

Year 2015

VCE

Specialist Mathematics

Trial Examination 1

Solutions



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Question 1

by partial fractions

$$\begin{aligned}\frac{2x-5}{x^2-25} &= \frac{A}{x-5} + \frac{B}{x+5} \\ &= \frac{A(x+5)+B(x-5)}{(x-5)(x+5)} \\ &= \frac{x(A+B)+5(A-B)}{x^2-25}\end{aligned}\quad \text{M1}$$

equating coefficients

$$(1) \ A+B=2 \quad (2) \ A-B=-1 \quad \text{adding} \ 2A=1 \Rightarrow A=\frac{1}{2} \quad B=\frac{3}{2} \quad \text{A1}$$

$$\begin{aligned}\int \frac{2x-5}{x^2-25} dx &= \int \frac{1}{2} \left(\frac{1}{x-5} + \frac{3}{x+5} \right) dx \\ &= \frac{1}{2} (\log_e(|x-5|) + 3\log_e(|x+5|)) + c \\ &= \frac{1}{2} \log_e(|x-5||x+5|^3) + c\end{aligned}\quad \text{A1}$$

Question 2

$$\text{a.} \quad z = 2 \operatorname{cis}\left(-\frac{\pi}{6}\right) = 2 \left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \right) = 2 \cos\left(\frac{\pi}{6}\right) - 2i \sin\left(\frac{\pi}{6}\right) = \sqrt{3} - i$$

Since c is a real constant, by the conjugate root theorem, $\bar{z} = \sqrt{3} + i$ is also a root,
now $z + \bar{z} = 2\sqrt{3}$ and $z\bar{z} = 3 - i^2 = 4$, M1

the quadratic factor is $z^2 - (\text{sum of the roots})z + \text{product of the roots}$ so $(z^2 - 2\sqrt{3}z + 4)$ is the quadratic factor A1

$$\begin{aligned}\text{b.} \quad f(z) &= z^3 + (2 - 2\sqrt{3})z^2 + cz + 8 \\ &= (z^2 - 2\sqrt{3}z + 4)(z + 2) = 0 \\ &= (z - \sqrt{3} + i)(z - \sqrt{3} - i)(z + 2) = 0\end{aligned}$$

expanding gives $c = 4 - 4\sqrt{3}$ A1and all the roots are $z = \sqrt{3} - i$, $\sqrt{3} + i$, $z = -2$ A1

Question 3

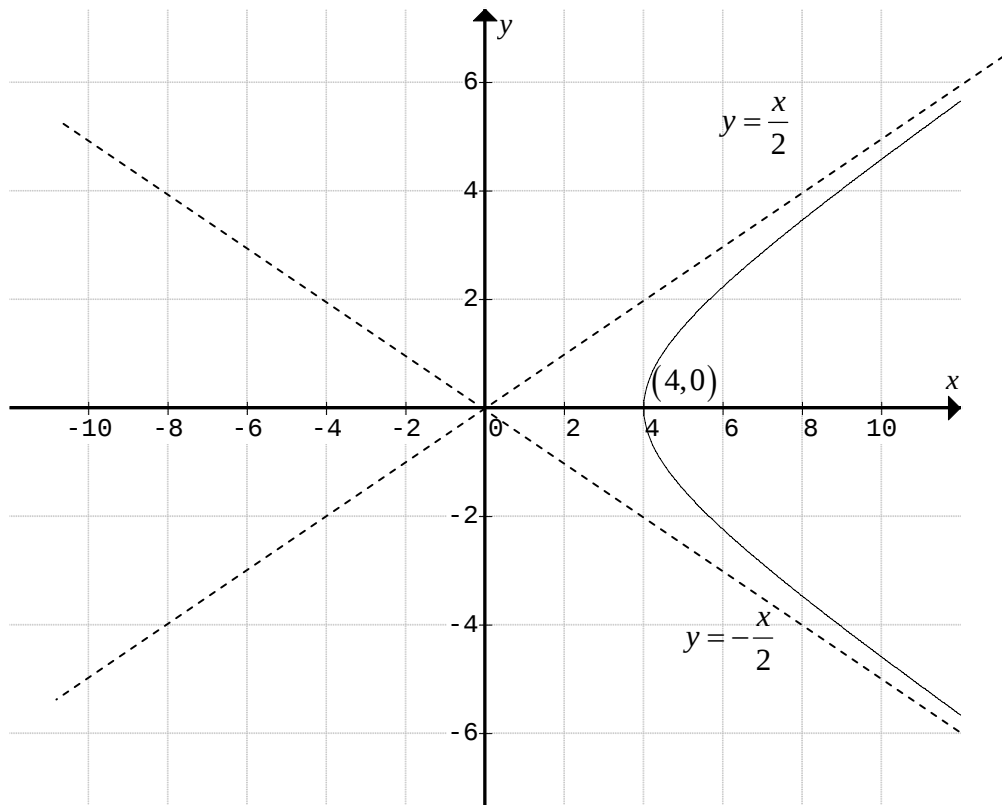
a. $\vec{r}(t) = 2\left(t + \frac{1}{t}\right)\vec{i} + \left(t - \frac{1}{t}\right)\vec{j} \quad t > 0$

