

The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2015

Trial Written Examination 1 - SOLUTIONS

Question 1

a. Method 1:

If $2z + i$ is a factor of $2z^3 + 9iz^2 + 10z + 7i$ then $z = -\frac{i}{2}$ is a root of $2z^3 + 9iz^2 + 10z + 7i$:

$$2\left(-\frac{i}{2}\right)^3 + 9i\left(-\frac{i}{2}\right)^2 + 10\left(-\frac{i}{2}\right) + 7i \quad [\text{M1}]$$

$$= 2\left(\frac{i}{8}\right) - 9i\left(\frac{1}{4}\right) + 10\left(-\frac{i}{2}\right) + 7i \quad [\text{M1}]$$

$$= \frac{i}{4} - \frac{9i}{4} - 5i + 7i = -\frac{8i}{4} - 5i + 7i = -2i - 5i + 7i$$

$$= 0.$$

There must be sufficient correct evidence that $2\left(-\frac{i}{2}\right)^3 + 9i\left(-\frac{i}{2}\right)^2 + 10\left(-\frac{i}{2}\right) + 7i = 0$.

Method 2:

IF $2z + i$ is a factor of $2z^3 + 9iz^2 + 10z + 7i$ then the accompanying quadratic factor must have the form $z^2 + az + 7$ so that $2z^3 + 9iz^2 + 10z + 7i = (2z + i)(z^2 + az + 7)$.

It follows that IF a value of a can be found then the quadratic factor $z^2 + az + 7$ exists and therefore $2z + i$ is a factor.

When $(2z + i)(z^2 + az + 7)$ is expanded, the coefficient of z must be equal to 10 by comparison with $2z^3 + 9iz^2 + 10z + 7i$.

Clear explanation of the method:

[M1]

Therefore:

$$14 + ai = 10 \quad \Rightarrow a = \frac{-4}{i} = 4i.$$

Expand to check: $(2z + i)(z^2 + 4iz + 7) = 2z^3 + 9iz^2 + 10z + 7i$.

Therefore $2z^3 + 9iz^2 + 10z + 7i = (2z + i)(z^2 + 4iz + 7)$

[A1]

which shows that $2z + i$ is a factor.

The benefit of this method is that the conclusion $2z^3 + 9iz^2 + 10z + 7i = (2z + i)(z^2 + 4iz + 7)$ can be recruited to answer **part b.**, which saves time. The benefit of using this method as a time investment for answering **part b.** could be recognised during reading time.

Method 3: Use polynomial long division.

$$\begin{array}{r}
 z^2 + 4iz + 7 \\
 2z + i \overline{) 2z^3 + 9iz^2 + 10z + 7i} \\
 \underline{2z^3 + iz^2} \\
 8iz^2 + 10z + 7i \\
 \underline{8iz^2 - 4z} \\
 14z + 7i \\
 \underline{14z + 7i} \\
 0
 \end{array}$$

[M1]

Correct polynomial long division:

The remainder is zero therefore $2z + i$ is a factor.

[A1]

The benefit of this method is that $2z^3 + 9iz^2 + 10z + 7i = (2z + i)(z^2 + 4iz + 7)$ readily follows. This result can be recruited to answer **part b.**, which saves time. The benefit of using this method as a time investment for answering **part b.**, which could be recognised during reading time.

b. Method 1: Construct the quadratic factor.

Clearly $2z^3 + 9iz^2 + 10z + 7i = (2z + i)(z^2 + az + 7)$.

If the value of a can be found then the quadratic factor is completely specified and its linear factors can be determined.

When $(2z + i)(z^2 + az + 7)$ is expanded, the coefficient of z must be equal to 10 by comparison with $2z^3 + 9iz^2 + 10z + 7i$.

