2015 Specialist Maths Trial Exam 2 Solutions

Section 1

1	2	3	4	5	6	7	8	9	10	11
D	B	A	D	C	C	D	E	C	A	C

12	13	14	15	16	17	18	19	20	21	22
A	B	B	D	D	D	C	A	C	D	A

Q1 $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, gradient of asymptote $=\pm\frac{b}{a}$

$(1,1)$ is on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, .: $\frac{1}{a^2}-\frac{1}{b^2}=1$,

.: $\frac{b}{a}=\pm\frac{1}{\sqrt{1-a^2}}$ or $\pm\sqrt{1+b^2}$ D

Q2 $\cos^{-1}(ax)-\frac{\pi}{2}=\pm\frac{\pi}{4}$, $\cos^{-1}(ax)=\frac{\pi}{2}\pm\frac{\pi}{4}$

.: $\cos^{-1}(ax)=\frac{\pi}{4}$ or $\frac{3\pi}{4}$, .: $ax=\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$,

.: $x=\frac{1}{\sqrt{2}a}$ or $-\frac{1}{\sqrt{2}a}$, .: the domain of f is $\left[-\frac{1}{\sqrt{2}a},\frac{1}{\sqrt{2}a}\right]$ B

Q3 For $b\in R^+$, $\{z:|z-ib\sqrt{3}|=|z-b|\}$ is a perpendicular bisector of the line joining $z=ib\sqrt{3}$ and $z=b$.

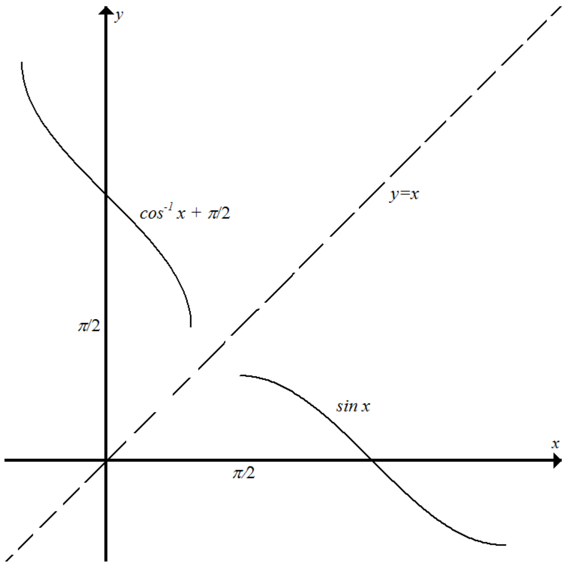
The midpoint is $z=\frac{b}{2}+\frac{ib\sqrt{3}}{2}$ and $\text{Arg}(z)=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$.

.: $\left\{z:\text{Arg}(z)=\frac{\pi}{3}\right\}\cap\{z:|z-ib\sqrt{3}|=|z-b|\}$ is $\left\{\frac{b}{2}+\frac{ib\sqrt{3}}{2}\right\}$ A

Q4 $z^5+i=z^5+i^5=(z+i)(z^4-iz^3+i^2z^2-i^3z+i^4)$

$=(z+i)(z^4-iz^3-z^2+iz+1)$ D

Q5 C



Q6 $\frac{d}{dx}\left[\cot^2\left(\frac{a+2x}{b\sqrt{x}+1}\right)-\text{cosec}^2\left(\frac{a+2x}{b\sqrt{x}+1}\right)\right]=\frac{d}{dx}[1]=0$ C

Q7 $\frac{z+1}{z+i}=i$, $z+1=iz-1$, $2=(i-1)z$, $z=\frac{2}{-1+i}=-1-i$

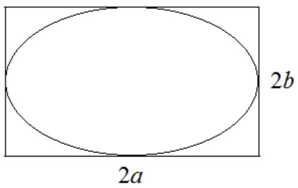
.: $\text{Arg}(z)=-\frac{3\pi}{4}$ D

Q8 $(\sin x+1)\left(\tan x+\frac{3}{2}\right)=0$, $\tan x$ is undefined when

$\sin x=-1$, .: $\sin x+1\neq 0$ and $\tan x+\frac{3}{2}=0$, .: $x=\tan^{-1}\left(-\frac{3}{2}\right)$