$x = 2\left(t + \frac{1}{t}\right), y = t - \frac{1}{t}, \frac{x}{2} = t + \frac{1}{t}, y = t - \frac{1}{t}$ squaring both equations

$\frac{x^2}{4} = t^2 + 2 + \frac{1}{t^2}, y^2 = t^2 - 2 + \frac{1}{t^2}$ subtracting to eliminate $t, \frac{x^2}{4} - y^2 = 4$ M1

$\frac{x^2}{16} - \frac{y^2}{4} = 1$ so that $b = 2$ A1

- b. since $t > 0 \Rightarrow x \geq 4$ it is only the right hand branch of the hyperbola,
crosses x-axis at $(4, 0)$ with asymptotes $y = \pm \frac{x}{2}$, correct shape G2



c. $\dot{\vec{r}}(t) = 2\left(1 - \frac{1}{t^2}\right)\vec{i} + \left(1 + \frac{1}{t^2}\right)\vec{j}$ A1

when it moves parallel to the y-axis $\dot{\vec{r}}(t) \cdot \vec{i} = 0$

$1 - \frac{1}{t^2} = 0 \Rightarrow t = 1$ since $t > 0$ A1

Question 4

$$y = \log_e(x + \sqrt{x^2 + 4}) = \log_e(u) \quad u = x + \sqrt{x^2 + 4} = x + (x^2 + 4)^{\frac{1}{2}}$$

$$\begin{aligned} \frac{dy}{du} &= \frac{1}{u} & \frac{du}{dx} &= 1 + \frac{1}{2} \times 2x(x^2 + 4)^{-\frac{1}{2}} \\ & & &= 1 + \frac{x}{\sqrt{x^2 + 4}} = \frac{\sqrt{x^2 + 4} + x}{\sqrt{x^2 + 4}} \end{aligned} \quad \text{M1}$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{x + \sqrt{x^2 + 4}} \times \frac{x + \sqrt{x^2 + 4}}{\sqrt{x^2 + 4}} = \frac{1}{\sqrt{x^2 + 4}} = (x^2 + 4)^{-\frac{1}{2}} \quad \text{A1}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2} \times 2x(x^2 + 4)^{-\frac{3}{2}} = \frac{-x}{\sqrt{(x^2 + 4)^3}}$$

$$(x^2 + 4) \frac{d^2y}{dx^2} - bx \frac{dy}{dx} = 0$$

$$(x^2 + 4) \times \frac{-x}{\sqrt{(x^2 + 4)^3}} - \frac{bx}{\sqrt{x^2 + 4}} = \frac{-x(b+1)}{\sqrt{x^2 + 4}} = 0$$

$$b = -1 \quad \text{A1}$$

Question 5

- a. $(x^2 + y^2)^2 = 4(x^2 - y^2)$ expanding $x^4 + 2x^2y^2 + y^4 = 4x^2 - 4y^2$
using implicit differentiation and the product rule on the second term.