Therefore:

$$14 + ai = 10 \quad \Rightarrow a = \frac{-4}{i} = 4i.$$

Therefore the quadratic factor is $z^2 + 4iz + 7$. [M1]

$$z^2 + 4iz + 7 = (z + 2i)^2 + 4 + 7$$

$$= (z + 2i)^2 + 11 \quad \text{[M1]}$$

$$= (z + 2i + i\sqrt{11})(z + 2i - i\sqrt{11}).$$

Answer: $z + (2 + \sqrt{11})i, \quad z + (2 - \sqrt{11})i$. [A1]

Method 2: Use polynomial long division to find the quadratic factor.

$$\begin{array}{r} z^2 + 4iz + 7 \\ 2z + i \overline{) 2z^3 + 9iz^2 + 10z + 7i} \\ \underline{2z^3 + iz^2} \\ 8iz^2 + 10z + 7i \\ \underline{8iz^2 - 4z} \\ 14z + 7i \\ \underline{14z + 7i} \\ 0 \end{array}$$

Therefore the quadratic factor is $z^2 + 4iz + 7$. [M1]

$$z^2 + 4iz + 7 = (z + 2i)^2 + 4 + 7$$

$$= (z + 2i)^2 + 11 \quad \text{[M1]}$$

$$= (z + 2i + i\sqrt{11})(z + 2i - i\sqrt{11}).$$

Answer: $z + (2 + \sqrt{11})i, \quad z + (2 - \sqrt{11})i$. [A1]

Question 2

a. Method 1: Use the formula $y - k = \pm \frac{b}{a}(x - h)$.

The asymptotes of the hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ are $y - k = \pm \frac{b}{a}(x - h)$.

Therefore complete the square to get the standard form and hence identify the values of a , b , h and k :

$$2x^2 - y^2 + 4y - 8 = 0$$

$$\Rightarrow 2x^2 - (y^2 - 4y) - 8 = 0$$

$$\Rightarrow 2x^2 - ([y - 2]^2 - 4) - 8 = 0$$

$$\Rightarrow 2x^2 - [y - 2]^2 + 4 - 8 = 0$$

$$\Rightarrow 2x^2 - (y - 2)^2 = 4$$

$$\Rightarrow \frac{x^2}{2} - \frac{(y - 2)^2}{4} = 1.$$

[A1]

Substitute $a = \sqrt{2}$, $b = 2$, $h = 0$ and $k = 2$ into $y - k = \pm \frac{b}{a}(x - h)$:

$$y - 2 = \pm \frac{2}{\sqrt{2}}x.$$

Answer: $y = \sqrt{2}x + 2$ and $y = -\sqrt{2}x + 2$.

[A1]

Also accept $y = \pm\sqrt{2}x + 2$.

Method 2: Solve for y as a function of x and then take the limit $x \rightarrow \infty$.

$$2x^2 - y^2 + 4y - 8 = 0$$

$$\Rightarrow y^2 - 4y + 8 - 2x^2 = 0$$

$$\Rightarrow y = \frac{4 \pm \sqrt{16 - 4(8 - 2x^2)}}{2}$$

$$= \frac{4 \pm \sqrt{8x^2 - 16}}{2}$$

[M1]

$$= 2 \pm \sqrt{2} \sqrt{x^2 - 2}.$$

Take the limit $x \rightarrow \infty$:

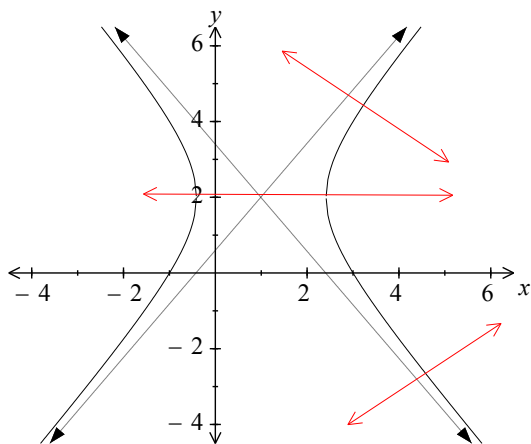
$$y \rightarrow 2 \pm \sqrt{2} |x|.$$

Answer: $y = \sqrt{2}x + 2$ and $y = -\sqrt{2}x + 2$.

[A1]

Also accept $y = \pm\sqrt{2}x + 2$.

- b. Sketching a rough graph of $\frac{x^2}{2} - \frac{(y-2)^2}{4} = 1$ (found in **part a. Method 1**) and drawing in a couple of normal at different points helps to see the answer:



Maximum value of m approaches the gradient of the line that is perpendicular to the diagonal asymptote $y = -\sqrt{2}x + 2$.

