2 = -64\cos^4(\theta)\sin^2(\theta)$ M1

So $\displaystyle\int \cos^4(\theta)\sin^2(\theta)\,d\theta = -\frac{1}{192}\sin(6\theta) - \frac{1}{64}\sin(4\theta) + \frac{1}{64}\sin(2\theta) + \frac{1}{16}\theta\ (+c)$. M1 A1

Question 4 (14 marks)

a. $12\underset{\sim}{i} = 12\cos\left(\dfrac{t}{4}\right)\underset{\sim}{i} + 6\sin\left(\dfrac{t}{4}\right)\underset{\sim}{j} \Rightarrow \cos\left(\dfrac{t}{4}\right) = 1$ M1

$\cos\left(\dfrac{t}{4}\right) = 1 \Rightarrow t = 0, 8\pi, 16\pi$

So the particle takes 8π seconds to return to its initial position. A1

b. The parametric equations are $x = 12\cos\left(\dfrac{t}{4}\right)$ and $y = 6\sin\left(\dfrac{t}{4}\right)$.

Using $\cos^2\left(\dfrac{t}{4}\right) + \sin^2\left(\dfrac{t}{4}\right) = 1$ we obtain $\dfrac{x^2}{144} + \dfrac{y^2}{36} = 1$. A1

$-12 \le 12\cos\left(\dfrac{t}{4}\right) \le 12 \Rightarrow -12 \le x \le 12$ A1

c. $\underset{\sim}{r}(t) = 12\cos\left(\frac{t}{4}\right)\underset{\sim}{i} + 6\sin\left(\frac{t}{4}\right)\underset{\sim}{j}$

$\underset{\sim}{r}'(t) = -3\sin\left(\frac{t}{4}\right)\underset{\sim}{i} + \frac{3}{2}\cos\left(\frac{t}{4}\right)\underset{\sim}{j}$ M1

$|\underset{\sim}{r}'(t)| = \sqrt{9\sin^2\left(\frac{t}{4}\right) + \frac{9}{4}\cos^2\left(\frac{t}{4}\right)}$ (or equivalent) A1

d. **Method 1:**

$|\underset{\sim}{r}'(t)| = \frac{3}{2}\sqrt{1 + 3\sin^2\left(\frac{t}{4}\right)}$ M1

The maximum occurs when $\sin^2\left(\frac{t}{4}\right) = 1$, that is, $\sin\left(\frac{t}{4}\right) = \pm 1$. A1

So the maximum speed is 3 (units/sec). A1

When $\sin\left(\frac{t}{4}\right) = \pm 1$, $t = 2\pi, 6\pi, 10\pi, 14\pi$. A1

Method 2:

$|\underset{\sim}{r}'(t)| = \frac{3}{2}\sqrt{4 - 3\cos^2\left(\frac{t}{4}\right)}$ M1

The maximum occurs when $\cos^2\left(\frac{t}{4}\right) = 0$, that is, $\cos\left(\frac{t}{4}\right) = 0$. A1

So the maximum speed is 3 (units/sec). A1

When $\cos\left(\frac{t}{4}\right) = 0$, $t = 2\pi, 6\pi, 10\pi, 14\pi$. A1

e. $\underset{\sim}{r}'(t) = -3\sin\left(\frac{t}{4}\right)\underset{\sim}{i} + \frac{3}{2}\cos\left(\frac{t}{4}\right)\underset{\sim}{j}$

$\underset{\sim}{r}''(t) = -\frac{3}{4}\cos\left(\frac{t}{4}\right)\underset{\sim}{i} - \frac{3}{8}\sin\left(\frac{t}{4}\right)\underset{\sim}{j}$ M1

So $\underset{\sim}{r}''(t) = -\frac{1}{16}\underset{\sim}{r}(t)$, that is, $k = -\frac{1}{16}$. A1

Method 1:

$|\underset{\sim}{r}''(t)| = \frac{3}{8}\sqrt{4-3\sin^2\left(\frac{t}{4}\right)}$ M1