$$\begin{aligned} \frac{d}{dx}(x^4) + \frac{d}{dx}(2x^2y^2) + \frac{d}{dx}(y^4) &= \frac{d}{dx}(4x^2) - \frac{d}{dx}(4y^2) \\ 4x^3 + 4xy^2 + 4x^2 \frac{dy}{dx} + 4y^3 \frac{dy}{dx} &= 8x - 8y \frac{dy}{dx} \end{aligned} \quad \text{M1}$$

$$\frac{dy}{dx}(4x^2y + 4y^3 + 8y) = 8x - 4x^3 - 4xy^2$$

$$\frac{dy}{dx} = \frac{x(2 - x^2 - y^2)}{y(x^2 + y^2 + 2)} \quad \text{A1}$$

- b. The tangent line is horizontal when $\frac{dy}{dx} = 0 \Rightarrow x^2 + y^2 = 2 \quad y^2 = 2 - x^2$

$$(x^2 + y^2)^2 = 4(x^2 - y^2) \Rightarrow 4 = 4(x^2 - (2 - x^2))$$

$$2x^2 - 2 = 1 \Rightarrow 2x^2 = 3 \Rightarrow x^2 = \frac{3}{2}$$

$$x = \pm \sqrt{\frac{3}{2}} = \pm \frac{\sqrt{6}}{2} \Rightarrow c = 6 \quad \text{A1}$$

Question 6

a. $\frac{dQ}{dt} = bg - \frac{fQ}{V_0 + (g-f)t} = 12 - \frac{4Q}{5+2t}$

$$V_0 = 5, f = 4, bg = 12, g - f = 2$$

$$g = 6, b = 2$$

A1

b. $\frac{dQ}{dt} = 12 - \frac{4Q}{5+2t}, Q(0) = 0$

using Euler's method, with $Q_0 = 0, h = \frac{1}{2}, t_0 = 0, f(Q, t) = 12 - \frac{4Q}{5+2t}$

$$Q_1 = Q_0 + hf(Q_0, t_0)$$

$$= 0 + \frac{1}{2} \left(12 - \frac{4 \times 0}{5+0} \right) = 6$$

M1

$$Q_2 = Q_1 + hf(Q_1, t_1)$$

$$= 6 + \frac{1}{2} \left(12 - \frac{4 \times 6}{5+2 \times 0.5} \right) = 10$$

A1

Question 7 $\underline{a} = 3\underline{i} - 2\underline{j} - 4\underline{k}$ and $\underline{b} = -2\underline{i} + \underline{j} + t\underline{k}$

a. If $\underline{a}$ and $\underline{b}$ are perpendicular, $\underline{a} \cdot \underline{b} = 0$

$$\underline{a} \cdot \underline{b} = -6 - 2 - 4t = 0 \Rightarrow 4t = -8$$

$$t = -2$$

A1

b. If the vectors $\underline{a}$ and $\underline{b}$ are equal in length, $|\underline{a}| = |\underline{b}|$

$$|\underline{a}| = \sqrt{(3)^2 + (-2)^2 + (-4)^2} = \sqrt{9+4+16} = \sqrt{29}$$

$$|\underline{b}| = \sqrt{(-2)^2 + 1^2 + t^2} = \sqrt{5+t^2}$$

$$|\underline{a}| = |\underline{b}| \Rightarrow \sqrt{29} = \sqrt{5+t^2}$$

M1

$$\text{squaring both sides, } \Rightarrow 29 = 5 + t^2 \Rightarrow t^2 = 24$$

$$t = \pm 2\sqrt{6} \text{ both answers are acceptable}$$

A1

c. If the vector $\underline{b}$ makes an angle of 150° with the z -axis.