Minimum value of m approaches the gradient of the line that is perpendicular to the diagonal asymptote $y = \sqrt{2}x + 2$.

m is continuous.

It follows that $\frac{-1}{\sqrt{2}} < m < \frac{1}{\sqrt{2}}$.

Answer: $\frac{-1}{\sqrt{2}} < m < \frac{1}{\sqrt{2}}$.

[A1]

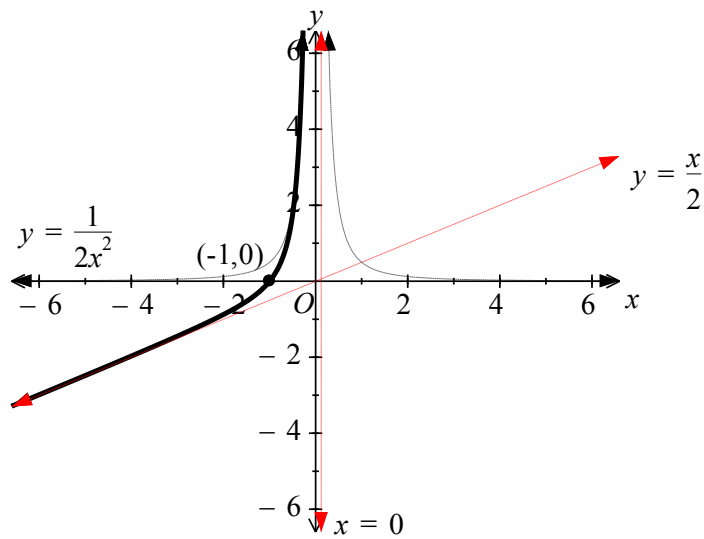
Also accept $|m| < \frac{1}{\sqrt{2}}$, $|m| < \frac{\sqrt{2}}{2}$, $\frac{-\sqrt{2}}{2} < m < \frac{\sqrt{2}}{2}$.

Consequential on answer to part a.: $\frac{-a}{b} < m < \frac{a}{b}$.

Question 3

- Use addition of ordinates to sketch a graph of $y = h(x)$:

$$y = f(x) + g(x) \text{ where } f(x) = \frac{x}{2} \text{ and } g(x) = \frac{1}{2x^2}.$$



Correct shape:

[M $\frac{1}{2}$]

Coordinates $(-1, 0)$ of x-intercept:

[M $\frac{1}{2}$]

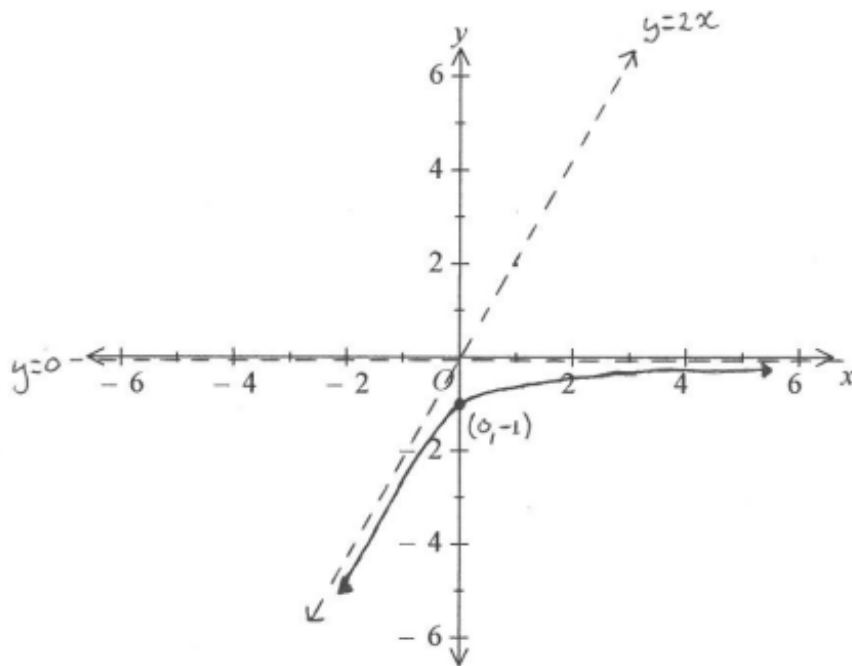
Vertical asymptote $x = 0$:

[M $\frac{1}{2}$]

Diagonal asymptote $y = \frac{x}{2}$:

[M $\frac{1}{2}$]

- Reflect the graph of $y = h(x)$ in the line $y = x$:



Correct shape:

[A $\frac{1}{2}$]

Coordinates $(0, -1)$ of y -intercept:

[A $\frac{1}{2}$]

Horizontal asymptote $y = 0$:

[A $\frac{1}{2}$]

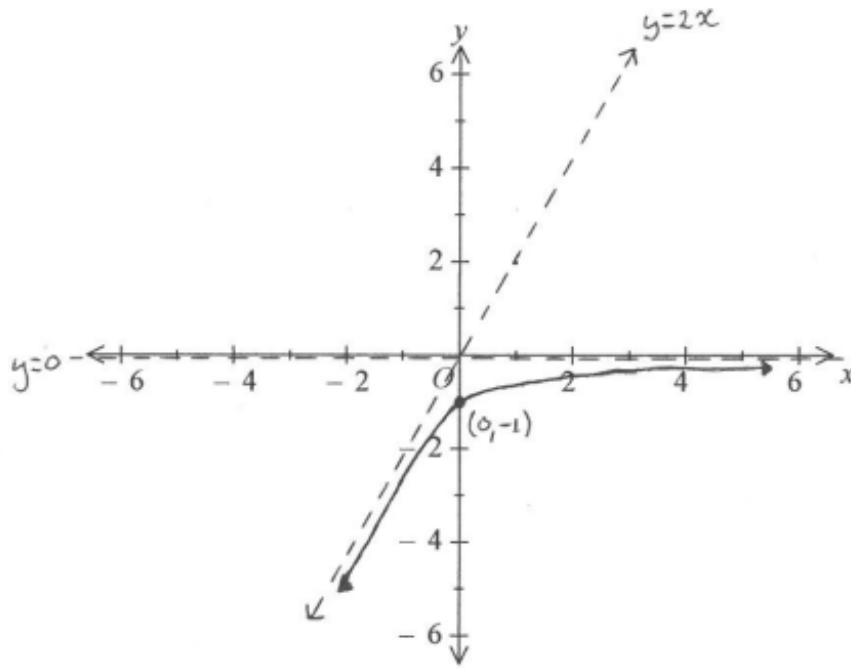
Diagonal asymptote $y = 2x$:

[A $\frac{1}{2}$]

Add up all half marks and round down

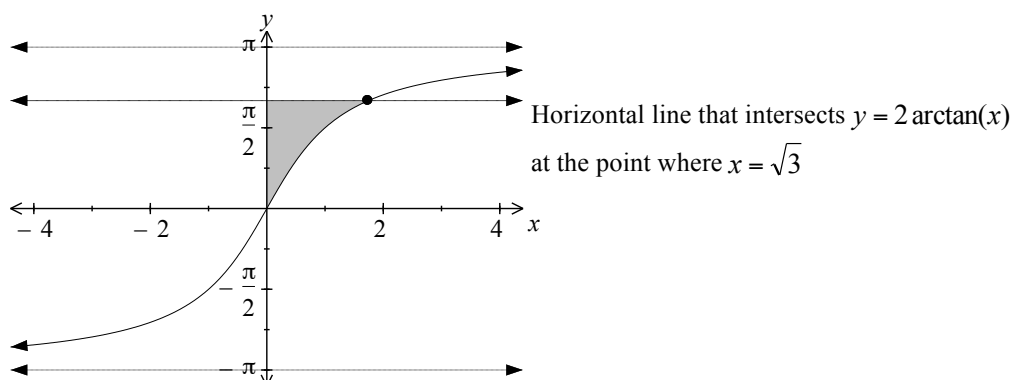
To find the equations of the asymptotes of $y = h^{-1}(x)$, swap x and y in the equations of the asymptotes of $y = h(x)$:

- Vertical asymptote $x = 0$ of $y = h(x)$ becomes horizontal asymptote $y = 0$ of $y = h^{-1}(x)$.
- Diagonal asymptote $y = \frac{x}{2}$ of $y = h(x)$ becomes diagonal asymptote $x = \frac{y}{2} \Rightarrow y = 2x$ of $y = h^{-1}(x)$.

