

The Mathematical Association of Victoria

SPECIALIST MATHEMATICS

SOLUTIONS - Trial Exam 2015

Written Examination 2

SECTION 1: Multiple Choice Questions

Question		Question	
1	B	12	B
2	D	13	D
3	C	14	E
4	D	15	D
5	D	16	A
6	E	17	C
7	B	18	A
8	D	19	C
9	B	20	C
10	E	21	A
11	C	22	D

Question 1 B

$$\frac{(y+2)^2}{4} - \frac{(x-1)^2}{16} = 1 \text{ has asymptotes at } (x-1) = \pm \frac{4}{2}(y+2) .$$

Simplifying gives, asymptotes are located at $x = 2y + 5$ and $x = -2y - 3$.

$x = 2y + 5$ intersects the line $y = 3$ at $(11, 3)$.

$x = -2y - 3$ intersects the line $y = 3$ at $(-9, 3)$. B

Question 2 D

By completing the square $4x^2 + 8x + y^2 - 6y + 12 = 0$ can be expressed as $4(x+1)^2 + (y-3)^2 = 1$.

Since $\cos^2(\theta) + \sin^2(\theta) = 1$ we can let $\cos(\theta) = 2(x+1)$ and $\sin(\theta) = y-3$.

Rearranging gives, $x = \frac{1}{2}\cos(\theta) - 1$ and $y = \sin(\theta) + 3$. D

Question 3 C

$$\overrightarrow{OB} + \overrightarrow{AC} = (\underset{\sim}{c} + \underset{\sim}{a}) + (-\underset{\sim}{a} + \underset{\sim}{c}) = 2\underset{\sim}{c} \quad C$$

Question 4 D

$$\sec(2\theta) = a \text{ so } \cos(2\theta) = \frac{1}{a}$$

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

$$\therefore \frac{1}{a} = 2\cos^2(\theta) - 1$$

$$\therefore \cos^2(\theta) = \frac{1+a}{2a}$$

$$\therefore \cos(\theta) = \pm \sqrt{\frac{a+1}{2a}}$$

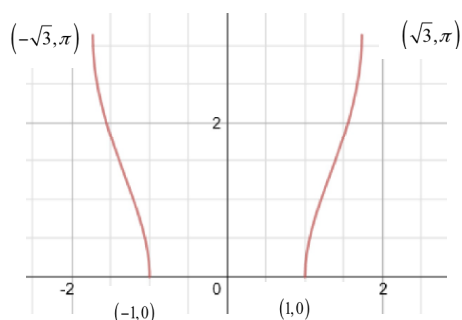
Only positive option given so D.

Question 5 D

$$-1 \leq 2 - x^2 \leq 1$$

Using CAS to solve $-\sqrt{3} \leq x \leq -1$ or $1 \leq x \leq \sqrt{3}$

So domain of $\arccos(2 - x^2)$ is $[-\sqrt{3}, -1] \cup [1, \sqrt{3}]$.



From the graph we can see that the range is $[0, \pi]$. So D.

Question 6 E

$$z = \sqrt{3} \operatorname{cis}\left(\frac{5\pi}{6}\right) \text{ so } \frac{1}{z^3} = z^{-3} = (\sqrt{3})^{-3} \operatorname{cis}\left(\frac{5\pi}{6} \times -3\right)$$

$$\text{Simplifying } z^{-3} = \frac{1}{3\sqrt{3}} \operatorname{cis}\left(-\frac{5\pi}{2}\right) = \frac{\sqrt{3}}{9} \operatorname{cis}\left(-\frac{\pi}{2}\right).$$

$$\text{Converting to Cartesian form gives } z^{-3} = -\frac{\sqrt{3}}{9}i. \quad \text{E}$$

Question 7 B

$\{z : |z+1| = |z-i|\}$ can be expressed by the Cartesian equation $y = -x$.

$\{z : |z| = 2\}$ can be expressed by the Cartesian equation $x^2 + y^2 = 4$.

Solving simultaneously gives $x = \pm\sqrt{2}, y = \mp\sqrt{2}$. So $A \cap B = \{\sqrt{2} - \sqrt{2}i, -\sqrt{2} + \sqrt{2}i\}$. B

Question 8 D

Let $z = x + yi$. So $w = -i^3 \bar{z} = -i \times i^2 (x - yi) = i(x - yi) = y + xi$.