$$\cos(150^\circ) = \frac{\underline{b} \cdot \underline{k}}{|\underline{b}|} \Rightarrow -\frac{\sqrt{3}}{2} = \frac{t}{\sqrt{5+t^2}} \text{ so that } t < 0$$

M1

$$-\sqrt{3}\sqrt{5+t^2} = 2t \text{ square both sides } 3(5+t^2) = 4t^2 \Rightarrow t^2 = 15 \text{ so } t = \pm\sqrt{15}$$

$$t = -\sqrt{15} \text{ only answer}$$

A1

Question 8

$$y = \frac{x^2 + 9}{3x} = \frac{x}{3} + \frac{3}{x} = \frac{x}{3} + 3x^{-1}$$

crosses x -axis when $y = 0 \Rightarrow x^2 + 9 = 0$ no real solutions,
does not cross the x -axis and does not cross the y -axis.

A1

$x = 0$ is a vertical asymptote and $y = \frac{x}{3}$ is an oblique asymptote.

A1

For turning points $\frac{dy}{dx} = \frac{1}{3} - 3x^{-2} = \frac{1}{3} - \frac{3}{x^2} = 0 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$,

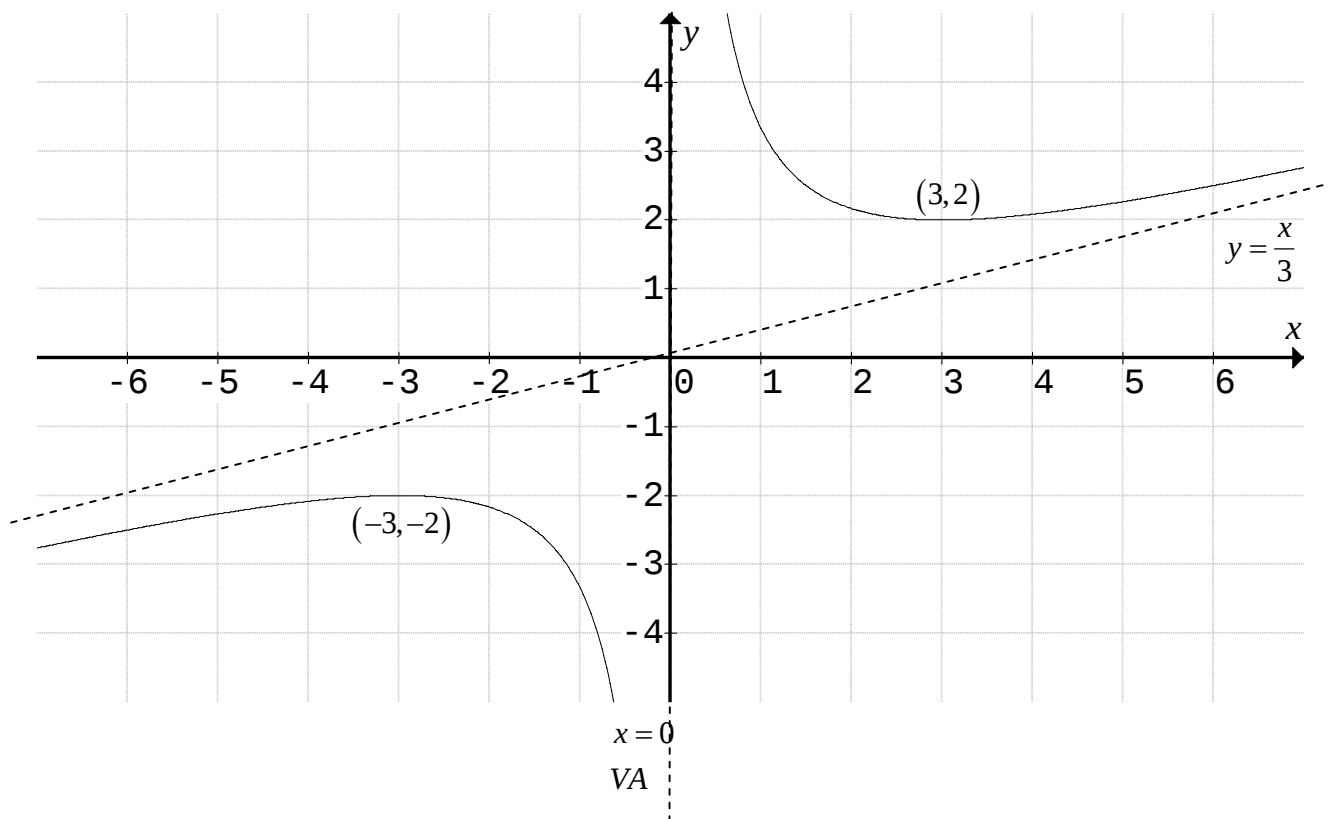
when $x = 3$, $y = 2$ and when $x = -3$, $y = -2$,

