

2015 VCAA Specialist Math Exam 2 Solutions

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SECTION 1

1	2	3	4	5	6	7	8	9	10	11
E	A	D	D	B	E	A	C	B/C	B	D

12	13	14	15	16	17	18	19	20	21	22
B	C	E	A	A	C	B	C	B	E	D

Q1 $\cos^2 t + \sin^2 t = 1$

Let $\frac{(x-2)^2}{9} = \cos^2 t$ and $\frac{(y-3)^2}{4} = \sin^2 t$

$x = 2 + 3\cos t$ and $y = 3 + 2\sin t$

E

Q2 Sketch the graph of $f(x) = (2-x)\sin^{-1}\left(\frac{x}{2}-1\right)$ by CAS.

A

Q3 $a^2x^2 + (1-a^2)y^2 = c^2$ is a circle when $a^2 = \frac{1}{2}$.

It is a hyperbola when $a > 1$. It is an ellipse when $a < 1$.

It is a pair of straight lines ($x = \pm c$) when $a = 1$.

D

Q4 $\frac{(x-2)^2}{9} - \frac{(y-1)^2}{4} = k$ passes through (5, 5)

$\therefore \frac{(5-2)^2}{9} - \frac{(5-1)^2}{4} = k, \therefore \frac{(x-2)^2}{9} - \frac{(y-1)^2}{4} = -3$

D

Q5 $z = \frac{1+i\sqrt{3}}{1+i} = \frac{2\operatorname{cis}\frac{\pi}{3}}{\sqrt{2}\operatorname{cis}\frac{\pi}{4}} = \sqrt{2}\operatorname{cis}\frac{\pi}{12} \therefore z^5 = 4\sqrt{2}\operatorname{cis}\frac{5\pi}{12}$

B

Q6 $(1+2i) + (1-2i) = 2$

E

Q7 $z = \sqrt{3} + 3i = 2\sqrt{3}\operatorname{cis}\frac{\pi}{3} \therefore z^{63} = (2\sqrt{3})^{63}\operatorname{cis}\frac{63\pi}{3}$

$= (2\sqrt{3})^{63}\operatorname{cis}21\pi = (2\sqrt{3})^{63}\operatorname{cis}\pi = -(2\sqrt{3})^{63}$

A

Q8 $|z-i| = |z+2|$ is a perpendicular bisector of the line segment joining $z = 0+i$ and $z = -2+0i$

C

Q9 Based on the locations of z_1 and z_1z_2 shown in the given

graph, $r_1 > r_1r_2$ and $\theta_2 > \theta_1, \therefore \frac{r_1}{r_2} > r_1$ and $\theta_1 < \theta_2$.

$\therefore \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right| > r_1$ and $\theta_1 < \theta_2$

B or C

Q10 Let $u = 3x+1, x = \frac{u-1}{3}, \frac{du}{dx} = 3$

$u = 1$ when $x = 0, u = 4$ when $x = 1 \therefore \int_0^1 x^2\sqrt{3x+1} dx$

$= \int_1^4 \frac{(u-1)^2\sqrt{u}}{9} \frac{du}{3} = \frac{1}{27} \int_1^4 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

B

Q11 By estimation, displacement = 0 when $t \approx 3.3$

D

Q12 $\frac{dy}{dx} = 1 - \frac{y}{3} = \frac{3-y}{3}, \frac{dx}{dy} = 3 \times \frac{1}{3-y}$

$\frac{x}{3} = \int \frac{1}{3-y} dy = -\log_e |3-y| + c$ and $y = 4$ when $x = 2$

$\therefore c = \frac{2}{3}$ and $|3-y| = e^{\frac{2-x}{3}} \therefore 3-y = \pm e^{\frac{2-x}{3}}, y = \pm e^{\frac{-(x-2)}{3}} + 3$

B

Q13 A smooth curve can be drawn joining $(-2.5, 1.5)$ and $(3, 1)$ with the slope field line segments as tangents to the curve.

C

Q14 $y = x \sin x, \frac{dy}{dx} = x \cos x + \sin x$

$\frac{d^2y}{dx^2} = -x \sin x + 2 \cos x \therefore \frac{d^2y}{dx^2} = -y + 2 \cos x$

E

Q15 $\hat{w} = \frac{\tilde{w}}{\sqrt{2}} = \frac{\tilde{i} + \tilde{j}}{\sqrt{2}}, \tilde{F} \cdot \tilde{w} = \frac{a+b}{\sqrt{2}}$

$(\tilde{F} \cdot \tilde{w})\hat{w} = (\tilde{F} \cdot \tilde{w})\frac{\tilde{w}}{\sqrt{2}} = \frac{a+b}{2}\tilde{w}$