Reflection in x -axis gives $x - yi$, followed by a rotation of $\frac{\pi}{2}$ anticlockwise about the origin (ie multiplication by i) gives $i(x - yi) = y + ix$. D

Question 9 B

$$V = \pi \int x^2 dy$$

Since $y = \sqrt[3]{x}$, $x^2 = y^6$, so $V = \pi \int_0^2 y^6 dy = \frac{128\pi}{7}$ B

Question 10 E

$$\frac{dS}{dt} = \text{input} - \text{output}$$

$$\frac{dS}{dt} = 0 \times 10 - \frac{S}{60000 + 4t} \times 6$$

$$\frac{dS}{dt} = -\frac{3S}{30000 + 2t} \quad \text{E}$$

Question 11 C

$$y_1 \approx y_0 + hf(x_0, y_0) \text{ where } f(x, y) = \frac{1}{xy}$$

$$= 1 + \frac{1}{1} \times 0.1$$

$$= 1.1 \text{ when } x_1 = 1.1$$

$$y_2 \approx 1.1 + \frac{1}{1.1 \times 1.1} \times 0.1 = 1.18264463.. = 1.183 \text{ (correct to 3 dec. places)} \quad \text{C}$$

Question 12 B

By rule: $y_2 = \int_{x_1}^{x_2} f(x) dx + y_1$ so $y_2 = \int_5^6 (3x^4 - 2x)^{\frac{2}{3}} dx + 10$ B

Question 13 D

$$\int_{-1}^0 \frac{2x+1}{\sqrt{1-2x}} dx$$

$$\text{let } u = 1 - 2x, \frac{du}{dx} = -2$$

Rearranging $u = 1 - 2x$ to give $2x + 1 = 2 - u$

Changing the terminals to u values,

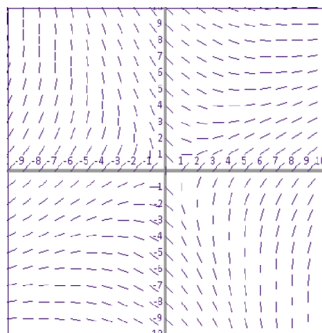
$$x = -1 \rightarrow u = 3$$

$$x = 0 \rightarrow u = 1$$

So $\int_{-1}^0 \frac{2x+1}{\sqrt{1-2x}} dx$ can be expressed as $\int_3^1 \frac{2-u}{\sqrt{u}} \times -\frac{1}{2} du$

Simplifying we get $-\frac{1}{2} \int_3^1 \left(\frac{2}{\sqrt{u}} - \sqrt{u} \right) du = \frac{1}{2} \int_1^3 \left(\frac{2}{\sqrt{u}} - \sqrt{u} \right) du = \int_1^3 \left(\frac{1}{\sqrt{u}} - \frac{\sqrt{u}}{2} \right) du$ D

Question 14 E



Consider $\frac{dy}{dx} = \frac{x-y}{x+y}$.

When $x = 0$, $\frac{dy}{dx} = -1$ as indicated on direction field.

When $y = 0$, $\frac{dy}{dx} = 1$ as indicated on direction field.

When $x = 5$ and $y = 5$, $\frac{dy}{dx} = 0$ as indicated on direction field.

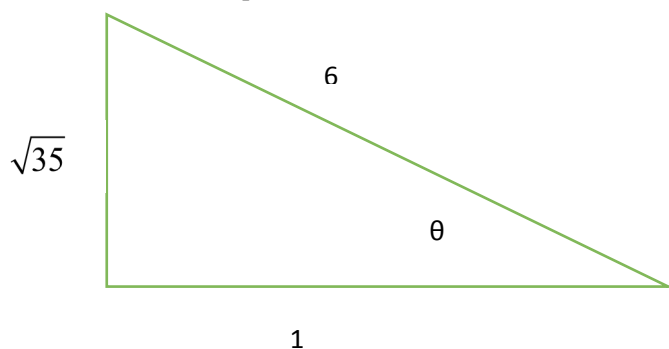
When $x = 5$ and $y = -5$, $\frac{dy}{dx}$ is ∞ , as indicated on direction field. E

Alternatively use CAS

Question 15 D

$$\cos(\theta) = \frac{\tilde{a} \cdot \tilde{b}}{|\tilde{a}| |\tilde{b}|} = \frac{2\sqrt{3} - 2\sqrt{3} - 2}{\sqrt{16} \times \sqrt{9}} = \frac{-2}{12} = \frac{-1}{6}$$

So θ is in the second quadrant.



$$\tan(2\theta) = \frac{2 \tan(\theta)}{1 - \tan^2(\theta)} = \frac{2 \times -\sqrt{35}}{1 - 35} = \frac{\sqrt{35}}{17} \quad D$$