Alternative marking scheme:**Correct shape:****[A1]****Coordinates $(0, -1)$ of x-intercept:****[A1]****Horizontal asymptote $y = 0$:****[A1]****Diagonal asymptote $y = 2x$:****[A1]**

Question 4

- a. A rough graph of $y = 2 \arctan(x)$ should be drawn and the required area shaded:



The equation of the horizontal line is found by substituting $x = \sqrt{3}$ into $y = 2 \arctan(x)$:

$$y = 2 \arctan(\sqrt{3}) = 2\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}.$$

Therefore the required area is $A = \int_0^{\sqrt{3}} \frac{2\pi}{3} - 2 \arctan(x) dx$.

The integral of $\arctan(x)$ can only be done by hand within the scope of the course by using integration by recognition. However, this technique has not been flagged by a prior question such as “Find the derivative of”. Therefore the technique of ‘definite integration using the inverse function’ is required. A rough sketch graph is essential for seeing this.

$$A = \int_0^{2\pi/3} \tan\left(\frac{y}{2}\right) dy.$$

Correct integrand $\tan\left(\frac{y}{2}\right)$:

[M1]

Correct lower and upper integral terminals $y = 0$ and $y = \frac{2\pi}{3}$:

[M1]

Integrate by recognition:

$$A = \left[-2 \log_e \left(\cos\left(\frac{y}{2}\right) \right) \right]_0^{2\pi/3}$$

[M1]

$$= -2 \left(\log_e \left(\cos\left(\frac{\pi}{3}\right) \right) - \log_e(\cos(0)) \right) = -2 \log_e \left(\frac{1}{2} \right) + 2 \log_e(1).$$

Answer: $-2 \log_e \left(\frac{1}{2} \right)$ or $2 \log_e(2)$ or $\log_e(4)$.

[A1]

Units are not required.

b. Required volume:

$$V = \pi \int_0^{2\pi/3} x^2 dy$$

$$= \pi \int_0^{2\pi/3} \tan^2\left(\frac{y}{2}\right) dy \quad [\text{H1}]$$

Consequential on their upper terminal found in part a.

Consequential on their integrand found in part a.

$$= \pi \int_0^{2\pi/3} \sec^2\left(\frac{y}{2}\right) - 1 dy \quad [\text{M1}]$$

Integrate by using the formula on the VCAA detachable formula sheet:

$$= \pi \left[2 \tan\left(\frac{y}{2}\right) - y \right]_0^{2\pi/3} \quad [\text{H1}]$$

$$= \pi \left(\left(2 \tan\left(\frac{\pi}{3}\right) - \frac{2\pi}{3} \right) - (\tan(0) - 0) \right)$$

$$= \pi \left(2\sqrt{3} - \frac{2\pi}{3} \right).$$

Expand to get required form.

$$\text{Answer: } 2\sqrt{3}\pi - \frac{2}{3}\pi^2. \quad [\text{A1}]$$

Units are not required.

Question 5

a. i. Require $-1 \leq 2x \leq 1$ AND $\frac{\pi}{4} - \arccos(2x) \neq 0$:

$$\bullet -1 \leq 2x \leq 1 \quad \Rightarrow -\frac{1}{2} \leq x \leq \frac{1}{2}.$$

$$\bullet \frac{\pi}{4} - \arccos(2x) \neq 0 \quad \Rightarrow \arccos(2x) \neq \frac{\pi}{4}$$

$$\Rightarrow 2x \neq \frac{1}{\sqrt{2}} \quad \Rightarrow x \neq \frac{1}{2\sqrt{2}}. \quad [\text{M1}]$$

$$\text{Answer: } -\frac{1}{2} \leq x < \frac{1}{2\sqrt{2}} \cup \frac{1}{2\sqrt{2}} < x \leq \frac{1}{2}. \quad [\text{A1}]$$

Also accept $-\frac{1}{2} \leq x < \frac{\sqrt{2}}{4} \cup \frac{\sqrt{2}}{4} < x \leq \frac{1}{2}$ **and answers expressed using bracket notation.**

ii. **Method 1:** Use the “Hence ...”, that is, the answer to **part a**.