A

Q16 Resultant force = $\tilde{0}$, i.e. $\tilde{T}_1 + \tilde{T}_2 + \tilde{W} = \tilde{0}$

A

Q17 $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = -\tilde{i} + 2\tilde{j} - \tilde{k}, |\overrightarrow{BA}| = \sqrt{6}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -3\tilde{i} - 2\tilde{j}, |\overrightarrow{BC}| = \sqrt{13}$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}| |\overrightarrow{BC}| \cos \angle ABC$

$\cos \angle ABC = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| |\overrightarrow{BC}|} = \frac{3-4}{\sqrt{6}\sqrt{13}} = \frac{-1}{\sqrt{6}\sqrt{13}}$

C

Q18 Let $\tilde{r}_1(t) = \tilde{r}_2(t)$.

$\therefore 2 + 4t^2 = 6t$ and $3t + 2 = 4 + t, \therefore t = 1, \tilde{r}(1) = 6\tilde{i} + 5\tilde{j}$

B

Q19 1 kg mass: $T - 9.8 = 1 \times 4.9, \therefore T = 14.7$

m kg mass: $m \times 9.8 - 14.7 = m \times 4.9, \therefore m = 3$

C

Q20 $|s| = \left|\frac{1}{2}(u+v)t\right| = \left|\frac{1}{2}(+5+^{-}11) \times 16\right| = 48$

B

Q21 Vertical: $F \sin \theta + N - Mg = 0, \therefore N = Mg - F \sin \theta$

Horizontal: $F \cos \theta - \mu N = Ma$

$\therefore F \cos \theta - \mu(Mg - F \sin \theta) = Ma$

E

Q22 $\frac{dv}{dt} = -(9.8 + 0.1v^2), \frac{dt}{dv} = -\frac{1}{9.8 + 0.1v^2}$

$t = \frac{-10}{\sqrt{98}} \tan^{-1} \frac{v}{\sqrt{98}} + c$ and given $v = 7\sqrt{6}$ when $t = 0$

$\therefore c = \frac{10}{\sqrt{98}} \tan^{-1} \frac{7\sqrt{6}}{\sqrt{98}} = \frac{10}{\sqrt{98}} \tan^{-1} \sqrt{3} = \frac{10}{\sqrt{98}} \times \frac{\pi}{3} = \frac{10\pi}{21\sqrt{2}}$

When $v = 0, t = c = \frac{10\pi}{21\sqrt{2}}$.

D

SECTION 2

Q1a $y = \sqrt{2 - \sin^2 x}$, $y^2 = 2 - \sin^2 x$

$y \frac{dy}{dx} = -2 \sin x \cos x = -\sin(2x)$, $\frac{dy}{dx} = -\frac{\sin(2x)}{y}$

Q1bi $x = 0$, $y = \sqrt{2 - \sin^2 0} = \sqrt{2}$; $x = \frac{\pi}{2}$, $y = \sqrt{2 - \sin^2 \frac{\pi}{2}} = 1$

Q1bii $x = 0$, $\frac{dy}{dx} = -\frac{\sin(0)}{y} = 0$; $x = \frac{\pi}{2}$, $\frac{dy}{dx} = -\frac{\sin \pi}{y} = 0$

Q1c $y = \sqrt{2 - \sin^2 x}$, $y^2 = 2 - \sin^2 x$, dom $\left[0, \frac{\pi}{2}\right]$, range $[1, \sqrt{2}]$

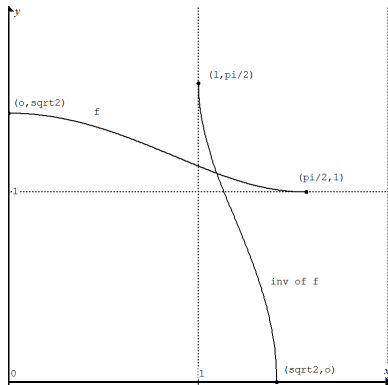
The inverse is $x^2 = 2 - \sin^2 y$, $\sin^2 y = 2 - x^2$

$\therefore 1 - 2 \sin^2 y = 1 - 2(2 - x^2) = 2x^2 - 3$

$\therefore \cos(2y) = 2x^2 - 3$, $y = \frac{1}{2} \cos^{-1}(2x^2 - 3)$,

Its domain is $[1, \sqrt{2}]$ and range is $\left[0, \frac{\pi}{2}\right]$.