Question 16 A

Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependant then there exists real numbers p and q such that

$$p\vec{a} + q\vec{c} = \vec{b}.$$

$$\text{So } p \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + q \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} m \\ n \\ 0 \end{bmatrix}$$

Therefore we obtain the simultaneous equations:

$$2p - q = m$$

$$p - 3q = n$$

$$-p + q = 0$$

The third equations gives $p = q$.

$$\text{So } m = p, n = -2p.$$

As $\vec{b}$ is a unit vector: $m^2 + n^2 = 1$.

$$\text{Therefore: } p^2 + (-2p)^2 = 1 \text{ gives } p = \pm \frac{1}{\sqrt{5}}.$$

$$\text{So either } m = \frac{1}{\sqrt{5}}, n = -\frac{2}{\sqrt{5}} \text{ or } m = -\frac{1}{\sqrt{5}}, n = \frac{2}{\sqrt{5}}.$$

$$m = -\frac{1}{\sqrt{5}}, n = \frac{2}{\sqrt{5}} \text{ is option A.}$$

Question 17 C

$$\vec{r}(t) = \int \vec{v}(t) dt$$

$$= \frac{3}{2} \sin(2t) \vec{i} + 2 \cos(t) \vec{j} - 2e^{-2t} \vec{k} + \vec{c}$$

$$\vec{r}(0) = 0 \vec{i} + 0 \vec{j} + 0 \vec{k} = 0 \vec{i} + 2 \vec{j} - 2 \vec{k} + \vec{c}$$

$$\text{So } \vec{c} = -2 \vec{j} + 2 \vec{k}$$

$$\text{Therefore } \vec{r}(t) = \frac{3}{2} \sin(2t) \vec{i} + (2 \cos(t) - 2) \vec{j} + (2 - 2e^{-2t}) \vec{k} \quad \text{C}$$

Question 18 A

$$a = v \frac{dv}{dx} = (1-x)^2 \times -2(1-x) = -2(1-x)^3$$

$$\text{Since } v = (1-x)^2 \text{ then } \frac{dx}{dt} = (1-x)^2. \text{ So } t = \int \frac{1}{(1-x)^2} dx$$

$$\therefore t = \frac{1}{1-x} + c$$

$$\text{Since } x=0 \text{ when } t=0, \text{ gives } c=-1, \text{ so } t = \frac{1}{1-x} - 1.$$

$$\text{Solving for when } t=2, \text{ gives } x = \frac{2}{3}.$$

Substituting into the acceleration equation gives $-\frac{2}{27} \text{ ms}^{-2}$. A

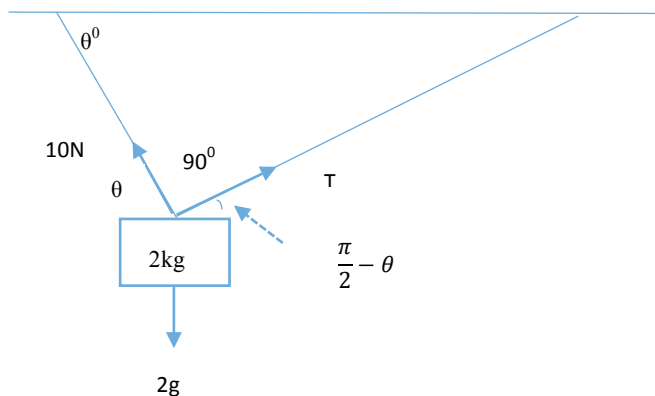
Question 19 C

Substituting $t = 4, v = 0, s = 12$ into $s = \frac{1}{2}(u + v)t$ gives $u = 6$.

Substituting into $v = u + at$ gives $a = -1.5$.

So net force that brings the particle to rest is $1.5 \times 5 = 7.5 \text{ N}$ C

Question 20 C



Let the angle made between the string and the ceiling be θ and the tension in the other section of the string be T .

Resolving the forces into components:

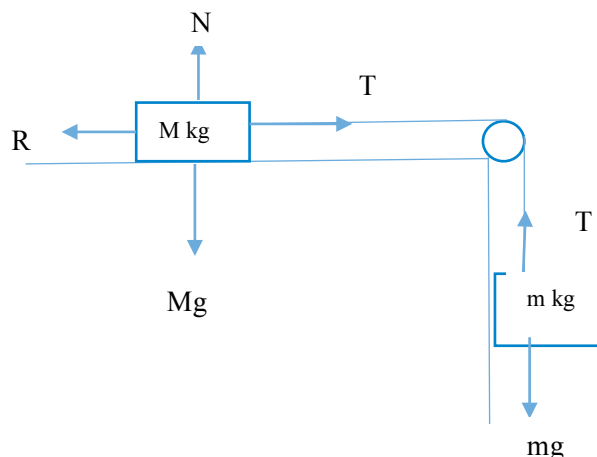
Horizontal components: $10 \cos(\theta) = T \sin(\theta)$