- $y = \frac{\pi}{4} - \arccos(2x)$ is a strictly increasing function.

It follows that $y = f(x)$ is a strictly decreasing function.

- The domain of $y = f(x)$ is $-\frac{1}{2} \leq x < \frac{1}{2\sqrt{2}} \cup \frac{1}{2\sqrt{2}} < x \leq \frac{1}{2}$.

- $f\left(-\frac{1}{2}\right) = -\frac{4}{3\pi}$ and $f\left(\frac{1}{2}\right) = \frac{4}{\pi}$.

[A1]

- $y = f(x)$ is undefined for $x = \frac{1}{2\sqrt{2}}$.

(Specifically, $\lim_{x \rightarrow \frac{1}{2\sqrt{2}}^-} f(x) = -\infty$ and $\lim_{x \rightarrow \frac{1}{2\sqrt{2}}^+} f(x) = +\infty$).

It follows from the above dot points that $y \leq -\frac{4}{3\pi} \cup y \geq \frac{4}{\pi}$.

Answer: $y \leq -\frac{4}{3\pi} \cup y \geq \frac{4}{\pi}$.

[A1]

Also accept $\left(-\infty, -\frac{4}{3\pi}\right] \cup \left[\frac{4}{3\pi}, +\infty\right)$.

Method 2: "... or otherwise, ...". Construct a rough sketch graph of $y = f(x)$ from a rough sketch graph of the reciprocal function $y = \frac{1}{f(x)} = \frac{\pi}{4} - \arccos(2x)$.

- Graph of $y = \frac{1}{f(x)} = \frac{\pi}{4} - \arccos(2x)$.

Domain:

$$-1 \leq 2x \leq 1$$

$$\Rightarrow -\frac{1}{2} \leq x \leq \frac{1}{2}.$$

Range:

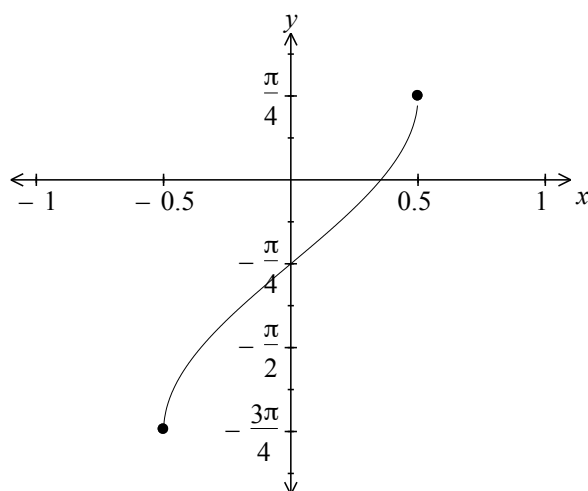
$$\frac{\pi}{4} - \arccos(-1) \leq y \leq \frac{\pi}{4} - \arccos(1)$$

$$\Rightarrow -\pi + \frac{\pi}{4} \leq y \leq 0 + \frac{\pi}{4}$$

$$\Rightarrow -\frac{3\pi}{4} \leq y \leq \frac{\pi}{4}.$$

Coordinates of endpoints: Get from the domain and range.

$$\left(-\frac{3\pi}{4}, -\frac{1}{2}\right) \text{ and } \left(\frac{\pi}{4}, \frac{1}{2}\right).$$