Q1d

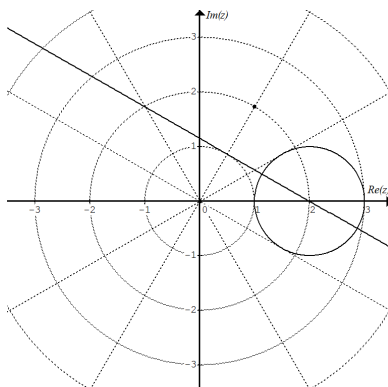


Q1e Let $\sqrt{2 - \sin^2 x} = x$, by CAS $a = x \approx 1.099$

Q1fi $V = \int_0^1 \pi y^2 dx = \int_0^1 \pi (2 - \sin^2 x) dx$

Q1fii By CAS, $V = 5.4$ cubic units

Q2ai and Q2aii



Q2aiii The line is a perpendicular bisector of the line segment joining $0 + 0i$ and $1 + i\sqrt{3}$, $\therefore$ its gradient $= -\frac{1}{\sqrt{3}}$

The equation: $\frac{y-0}{x-2} = -\frac{1}{\sqrt{3}}$, i.e. $x + \sqrt{3}y = 2$

Q2aiiv Circle: $(x-2)^2 + y^2 = 1$, line: $x-2 = -\sqrt{3}y$

$\therefore (-\sqrt{3}y)^2 + y^2 = 1$, $3y^2 + y^2 = 1$, $4y^2 = 1$, $y = \pm \frac{1}{2}$

$\therefore x = 2 - \sqrt{3}\left(\pm \frac{1}{2}\right) = 2 \mp \frac{\sqrt{3}}{2}$

The points of intersection are $\left(2 - \frac{\sqrt{3}}{2}\right) + \frac{1}{2}i$ and $\left(2 + \frac{\sqrt{3}}{2}\right) - \frac{1}{2}i$.

Q2bi $z^2 - (4 \cos \alpha)z + 4 = 0$,

$z = \frac{4 \cos \alpha \pm \sqrt{16 \cos^2 \alpha - 16}}{2}$

$= \frac{4 \cos \alpha \pm i 4 \sin \alpha}{2} = 2(\cos \alpha \pm i \sin \alpha)$

$\therefore z_1 = 2 \operatorname{cis} \alpha$ and $z_2 = 2 \operatorname{cis}(-\alpha)$

Q2bii $\frac{z_1}{z_2} = \operatorname{cis}(2\alpha)$, $\left| \operatorname{Arg}\left(\frac{z_1}{z_2}\right) \right| = 2\alpha = \frac{5\pi}{6}$, $\therefore \alpha = \frac{5\pi}{12}$

Q3a $x = \sin t$, $\frac{dx}{dt} = \cos t$

$y = \frac{1}{2} \sin t \tan t$, $\frac{dy}{dt} = \frac{1}{2} (\sin t \sec^2 t + \cos t \tan t)$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{2} \left(\frac{\sin t \sec^2 t + \cos t \tan t}{\cos t} \right) = \frac{1}{2} \tan t (\sec^2 t + 1)$

Q3b At $t = \frac{\pi}{6}$, the slope $= \frac{dy}{dx} = \frac{1}{2} \tan \frac{\pi}{6} \left(\sec^2 \frac{\pi}{6} + 1 \right) = \frac{7\sqrt{3}}{18}$

Q3ci $\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - x^2}$

$y = \frac{1}{2} \sin t \tan t = \frac{1}{2} x \frac{x}{\sqrt{1 - x^2}} = \frac{x^2}{2\sqrt{1 - x^2}}$

Q3cii $t \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, $x = \sin t \in \left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$, the domain.

Q3d RHS $= \frac{2x^2}{\sqrt{1 - x^2}} + \frac{d}{dx} \left(x\sqrt{1 - x^2} \right)$

$= \frac{2x^2}{\sqrt{1 - x^2}} + \sqrt{1 - x^2} - \frac{x^2}{\sqrt{1 - x^2}} = \frac{1}{\sqrt{1 - x^2}} = \frac{d}{dx} (\arcsin x) = \text{LHS}$

Q3e Area $= 4 \int_0^{\frac{\sqrt{3}}{2}} \frac{x^2}{2\sqrt{1 - x^2}} dx = \int_0^{\frac{\sqrt{3}}{2}} \frac{2x^2}{\sqrt{1 - x^2}} dx$

$= \left[\arcsin x - x\sqrt{1 - x^2} \right]_0^{\frac{\sqrt{3}}{2}} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

Q4ai $\frac{2t}{5} = 60$, $t = 150$, the time required is 150 seconds

Q4aii $\tilde{r}(150) = (50 + 25 \cos 5\pi) \tilde{i} + (50 + 25 \sin 5\pi) \tilde{j} + 60 \tilde{k}$
 $= 25 \tilde{i} + 50 \tilde{j} + 60 \tilde{k}$