E

Q9 Area $=(2a)(2b)=4ab$ C

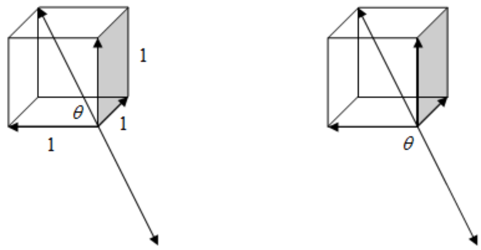


Q10 $y=\tan^{-1}x-\frac{\pi}{4}$, $\frac{dy}{dx}=\frac{1}{1+x^2}$

$y=mx$, $m=\frac{y}{x}=\frac{\tan^{-1}x-\frac{\pi}{4}}{x}$. Let $\frac{dy}{dx}=\frac{\tan^{-1}x-\frac{\pi}{4}}{x}$

.: $\frac{1}{1+x^2}=\frac{\tan^{-1}x-\frac{\pi}{4}}{x}$, by CAS, $x\approx 0.066$ A

Q11



$\cos\theta=\pm\frac{1}{\sqrt{3}}$, .: $\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ or $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ C

Q12 A possible vector on the plane defined by $\tilde{a}$ and $\tilde{b}$ is given by $\tilde{p}=m\tilde{a}+n\tilde{b}$ where $m,n\in R$.

.: $\tilde{p}=(m-\sqrt{2}n)\tilde{i}+(\sqrt{2}m+2n)\tilde{j}+(-m+\sqrt{2}n)\tilde{k}$

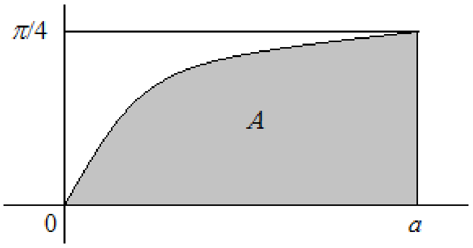
Let $m=\frac{\sqrt{2}}{8}$ and $n=\frac{1}{8}$, $\tilde{p}=\frac{1}{2}\tilde{j}$ A

Q13 $|\overrightarrow{AB}|=\sqrt{32}$, $|\overrightarrow{AC}|=\sqrt{145}$, $\overrightarrow{AB}.\overrightarrow{AC}=|\overrightarrow{AB}||\overrightarrow{AC}|\cos\theta$

$\cos\theta=\frac{4}{\sqrt{32}\sqrt{145}}\approx 0.0587$, .: $\sin\theta=\sqrt{1-\cos^2\theta}\approx 0.9983$

Shortest distance $=\sqrt{145}\sin\theta\approx 12.02$ B

Q14



Area of the required region A

$= \frac{\pi}{4}\times a - \int_0^{\frac{\pi}{4}} x\,dy = \frac{a\pi}{4} - \int_0^{\frac{\pi}{4}} a\tan y\,dy = a\left(\frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \tan x\,dx\right)$ B

Q15 $V = \int_0^4 \pi\left(\sin^{-1}\frac{x}{x^2+1}\right)^2 dx \approx 1.8$ by CAS D

Q16 $\int_0^2 f'(x)dx = \int_0^2 \log_e \sqrt{x^2+1}\,dx = f(2) - f(0)$

$\therefore\ 0.7166 \approx 2 - f(0)$, $f(0) \approx 1.3$ D

Q17 $f'(x) = \dfrac{-1}{\sqrt{1-x^2}\cos^{-1} x}$,

$f'\left(\frac{1}{2}\right) = \dfrac{-1}{\sqrt{1-\frac{1}{4}}\cos^{-1}\left(\frac{1}{2}\right)} = -\dfrac{2\sqrt{3}}{\pi}$ D

Q18 $\tilde{r}(2) - \tilde{r}(1) = \tilde{s} = \int_1^2 \left(2t\,\tilde{i} - \tilde{j}\right)dt = \left[t^2\,\tilde{i} - t\,\tilde{j}\right]_1^2 = 3\tilde{i} - \tilde{j}$