Vertical components: $2g = T \cos(\theta) + 10 \sin(\theta)$

Solving simultaneously gives $\theta = 30.67742$

So closest to 31° . C

Question 21 A



Mass $m \text{ kg}$: $mg - T = ma$

So $T = m(g - a)$

Mass $M \text{ kg}$: $T - R = Ma$

But $R = \mu N = \frac{1}{3}Mg$

So $T = M\left(a + \frac{1}{3}g\right)$

Equating the tension equations gives $m(g - a) = M\left(a + \frac{1}{3}g\right)$

Therefore $\frac{M}{m} = \frac{g - a}{a + \frac{1}{3}g}$

Substituting $a = 2$ gives $\frac{M}{m} = \frac{g - 2}{2 + \frac{1}{3}g} = \frac{3(g - 2)}{6 + g} = \frac{3(g - 2)}{g + 6}$ A

Question 22 D

When $t = 9, v = -10$.

$$\int_0^9 \left(-\frac{1}{2}(t-3)^2 + 8\right) dt = (t-9) \times 10$$

Solving gives $t = \frac{243}{20}$ seconds. D

SECTION 2

Question 1

a. $x = -1, x = 3$ $\frac{1}{2}$ mark each [round down] A1
 $y = 1$ A1

b. Stationary points occur when $f'(x) = 0$. A1

$$f'(x) = \frac{-2x(x+3)}{(x-3)^2(x+1)^2} = 0$$

So $x = 0, x = -3$

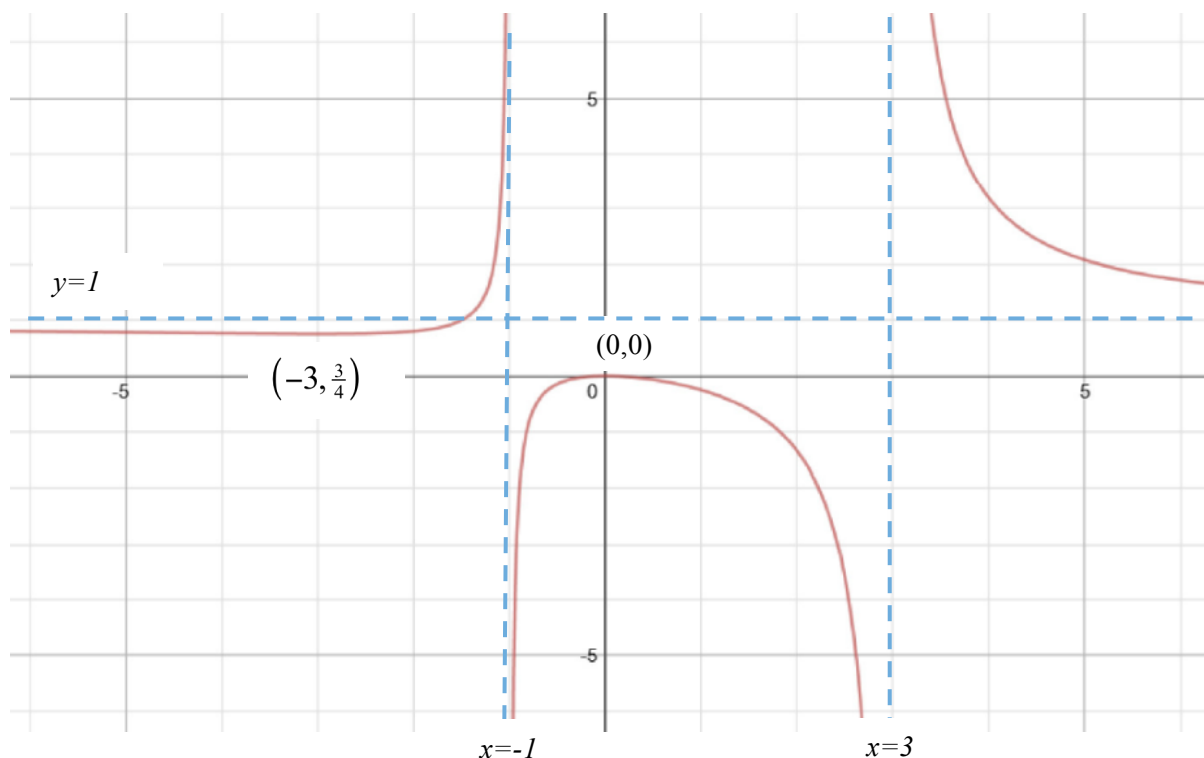
$(0, 0)$ and $\left(-3, \frac{3}{4}\right)$ A1

c. $f''(x) = 0$ or $\frac{2(2x^3 + 9x^2 + 9)}{(x-3)^3(x+1)^3} = 0$ M1

$x = -4.703416..... \approx -4.70$ (correct to 2 dec. places)

$(-4.70, 0.78)$ A1

d.