$\therefore \tan \theta = \frac{60}{\sqrt{25^2 + 50^2}}$, $\theta \approx 47^\circ$

Q4b After one period $= \frac{2\pi}{\frac{\pi}{30}} = 60$ seconds

Q4c $\tilde{r}(t) = \left(50 + 25 \cos \frac{\pi t}{30}\right) \tilde{i} + \left(50 + 25 \sin \frac{\pi t}{30}\right) \tilde{j} + \frac{2t}{5} \tilde{k}$

$\tilde{v}(t) = \dot{\tilde{r}}(t) = \left(-\frac{5\pi}{6} \sin \frac{\pi t}{30}\right) \tilde{i} + \left(\frac{5\pi}{6} \cos \frac{\pi t}{30}\right) \tilde{j} + \frac{2}{5} \tilde{k} \neq \tilde{0}$

$\tilde{a}(t) = \ddot{\tilde{r}}(t) = \left(-\frac{\pi^2}{36} \cos \frac{\pi t}{30}\right) \tilde{i} + \left(-\frac{\pi^2}{36} \sin \frac{\pi t}{30}\right) \tilde{j} \neq \tilde{0}$

$\tilde{v}(t) \cdot \tilde{a}(t) = \frac{5\pi^3}{246} \sin \frac{\pi t}{30} \cos \frac{\pi t}{30} - \frac{5\pi^3}{246} \sin \frac{\pi t}{30} \cos \frac{\pi t}{30} = 0$

$\therefore \tilde{v}(t) \perp \tilde{a}(t)$, i.e. the velocity of the helicopter is perpendicular to its acceleration

Q4d Speed $= |\tilde{v}(t)| = \sqrt{\left(-\frac{5\pi}{6} \sin \frac{\pi t}{30}\right)^2 + \left(\frac{5\pi}{6} \cos \frac{\pi t}{30}\right)^2 + \left(\frac{2}{5}\right)^2}$
 $= \sqrt{\left(\frac{5\pi}{6}\right)^2 + \left(\frac{2}{5}\right)^2} \approx 2.65 \text{ ms}^{-1}$

Q4e $\tilde{r}_h(45) = \left(50 + 25 \cos \frac{3\pi}{2}\right) \tilde{i} + \left(50 + 25 \sin \frac{3\pi}{2}\right) \tilde{j} + 18 \tilde{k}$

$= 50 \tilde{i} + 25 \tilde{j} + 18 \tilde{k}$

$\tilde{r}_i = 60 \tilde{i} + 40 \tilde{j} + 8 \tilde{k}$

Distance $= |\tilde{r}_h - \tilde{r}_i| = \sqrt{(-10)^2 + (-15)^2 + 10^2} \approx 20.6 \text{ m}$

Q5a $F = 250g \sin 10^\circ \approx 425.438 \approx 425 \text{ N}$

Q5b $250a = 425.438 - 200$, $a \approx 0.902 \text{ ms}^{-2}$

Q5c $v^2 = u^2 + 2as \approx 0 + 2(0.902)(30)$, $|v| \approx 7.36 \text{ ms}^{-1}$

Q5di $a = 1.4(7 - v)$, $v \frac{dv}{dx} = 1.4(7 - v)$, $\frac{dv}{dx} = 1.4 \left(\frac{7 - v}{v}\right)$

$\therefore 1.4 \frac{dx}{dv} = \frac{v}{7 - v} = \frac{7 - (7 - v)}{7 - v}$, $\therefore 1.4 \frac{dx}{dv} = -1 + \frac{7}{7 - v}$

Q5dii $1.4 \int \frac{dx}{dv} dv = \int \left(-1 + \frac{7}{7 - v}\right) dv$

$1.4x = -v - 7 \log_e(7 - v) + c$ and $v = 0$ when $x = 0$, $\therefore c = 7 \log_e 7$

$\therefore 1.4x = -v - 7 \log_e(7 - v) + 7 \log_e 7$

Q5diii When $x = D$, $v = 5$

$\therefore 1.4D = -5 - 7 \log_e(7 - 5) + 7 \log_e 7$, $D \approx 2.7 \text{ m}$

Q5div $a = 1.4(7 - v)$, $\frac{dv}{dt} = 1.4(7 - v)$, $\frac{dt}{dv} = \frac{1}{1.4} \times \frac{1}{7 - v}$

$v = 0$ when $t = 0$, $\therefore v = 5$ when

$t = \frac{1}{1.4} \times \int_0^5 \frac{1}{7 - v} dv = \frac{1}{1.4} \times [-\log_e(7 - v)]_0^5 \approx 0.9 \text{ s}$

Please inform mathline@itute.com re conceptual and/or mathematical errors