When $\sin\left(\frac{t}{4}\right) = 0$, $t = 0, 4\pi, 8\pi, 12\pi, 16\pi$. A1

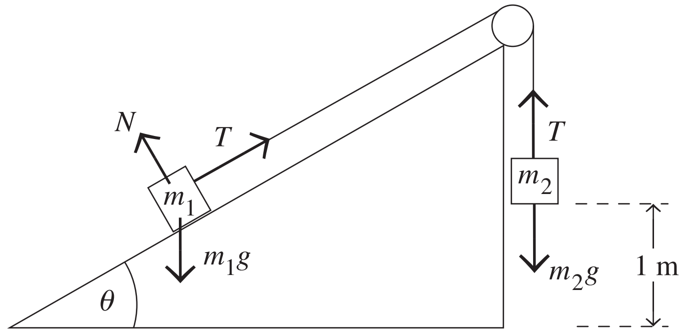
Method 2:

$|\underset{\sim}{r}''(t)| = \frac{3}{8}\sqrt{1 + 3\cos^2\left(\frac{t}{4}\right)}$ M1

When $\cos\left(\frac{t}{4}\right) = \pm 1$, $t = 0, 4\pi, 8\pi, 12\pi, 16\pi$. A1

Question 5 (12 marks)

a.



A1

Award A1 for the forces acting on both the m_1 kg block and the m_2 kg block.

b. m_1 kg block: $T - m_1 g\sin(\theta) = m_1 a$; m_2 kg block: $m_2 g - T = m_2 a$ A1

Either adds the two equations to give $m_2 g - m_1 g\sin(\theta) = (m_1 + m_2)a$ (eliminating T), or attempts to solve the above two equations simultaneously for a and T. M1

$$a = \frac{(m_2 - m_1\sin(\theta))g}{(m_1 + m_2)}$$ A1

c. **Method 1:**

Attempts to solve the above two equations simultaneously for a and T. M1

$$T = \frac{m_1 m_2 g(1 + \sin(\theta))}{m_1 + m_2}$$ (expressed as a single fraction) A1

Method 2:

$$T = m_2 g - \frac{m_2 g(m_2 - m_1\sin(\theta))}{m_1 + m_2}$$

$$= \frac{m_2 g(m_1 + m_2)}{m_1 + m_2} - \frac{m_2 g(m_2 - m_1\sin(\theta))}{m_1 + m_2}$$ M1

$$= \frac{(m_1 m_2 + m_2^2 - m_2^2 + m_1 m_2\sin(\theta))g}{m_1 + m_2}$$

$$= \frac{m_1 m_2 g(1 + \sin(\theta))}{m_1 + m_2}$$ (expressed as a single fraction) A1

d. Using $v^2 = u^2 + 2as$ with $u = 0$, $a = \dfrac{(m_2 - m_1\sin(\theta))g}{(m_1 + m_2)}$ and $s = 1$ gives:

$$v = \sqrt{\frac{2g(m_2 - m_1\sin(\theta))}{m_1 + m_2}}$$ (since $v > 0$) M1 A1

e. With the string slack, the m_1 kg block will move up the plane, come to rest instantaneously, and then move down the plane to the same position corresponding to when the m_2 kg block hit the ground.

$m_1 a = -m_1 g \sin(\theta) \Rightarrow a = -g \sin(\theta)$ A1

Using $v = u + at$ with $u = \sqrt{\dfrac{2g(m_2 - m_1 \sin(\theta))}{m_1 + m_2}}$ and $a = -g\sin(\theta)$ gives:

$0 = \sqrt{\dfrac{2g(m_2 - m_1 \sin(\theta))}{m_1 + m_2}} - (g\sin(\theta))t$ M1

Solving for t gives $t = \dfrac{1}{g\sin(\theta)}\sqrt{\dfrac{2g(m_2 - m_1 \sin(\theta))}{m_1 + m_2}}$. M1

The m_1 kg block will take the same amount of time to travel down the plane.

So $t = \dfrac{2}{g\sin(\theta)}\sqrt{\dfrac{2g(m_2 - m_1 \sin(\theta))}{m_1 + m_2}}$ (or equivalent). A1