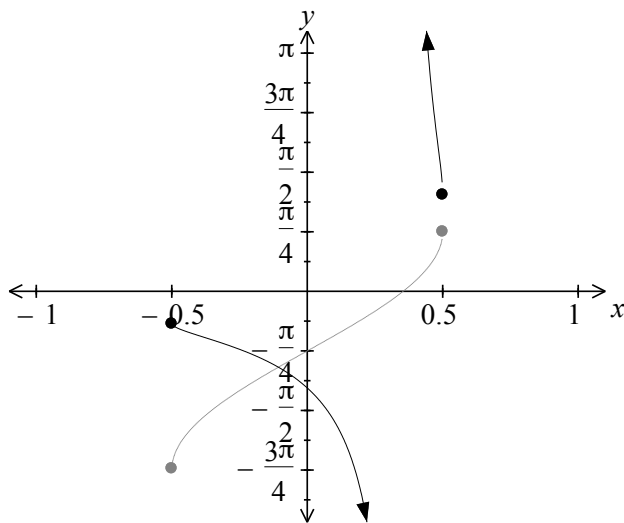


[M1]

There is an **x-intercept** which means that the graph of $y = f(x)$ will have a vertical asymptote but the location is not relevant.

Note: A graph of $y = \frac{\pi}{4} - \arccos(2x)$ can be easily sketched from a knowledge of its domain and range.

- Construct a graph of $y = f(x)$:



Coordinates of endpoints: $\left(-\frac{1}{2}, -\frac{4}{3\pi}\right)$ and $\left(\frac{1}{2}, \frac{4}{\pi}\right)$.

The range is found from the y -coordinates of the endpoints and the shape of this graph.

Answer: $y \leq -\frac{4}{3\pi} \cup y \geq \frac{4}{\pi}$.

[A1]

Also accept $\left(-\infty, -\frac{4}{3\pi}\right] \cup \left[\frac{4}{3\pi}, +\infty\right)$.

b. Use the chain rule.

Let $u = \frac{\pi}{4} - \arccos(2x)$ so that $f(u) = \frac{1}{u}$.

$$f'(x) = f'(u) \times \frac{du}{dx}$$

$$= -\frac{1}{u^2} \times (-1) \frac{-1}{\sqrt{\left(\frac{1}{2}\right)^2 - x^2}}$$

$$= -\frac{1}{u^2} \times \frac{1}{\sqrt{\frac{1}{4} - x^2}} \quad [\text{M1}]$$

where $u = \frac{\pi}{4} - \arccos(2x)$.

There is no need to simplify (and it is more efficient not to) since only a value $f'(0)$ is required.

Substitute $x = 0$:

$$u(0) = \frac{\pi}{4} - \arccos(0) = \frac{\pi}{4} - \frac{\pi}{2}$$

$$= \frac{-\pi}{4} \quad [\text{A1}]$$

$$\Rightarrow \frac{1}{u^2} = \frac{16}{\pi^2}.$$

Therefore:

$$f'(0) = -\frac{16}{\pi^2} \times \frac{1}{\sqrt{\frac{1}{4}}} = -\frac{32}{\pi^2}.$$

$$\text{Answer: } -\frac{32}{\pi^2}. \quad [\text{A1}]$$

Question 6

Let $\underline{u} = (b - a^2)\underline{i} + \underline{j} - b\underline{k}$ and $\underline{v} = 2(a - b)\underline{i} + 4\underline{j} + 3(a - b)\underline{k}$.

$\underline{u}$ and $\underline{v}$ are parallel if $\underline{u} = \lambda \underline{v}$ where $\lambda \in \mathbb{R}$.

By considering the ratio of the components on each side of $\underline{u} = \lambda \underline{v}$ it follows that

$$\underbrace{\frac{b - a^2}{2(a - b)}}_{\text{Ratio of } \underline{i}\text{-components}} = \underbrace{\frac{1}{4}}_{\text{Ratio of } \underline{j}\text{-components}} = \underbrace{\frac{-b}{3(a - b)}}_{\text{Ratio of } \underline{k}\text{-components}}$$

from which it follows that:

$$\bullet \frac{b - a^2}{2(a - b)} = \frac{1}{4} \quad \text{[M1]}$$

$$\Rightarrow 2(b - a^2) = a - b$$

$$\Rightarrow 3b - a - 2a^2 = 0. \quad \dots (1)$$

$$\bullet \frac{-b}{a - b} = \frac{1}{4} \quad \text{[M1]}$$

$$\Rightarrow -4b = 3a - 3b$$

$$\Rightarrow b = -3a. \quad \dots (2)$$

Substitute equation (2) into equation (1):

$$\Rightarrow 3(-3a) - a - 2a^2 = 0$$

$$\Rightarrow a^2 + 5a = 0 \quad \text{[M1]}$$

$$\Rightarrow a(a + 5) = 0$$

$$\Rightarrow a = 0, -5.$$

$a = 0$ is rejected since $a, b \in \mathbb{R} \setminus \{0\}$.