½ mark each asymptote indicated, labelled & graph approaching such

½ mark each, coordinates of turning points indicated & in correct position

½ mark location of sections of curve, especially curve crossing asymptote & approaching $y=1$ from below. Round down.

e. i. Solving $\frac{x^2}{(x-1)(x+3)} = -\frac{1}{4}$ to give $x = 1, -\frac{3}{5}$.

But since $x \geq 0$ then $x=1$. A1

$$\text{Area} = \frac{1}{4} \times 1 - \left| \int_0^1 \frac{x^2}{(x+1)(x-3)} dx \right| \quad \text{M1 (allocated for the absolute value)}$$

$$\text{Simplifying we get: Area} = \frac{1}{4} + \int_0^1 \frac{x^2}{(x+1)(x-3)} dx \quad \text{A1}$$

Alternatively,

$$\left| \int_0^1 \left(-\frac{1}{4} - f(x) \right) dx \right|. \quad \text{A1 terminals, A1 absolute value, A1 integrand expression}$$

ii. Area = 0.1644167.... ie Area = 0.16 square units (correct to two decimal places) A1

Question 2

$$\text{a. } \vec{AB} = \vec{OB} - \vec{OA} = \sqrt{2}i - \sqrt{2}j \quad \text{A1}$$

$$|\vec{AB}| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$$

$$|\vec{OA}| = \sqrt{(\sqrt{3})^2 + 1^2} = 2 \quad \text{M1}$$

$$|\vec{OB}| = \sqrt{(\sqrt{2} + \sqrt{3})^2 + (1 - \sqrt{2})^2} = \sqrt{2\sqrt{6} - 2\sqrt{2} + 8}$$

So OAB is an isosceles triangle.

$$\text{b. } \cos(\theta) = \frac{\vec{OA} \cdot \vec{AB}}{|\vec{OA}| |\vec{AB}|} = \frac{(\sqrt{3}i + j) \cdot (\sqrt{2}i - \sqrt{2}j)}{2 \times 2} = \frac{\sqrt{6} - \sqrt{2}}{4} \quad \text{A1}$$

$$\text{c. i. } u = \sqrt{3} + i = 2\text{cis}\left(\frac{\pi}{6}\right) \quad \text{A1}$$

$$v = \sqrt{2} - \sqrt{2}i = 2\text{cis}\left(-\frac{\pi}{4}\right) \quad \text{A1}$$

$$\text{ii. } \therefore \phi = \frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12} \quad \text{A1}$$

$$\text{iii. } \cos(\phi) = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$$

M1 (use of compound angle theorem)

$$\begin{aligned} &= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

$$\text{From b. } \cos(\theta) = \frac{\sqrt{6} - \sqrt{2}}{4} \therefore \theta = \pm \frac{5\pi}{12}$$

$$\text{but } \theta \text{ is the angle between vectors so } \theta \geq 0, \therefore \theta = \frac{5\pi}{12} \quad \text{A1}$$

$$\phi = \frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12} \therefore \phi = \theta.$$

- d. Let $w = i\left(\frac{u}{v}\right)$. Express w in polar form.

In polar form $i = 0 + i = \text{cis}\left(\frac{\pi}{2}\right)$ A1

$$w = i\left(\frac{u}{v}\right) = \text{cis}\left(\frac{\pi}{2}\right) \times \frac{2\text{cis}\left(\frac{\pi}{6}\right)}{2\text{cis}\left(-\frac{\pi}{4}\right)} = \text{cis}\left(\frac{\pi}{2} + \frac{\pi}{6} - \left(-\frac{\pi}{4}\right)\right) = \text{cis}\left(\frac{11\pi}{12}\right) \quad \text{A1}$$

e. $w = \text{cis}\left(\frac{11\pi}{12}\right)$ so $w^n = \text{cis}\left(\frac{11\pi n}{12}\right) = \cos\left(\frac{11\pi n}{12}\right) + i\sin\left(\frac{11\pi n}{12}\right)$