$(3, 2)$, $(-3, -2)$ are turning points

A1

correct graph, shape asymptotes

G1



Question 9

resolving downwards around the M kg mass

$$(1) \quad Ma = Mg - T$$

resolving upwards around the 4 kg mass

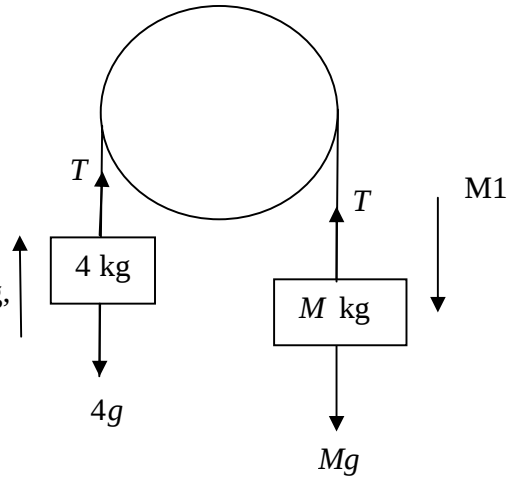
$$(2) \quad T - 4g = 4a$$

adding to eliminate the tension T in the string,

to find the acceleration a , of the system

$$(M - 4)g = (M + 4)a$$

$$a = \frac{(M - 4)g}{M + 4}$$



Now using $v^2 = u^2 + 2as$ with $s = 1$ $u = 0$ $v = \sqrt{g}$ and $a = \frac{(M - 4)g}{M + 4}$ A1

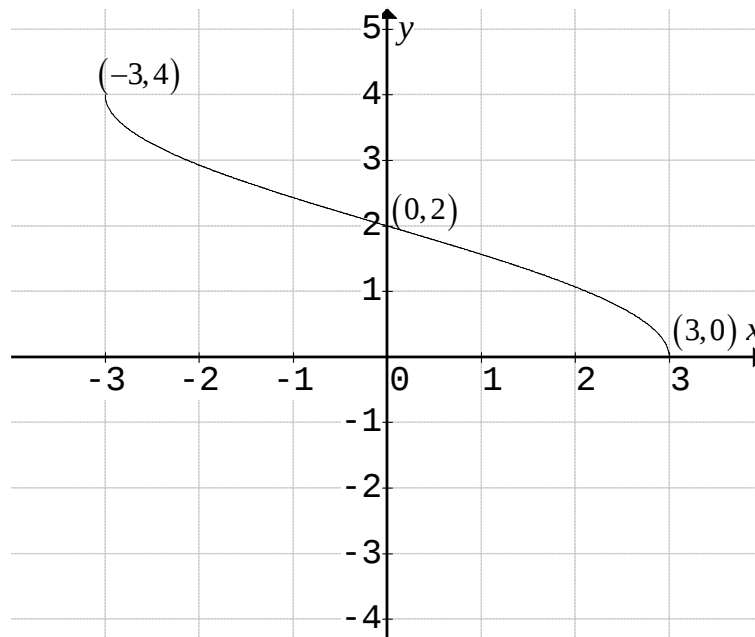
$$g = 0 + 2 \times \frac{(M - 4)}{(M + 4)} \times g \times 1$$