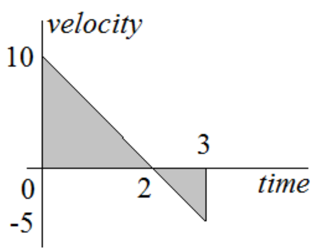
$|\,\tilde{s}\,| = \sqrt{10}$ C

Q19 $\tilde{v} = 4.9\left(\tilde{i} + \left(\sqrt{3} - 2t\right)\tilde{j}\right)$, $|\,\tilde{v}\,| = 4.9\sqrt{1 + \left(\sqrt{3}-2t\right)^2}$

$|\,\tilde{v}\,| = 4.9$ is the minimum when $\left(\sqrt{3} - 2t\right)^2 = 0$. A

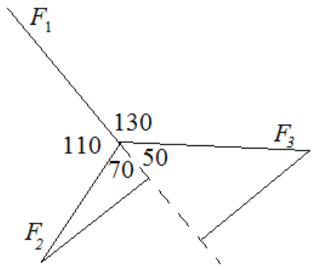
Q20 The particle moves in a straight line under constant acceleration.

$s = \frac{1}{2}(u+v)t$, $t = \frac{2s}{u+v} = \frac{2(7.5)}{10-5} = 3$



Total distance $=$ shaded area $= \frac{1}{2}(10\times 2 + 5\times 1) = 12.5$ C

Q21 $|\,F_3\,|\sin 50° = |\,F_2\,|\sin 70°$ D



Q22 $F = 0.5\times 9.8 = 4.9$ A

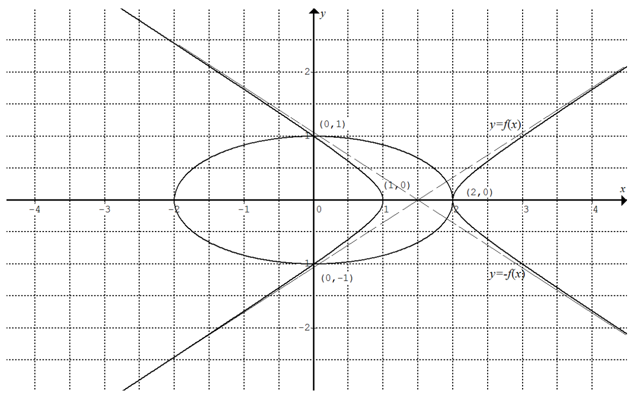
Section 2

Q1a Solve $x^2 + 4y^2 = 4$ and $4\left(x - \frac{3}{2}\right)^2 - 8y^2 = 1$ simultaneously.

$4\left(x-\frac{3}{2}\right)^2 - 2\left(4 - x^2\right) = 1$, $x = 0$ or 2, $\therefore\ y = \pm 1$ or 0

The intersecting points are $(0,-1)$, $(0,1)$ and $(2,0)$

Q1b



In the graph above, $f(x) = \frac{1}{\sqrt{2}}\left(x - \frac{3}{2}\right)$.

Q1c By implicit differentiation, $8\left(x - \frac{3}{2}\right) - 16y\frac{dy}{dx} = 0$

$\therefore\ \frac{dy}{dx} = \frac{x - \frac{3}{2}}{2y}$

At $(0,-1)$, $\frac{dy}{dx} = \frac{3}{4}$, $\therefore$ the tangent is $y = \frac{3}{4}x - 1$

At $(0,1)$, $\frac{dy}{dx} = -\frac{3}{4}$, $\therefore$ the tangent is $y = -\frac{3}{4}x + 1$

Q1d The tangents cut the x-axis at $x = \frac{4}{3}$

Volume of the cone formed by the tangents $= \frac{1}{3}\pi\left(1^2\right)\frac{4}{3} = \frac{4\pi}{9}$

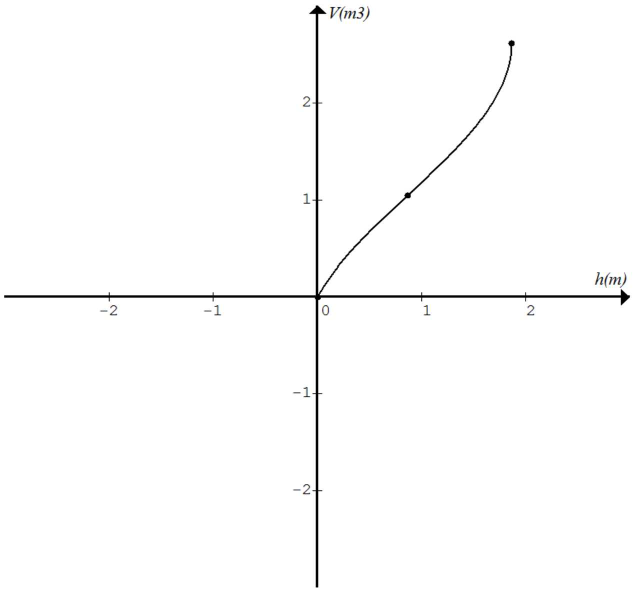
The curve $4\left(x - \frac{3}{2}\right)^2 - 8y^2 = 1$, $\therefore\ y^2 = \frac{1}{2}\left(x - \frac{3}{2}\right)^2 - \frac{1}{8}$