For w^n to be a real number $\sin\left(\frac{11\pi n}{12}\right) = 0$ H1

$$\therefore \frac{11\pi n}{12} = \pi k, k \in \mathbb{Z}^+$$

$$n = \frac{12k}{11}$$

Therefore $n = 12$. A1

Question 3

a. $\frac{x^2}{100} + \frac{(y-5)^2}{25} = 1$ A1

b. i $V = \pi \int x^2 dy$

Rearranging the ellipse equation to make x^2 the subject gives:

$$x^2 = 100 \left(1 - \frac{(y-5)^2}{25} \right) = 4(25 - (y-5)^2) \quad \text{A1}$$

$$\therefore V = 4\pi \int_0^{10} (25 - (y-5)^2) dy \quad \text{A1}$$

ii $V = \frac{2000\pi}{3}$ cubic units A1

c. Solving $1500 = 4\pi \int_0^d (25 - (y-5)^2) dy$ gives $d = 6.4849763$

$$\frac{dy}{dt} = \frac{dy}{dV} \times \frac{dV}{dt} \text{ where } \frac{dV}{dt} = 100 \text{ cm}^3/\text{hr}$$

From **b. ii** $\frac{dV}{dy} = 4\pi(25 - (y-5)^2)$ H1

So $\frac{dy}{dt} = \frac{1}{4\pi(25 - (y-5)^2)} \times 100$ M1

When $y = 6.4849763$ then $\frac{dy}{dt} = 0.34910..$

So rate of change of the depth, correct to two decimal places is 0.35 m/hr A1

d.

$$\frac{dC}{dt} = 0.1(10 - C) = \frac{10 - C}{10} \text{ so } \frac{dt}{dC} = \frac{-10}{C - 10}$$

$$t = -10 \int \frac{1}{C - 10} dC$$

$$t = -10 \log_e(C - 10) + c \quad \text{M1}$$

When $t=0$, $C=50$ so $c = 10 \log_e(40)$

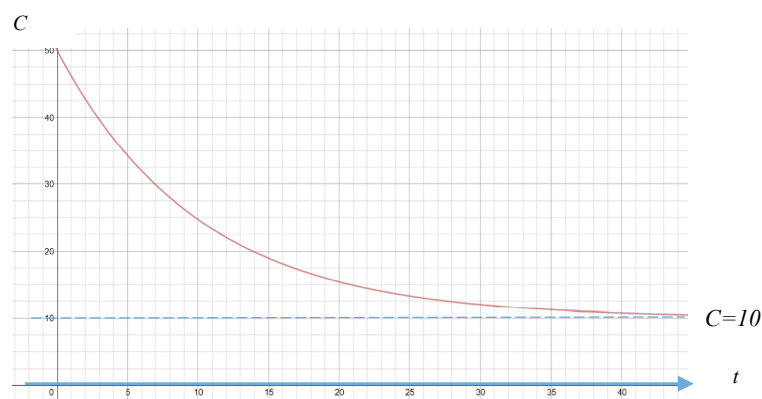
$$t = 10 \log_e \left(\frac{40}{C - 10} \right)$$

Rearranging to make C the subject $C = 40e^{-0.1t} + 10$ A1

Alternatively, using deSolve to get $C = ke^{-0.1t} + 10$ M1

Substituting $t=0$, $C=50$ to get $k=40$, so $C = 40e^{-0.1t} + 10$ A1

e. i.



Asymptote at $C=10$, so chemical concentration settles at 10 kg/m³. A1

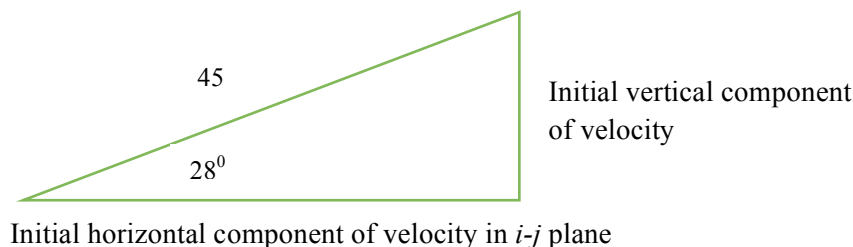
ii. Solve $15 = 40e^{-0.1t} + 10$ to give $t = 20.7944..$ So 21 hours A1