Substitute $a = -5$ into equation (2): $b = 15$.

Answer: $a = -5, b = 15$. [A1]

Question 7

The direction of motion is given by the direction of the velocity vector.

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{r} = 60\vec{i} - 80\vec{j} - 8\vec{k}. \quad [\text{A1}]$$

$$\begin{aligned} \tan(\theta) &= \frac{|\vec{-8k}|}{|60\vec{i} - 80\vec{j}|} \\ &= \frac{|\text{k-component}|}{\sqrt{(\text{i-component})^2 + (\text{j-component})^2}} \\ &= \frac{8}{\sqrt{(60)^2 + (-80)^2}} = \frac{8}{\sqrt{10,000}} = \frac{8}{100} = \frac{2}{25}. \end{aligned}$$

Answer: $\frac{2}{25}$. [A1]

Question 8

$$p = mv$$

therefore $p = 5v$

therefore the value of v when $x = \frac{5}{2}$ is required

therefore $v = v(x)$ is required.

$$a = \sqrt{4 - v^2}$$

$$\Rightarrow v \frac{dv}{dx} = \sqrt{4 - v^2} \quad [\text{M1}]$$

$$\Rightarrow \frac{dv}{dx} = \frac{\sqrt{4 - v^2}}{v}$$

$$\Rightarrow \frac{dx}{dv} = \frac{v}{\sqrt{4 - v^2}}$$

$$\Rightarrow x = -\sqrt{4 - v^2} + C \quad [\text{M1}]$$

either by recognition or substitution.

Substitute $v = 0$ when $x = 2$ to find C :

$$2 = -\sqrt{4} + C$$

$$\Rightarrow C = 4.$$

$$\text{Therefore } x = -\sqrt{4 - v^2} + 4. \quad [\text{M1}]$$

Substitute $x = \frac{5}{2}$:

$$\frac{5}{2} = -\sqrt{4 - v^2} + 4$$

$$\Rightarrow v = \pm \frac{\sqrt{7}}{2}.$$

$$\text{Answer: } |p| = \frac{5\sqrt{7}}{2} \text{ kg m/s.} \quad [\text{A1}]$$

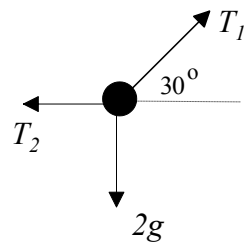
Units are not required.

Question 9

Let T_1 be the tension in the wire attached to the roof at A .

Let T_2 be the tension in the wire attached to the wall at B .

Weight force = $2g$.



Resolve forces acting on the object in the vertical and horizontal directions.

Take the upwards direction as positive.

Vertical direction: $2a = T_1 \sin(30^\circ) - 2g$

$$\Rightarrow 2a = \frac{T_1}{2} - 2g. \quad \dots (1) \quad \text{[A1]}$$

Horizontal direction: $0 = T_2 - T_1 \cos(30^\circ)$

$$\Rightarrow T_2 = \frac{\sqrt{3}}{2} T_1. \quad \dots (2) \quad \text{[A1]}$$

From equation (2) it follows that $T_2 < T_1$.

It follows that if the wire attached to the roof at A breaks then both wires will break.

It is therefore sufficient to find the maximum acceleration so that the wire attached to the ceiling does not break.

The restriction $T_1 < 9g$ is therefore required.

Substitute $T_1 < 9g$ into equation (1):

$$2a < \frac{9g}{2} - 2g$$

$$\Rightarrow a < \frac{5g}{4}.$$

Answer: $\frac{5g}{4} \text{ m/s}^2. \quad \text{[A1]}$

Units are not required.