$$\frac{2(M - 4)}{M + 4} = 1 \quad \Rightarrow \quad 2M - 8 = M + 4$$

$$M = 12 \text{ kg} \quad \text{A1}$$

Question 10 $f(x) = \frac{4}{\pi} \arccos\left(\frac{x}{3}\right) = \frac{4}{\pi} \cos^{-1}\left(\frac{x}{3}\right)$

a. domain $[-3, 3]$ range $[0, 4]$, endpoints $(3, 0)$, $(-3, 4)$ y-intercept $(0, 2)$ G2



$$\text{b.} \quad x \arccos\left(\frac{x}{3}\right) = x \cos^{-1}\left(\frac{x}{3}\right)$$

$$\frac{d}{dx}\left(x \cos^{-1}\left(\frac{x}{3}\right)\right) = \cos^{-1}\left(\frac{x}{3}\right) \frac{d}{dx}(x) + x \frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{3}\right)\right) \quad \text{using the product rule}$$

$$\frac{d}{dx}\left(x \cos^{-1}\left(\frac{x}{3}\right)\right) = \cos^{-1}\left(\frac{x}{3}\right) - \frac{x}{\sqrt{9-x^2}} \quad \text{A1}$$

$$\text{The required area is } A = \int_0^3 \frac{4}{\pi} \arccos\left(\frac{x}{3}\right) dx = \frac{4}{\pi} \int_0^3 \cos^{-1}\left(\frac{x}{3}\right) dx \quad \text{A1}$$

$$\begin{aligned} \text{Hence } \int \left(\cos^{-1}\left(\frac{x}{3}\right) - \frac{x}{\sqrt{9-x^2}} \right) dx &= x \cos^{-1}\left(\frac{x}{3}\right) \\ \int \cos^{-1}\left(\frac{x}{3}\right) dx - \int \frac{x}{\sqrt{9-x^2}} dx &= x \cos^{-1}\left(\frac{x}{3}\right) \\ \int \cos^{-1}\left(\frac{x}{3}\right) dx &= x \cos^{-1}\left(\frac{x}{3}\right) + \int \frac{x}{\sqrt{9-x^2}} dx \end{aligned}$$

$$\text{Consider } \int \frac{x}{\sqrt{9-x^2}} dx \quad \text{let } u = 9-x^2 \quad \frac{du}{dx} = -2x \quad \text{M1}$$

$$\begin{aligned} \int \frac{x}{\sqrt{9-x^2}} dx &= \int x u^{-\frac{1}{2}} \frac{dx}{du} du = \int x u^{-\frac{1}{2}} \frac{1}{-2x} du \\ &= -\frac{1}{2} \int u^{-\frac{1}{2}} du = -u^{\frac{1}{2}} = -\sqrt{9-x^2} \end{aligned}$$

$$\int \cos^{-1}\left(\frac{x}{3}\right) dx = x \cos^{-1}\left(\frac{x}{3}\right) - \sqrt{9-x^2}$$

$$\begin{aligned} A &= \int_0^3 \frac{4}{\pi} \arccos\left(\frac{x}{3}\right) dx = \frac{4}{\pi} \int_0^3 \cos^{-1}\left(\frac{x}{3}\right) dx \\ &= \frac{4}{\pi} \left[x \cos^{-1}\left(\frac{x}{3}\right) - \sqrt{9-x^2} \right]_0^3 \\ &= \frac{4}{\pi} \left[\left(3 \cos^{-1}(1) - \sqrt{9-9} \right) - \left(0 \times \cos^{-1}(0) - \sqrt{9} \right) \right] \\ &= \frac{4}{\pi} \left[(3 \times 0 - 0) + 3 \right] \\ &= \frac{12}{\pi} \end{aligned} \quad \text{A1}$$

END OF SUGGESTED SOLUTIONS