Question 4

a. $\underline{a} = -9.8\underline{k}$

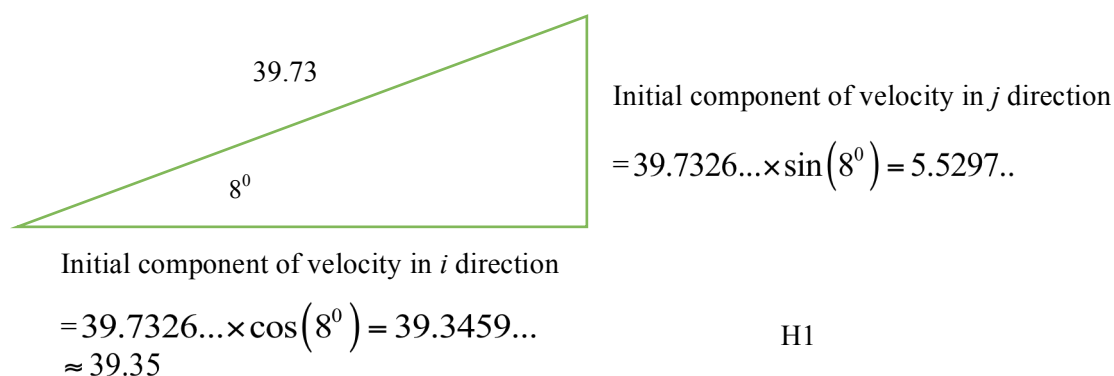
$$\therefore \underline{v} = \int -9.8\underline{k} dt = -9.8t\underline{k} + \underline{c} \quad \text{A1}$$

$$|\underline{v}(0)| = 45 \text{ so resolving in the given directions gives:}$$



$$\begin{aligned} &\text{Initial vertical component of velocity} \\ &= 45 \sin(28^\circ) = 21.126... \approx 21.13 \text{ (correct to 2 dec. pl.)} \quad \text{A1} \end{aligned}$$

$$\text{Horizontal component of velocity in } i\text{-}j \text{ plane} = 45 \cos(28^\circ) = 39.7326... \approx 39.73$$



$$\text{Therefore we get, correct to two decimal places: } \underline{v}(t) = 39.35\underline{i} + 5.53\underline{j} + (21.13 - 9.8t)\underline{k}$$

$$\begin{aligned} \text{b. } \underline{r}(t) &= \int \underline{v} dt = 39.35t\underline{i} + 5.53t\underline{j} + (21.13t - 4.9t^2)\underline{k} + \underline{c}_1 \\ \underline{r}(0) &= \underline{0} \text{ so } \underline{r}(t) = 39.35t\underline{i} + 5.53t\underline{j} + (21.13t - 4.9t^2)\underline{k} \quad \text{A1} \end{aligned}$$

c. Ball hits ground when vertical component equals zero.
 Solving $21.13t - 4.9t^2 = 0$ where $t > 0$ gives 4.31s (correct to 2 dec. pl.) A1

d. Position at $t=4.31$ s is $\underline{r}(4.31) = 169.58\underline{i} + 23.83\underline{j}$ A1

For the ball to land on the fairway the $\underline{j}$ component must be between -25 and 25.
 $-25 < 23.83 < 25$ so ball lands on fairway. A1

e.

$$\text{Distance} = \sqrt{(185 - 169.58)^2 + (23.83 - 25)^2} = 15.4643\dots \approx 15.5m \quad \text{A1}$$

f. i. $a = 1 - \frac{1}{4}v^2$ so $v \frac{dv}{dx} = 1 - \frac{1}{4}v^2$

$$\frac{dv}{dx} = \frac{4 - v^2}{4v} \quad \text{M1}$$

$$\therefore x = \int \frac{4v}{4 - v^2} dv$$

$$x = -2 \log_e |4 - v^2| + c$$

$$x = 0, v = 7 \text{ so } c = 2 \log_e (45) \quad \text{H1}$$

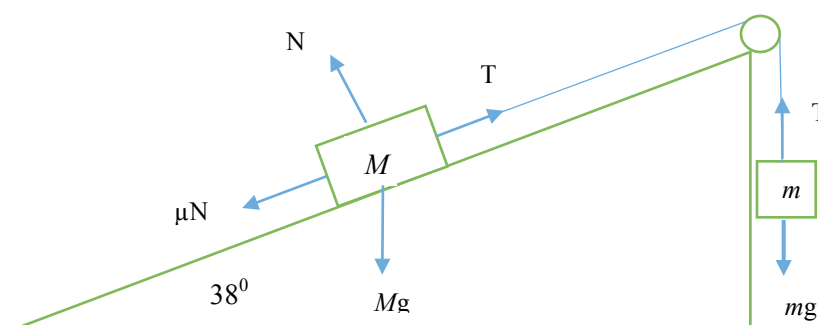
$$x = 2 \log_e \left| \frac{45}{4 - v^2} \right| \quad \text{A1}$$

ii. Ball stops when $v=0$ so distance travelled $x = 2 \log_e \left| \frac{45}{4} \right| \approx 4.84m \quad \text{A1}$

So ball does not roll into the hole which is 5 metres away.