Volume of the solid $= \frac{4\pi}{9} - \int_0^1 \pi\, y^2 dx = \frac{\pi}{36}$

Q2a The volume is at its maximum when $\cos^{-1}\left(h-\frac{\sqrt{3}}{2}\right)$.

.: $V_{\max}=\frac{5\pi}{6}$ when $h-\frac{\sqrt{3}}{2}=1$, i.e. $h=1+\frac{\sqrt{3}}{2}$

Q2b



In the graph above, the left endpoint is $(0,0)$, the point of inflection is $\left(\frac{\sqrt{3}}{2},\frac{\pi}{3}\right)$ and the right endpoint is $\left(1+\frac{\sqrt{3}}{2},\frac{5\pi}{6}\right)$.

Q2c 50 litres $=0.05\text{ m}^3$, $\frac{dV}{dt}=0.05\text{ m}^3$ per minute

.: $V=0.05t$, .: $\frac{5\pi}{6}-\cos^{-1}\left(h-\frac{\sqrt{3}}{2}\right)=0.05t$

.: $h=\cos\left(\frac{5\pi}{6}-\frac{t}{20}\right)+\frac{\sqrt{3}}{2}$

Q2d $\frac{dV}{dt}=0.05\text{ m}^3$ per minute is a constant rate

.: time required to fill the tank $=\frac{5\pi}{6}\div 0.05=\frac{50\pi}{3}$ minutes

Q2e $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$, $0.05=\frac{dV}{dh}\times\frac{dh}{dt}$

.: $\frac{dh}{dt}=\frac{0.05}{\frac{dV}{dh}}$ where $\frac{dV}{dh}=\frac{1}{\sqrt{1-\left(h-\frac{\sqrt{3}}{2}\right)^2}}$

When $h=\sqrt{3}$, $\frac{dV}{dh}=2$, .: $\frac{dh}{dt}=\frac{0.05}{2}=0.025$ m per minute

Q3a $x=10-100\cos\frac{\pi t}{15}$ and $y=160+150\sin\frac{\pi t}{15}$

.: $\cos\frac{\pi t}{15}=\frac{10-x}{100}$ and $\sin\frac{\pi t}{15}=\frac{y-160}{150}$

.: $\left(\frac{10-x}{100}\right)^2+\left(\frac{y-160}{150}\right)^2=1$, .: $\frac{(x-10)^2}{100^2}+\frac{(y-160)^2}{150^2}=1$

Q3b Time to complete one round $=\frac{2\pi}{\frac{\pi}{15}}=30$ seconds

Q3c $\tilde{v}_c=\frac{d\tilde{r}_c}{dt}=\frac{100\pi}{15}\sin\frac{\pi t}{15}\tilde{i}+\frac{150\pi}{15}\cos\frac{\pi t}{15}\tilde{j}$

$|\tilde{v}_c|^2=\left(\frac{100\pi}{15}\sin\frac{\pi t}{15}\right)^2+\left(\frac{150\pi}{15}\cos\frac{\pi t}{15}\right)^2$

$=\left(\frac{100\pi}{15}\right)^2\left(\sin^2\frac{\pi t}{15}+1.5^2\cos^2\frac{\pi t}{15}\right)$

$=\left(\frac{100\pi}{15}\right)^2\left(1+1.25\cos^2\frac{\pi t}{15}\right)$.: $|\tilde{v}_c|=\frac{100\pi}{15}\sqrt{1+1.25\cos^2\frac{\pi t}{15}}$