Question 5

a. Mass m is on the point of moving down.



M1 indication of forces on diagram or algebraic equivalent.

In equilibrium so $\sum \vec{F} = 0$.

Therefore, for mass m : $mg - T = 0$

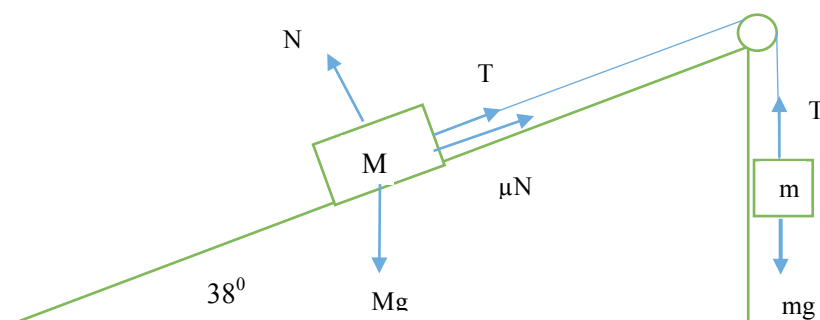
$$\text{and for mass } M: T - Mg \sin(38^\circ) - \mu Mg \cos(38^\circ) = 0 \quad \text{M1}$$

$$\text{Substituting for } T \text{ and } \mu, \text{ gives: } mg - Mg \sin(38^\circ) - 0.15Mg \cos(38^\circ) = 0$$

$$\text{So } \frac{m}{M} = \sin(38^\circ) + 0.15 \cos(38^\circ) = 0.733863\dots$$

$$\text{So } p = 0.73 \text{ (correct to 2 dec. places)} \quad \text{A1}$$

Mass m is on the point of moving up.



M1 indication of forces on diagram or algebraic equivalent.

In equilibrium, so $\sum \vec{F} = 0$.

Therefore, for mass m : $mg - T = 0$

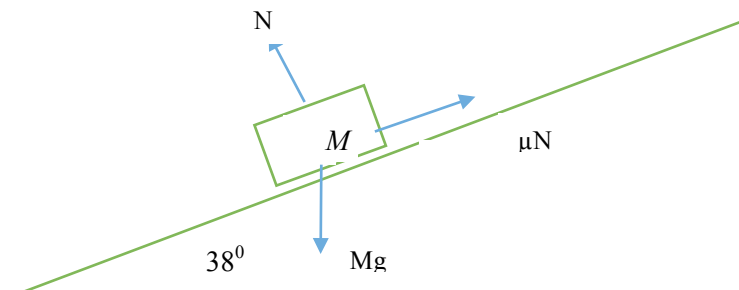
and for mass M : $Mg \sin(38^\circ) - \mu Mg \cos(38^\circ) - T = 0$ M1

Substituting for T and μ , and solving for $\frac{m}{M}$ gives:

$$\text{So } \frac{m}{M} = \sin(38^\circ) - 0.15 \cos(38^\circ) = 0.497\dots$$

So $q = 0.50$ (correct to 2 dec.places) A1

b. i



$$\sum \vec{F} = Ma$$

$$\therefore Mg \sin(38^\circ) - 0.15Mg \cos(38^\circ) = Ma$$

$$a = 4.8751\dots \text{A1}$$

$$v^2 = u^2 + 2as \text{ where } u = 0, s = 2.5, a = 4.8751\dots$$

$$\therefore v = \sqrt{2 \times 2.5 \times 4.8751\dots} = 4.937\dots \approx 4.94 \text{ ms}^{-1} \text{ A1}$$

ii Time to get to end of inclined plane:

$$s = ut + \frac{1}{2}at^2 \text{ where } u = 0, s = 2.5, a = 4.8751\dots$$

$$\text{So } t = 1.0127\dots \text{s}$$

Therefore time to come to rest along the horizontal plane is $2 - 1.0127\dots$ s A1

$$s = \frac{1}{2}(u + v)t \text{ where } u = 4.937\dots, v = 0, t = 2 - 1.0127\dots$$

So distance travelled = 2.21 m (correct to 2 dec. places) A1