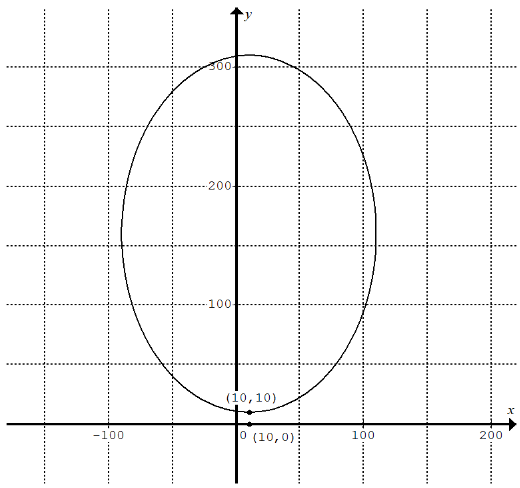
$|\tilde{v}_c|$ is maximum when $\cos\frac{\pi t}{15}=1$

.: the maximum speed is 10π m per s

Q3d i The distance to the spectator is closest when the motorcyclist is at $(10,10)$.

It occurs when $10-100\cos\frac{\pi t}{15}=10$ and $160+150\sin\frac{\pi t}{15}=10$.

.: It first occurs when $\frac{\pi t}{15}=\frac{3\pi}{2}$, i.e. $t=\frac{45}{2}$



Q3d ii The closest distance $=\sqrt{10^2+\left(\frac{40}{3}\right)^2}=\frac{50}{3}$ metres

Q3e $\tilde{v}_c=\frac{d\tilde{r}_c}{dt}=\frac{100\pi}{15}\sin\frac{\pi t}{15}\tilde{i}+\frac{150\pi}{15}\cos\frac{\pi t}{15}\tilde{j}$

$\tilde{a}=\frac{d\tilde{v}_c}{dt}=\frac{100\pi^2}{15^2}\cos\frac{\pi t}{15}\tilde{i}-\frac{150\pi^2}{15^2}\sin\frac{\pi t}{15}\tilde{j}$

$=-\left(\frac{\pi}{15}\right)^2\left(-100\cos\frac{\pi t}{15}\tilde{i}+150\sin\frac{\pi t}{15}\tilde{j}\right)=k(\tilde{r}_c-\tilde{r}_0)$

where $k=-\left(\frac{\pi}{15}\right)^2$

Q4a $z-\frac{1}{z}=x+yi-\frac{1}{x+yi}=x+yi-\frac{x-yi}{x^2+y^2}$

$=\left(x-\frac{x}{x^2+y^2}\right)+\left(y+\frac{y}{x^2+y^2}\right)i$

.: $\mathrm{Re}\left(z-\frac{1}{z}\right)=x-\frac{x}{x^2+y^2}$ and $\mathrm{Im}\left(z-\frac{1}{z}\right)=y+\frac{y}{x^2+y^2}$

Q4b i $2yi=\left(x-\frac{x}{x^2+y^2}\right)+\left(y+\frac{y}{x^2+y^2}\right)i$

.: $-x\left(1-\frac{1}{x^2+y^2}\right)+y\left(1-\frac{1}{x^2+y^2}\right)i=0$

.: $-\left(1-\frac{1}{x^2+y^2}\right)(x-yi)=0$

Since $z-\frac{1}{z}\Rightarrow z\neq 0\Rightarrow \bar{z}\neq 0$, i.e. $x-yi\neq 0$

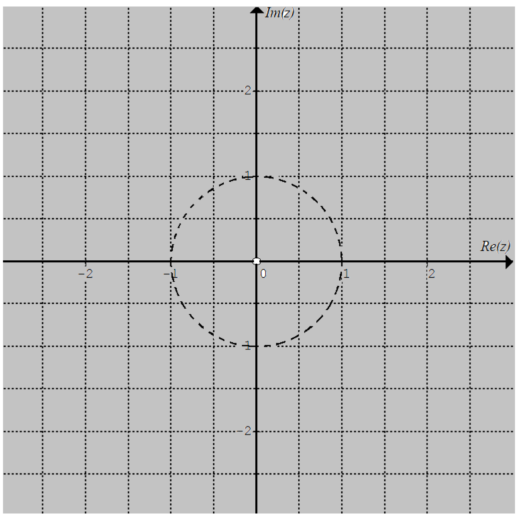
.: $1-\frac{1}{x^2+y^2}=0$, i.e. $x^2+y^2=1$

Q4b ii $\left|i\,2\,\mathrm{Im}(z)-\left(z-\frac{1}{z}\right)\right|>0$,

$\left|-x\left(1-\frac{1}{x^2+y^2}\right)+y\left(1-\frac{1}{x^2+y^2}\right)i\right|>0$

$\sqrt{x^2\left(1-\frac{1}{x^2+y^2}\right)^2+y^2\left(1-\frac{1}{x^2+y^2}\right)^2}>0$

$\sqrt{\left(x^2+y^2\right)\left(1-\frac{1}{x^2+y^2}\right)^2}>0$ which is true for $x, y\in R$ and $x^2+y^2\neq 1$ and $x=y\neq 0$, .: the required region is the shaded region not including the dotted circle and the origin.



Q4c $2x-\left(\left(x+\frac{x}{x^2+y^2}\right)+\left(y-\frac{y}{x^2+y^2}\right)i\right)=0$

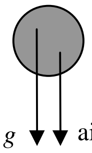
$x\left(1-\frac{1}{x^2+y^2}\right)-y\left(1-\frac{1}{x^2+y^2}\right)i=0$, $\left(1-\frac{1}{x^2+y^2}\right)(x-yi)=0$

.: $x^2+y^2=1$

Q4d An empty set

Q4e $2\,\mathrm{Re}(z-1-i)=(z-1-i)+\frac{1}{(z-1-i)}$, the centre of $x^2+y^2=1$ is translated to $(1,1)$, the Cartesian equation is $(x-1)^2+(y-1)^2=1$.

Q5a



Q5b $ma=R$, $1\times\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-\left(9.8+0.01v^2\right)$

Q5c i $\frac{1}{2}\times\frac{d\left(v^2\right)}{dx}=-\left(9.8+0.01v^2\right)$, $\frac{dx}{d\left(v^2\right)}=-\frac{1}{2}\times\frac{1}{9.8+0.01v^2}$

$x=-\frac{1}{2}\int\frac{1}{9.8+0.01v^2}d\left(v^2\right)$, $x=-50\log_e\left(9.8+0.01v^2\right)+c$

Let $v=20$ at $x=0$ when $t=0$.

.: $c=50\log_e(13.8)$, .: $x=50\log_e\left(\frac{13.8}{9.8+0.01v^2}\right)$

Q5c ii Maximum height is reached when $v=0$,

$x=50\log_e\left(\frac{13.8}{9.8}\right)\approx 17.11$, .: max height $\approx 17.11\,\mathrm{m}$

Q5d $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=9.8-0.01v^2$

Q5e i $\frac{dx}{d\left(v^2\right)}=\frac{1}{2}\times\frac{1}{9.8-0.01v^2}$, $x=\frac{1}{2}\int\frac{1}{9.8-0.01v^2}d\left(v^2\right)$

.: $x=-50\log_e\left(9.8-0.01v^2\right)+c$

Let $v=0$ at $x=0$ when $t=0$. .: $c=50\log_e 9.8$

.: $x=50\log_e\frac{9.8}{9.8-0.01v^2}$, .: $e^{\frac{x}{50}}=\frac{9.8}{9.8-0.01v^2}$

$v^2=\frac{9.8\left(e^{\frac{x}{50}}-1\right)}{0.01e^{\frac{x}{50}}}=980\left(1-e^{-\frac{x}{50}}\right)$

Since downward motion is taken as positive, $v=\sqrt{980\left(1-e^{-\frac{x}{50}}\right)}$

Q5e ii When $x=50\log_e\left(\frac{13.8}{9.8}\right)$, $v\approx 16.85$

Q5f $v=\sqrt{980\left(1-e^{-\frac{x}{50}}\right)}$, $\frac{dx}{dt}=\sqrt{980\left(1-e^{-\frac{x}{50}}\right)}$

$t=\int_0^{17.11}\frac{1}{\sqrt{980\left(1-e^{-\frac{x}{50}}\right)}}dx\approx 1.92$ by CAS

Alternatively, $a=\frac{dv}{dt}=9.8-0.01v^2$,

$t=\int_0^{16.85}\frac{1}{9.8-0.01v^2}dv\approx 1.92$ by CAS

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