

Year 12 Trial Exam Paper

2015

SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

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SECTION 1

Question 1

Answer is C.

Worked solution

$$y = \frac{9}{x^2 + bx + c}$$

$$y = 9(x^2 + bx + c)^{-1}$$

$$\frac{dy}{dx} = -9(x^2 + bx + c)^{-2}(2x + b)$$

$$\frac{dy}{dx} = \frac{-9(2x + b)}{(x^2 + bx + c)^2}$$

When $x = \frac{5}{2}$, $y = -4$ and $\frac{dy}{dx} = 0$ gives

$$-4 = \frac{9}{(\frac{5}{2})^2 + b(\frac{5}{2}) + c} \quad \text{and} \quad 0 = \frac{-9(2(\frac{5}{2}) + b)}{((\frac{5}{2})^2 + b(\frac{5}{2}) + c)^2}$$

Solving simultaneously (using CAS or otherwise) gives $b = -5$ and $c = 4$.



Tip

- It is easier if the derivative equation is first simplified to $0 = 2\left(\frac{5}{2}\right) + b$ before entering it into a CAS calculator.

Question 2

Answer is E.

Worked solution

A domain of $[-6, 2]$ and a range of $[-4, 12]$ gives a centre of $(-2, 4)$ for this ellipse.

The semi-major axis is parallel to the y-axis and has a length of 8 units.

The semi-minor axis is parallel to the x-axis and has a length of 4 units.

Therefore, $a = 4$ and $b = 8$.

The equation of the ellipse is

$$\frac{(x+2)^2}{16} + \frac{(y-4)^2}{64} = 1$$

$$\frac{4(x+2)^2}{64} + \frac{(y-4)^2}{64} = 1$$

$$4(x^2 + 4x + 4) + (y^2 - 8y + 16) = 64$$

$$4x^2 + 16x + 16 + y^2 - 8y + 16 = 64$$

$$4x^2 + y^2 + 16x - 8y - 32 = 0$$

Question 3**Answer is E.****Worked solution**

$$\tan^2(2\theta) + 1 = \sec^2(2\theta)$$

$$\tan^2(2\theta) = \left(\frac{-5}{4}\right)^2 - 1$$

$$\tan^2(2\theta) = \frac{25}{16} - 1$$

$$\tan^2(2\theta) = \frac{9}{16}$$

$$\tan(2\theta) = \pm \frac{3}{4}$$

Because $2\theta \in (\frac{\pi}{2}, \pi)$, $\tan(2\theta) = \frac{-3}{4}$.

Now

$$\tan(2\theta) = \frac{2 \tan(\theta)}{1 - \tan^2(\theta)}$$

$$\frac{-3}{4} = \frac{2 \tan(\theta)}{1 - \tan^2(\theta)}$$

Because $\tan(\theta) = y$:

$$\frac{-3}{4} = \frac{2y}{1 - y^2}$$

$$-3(1 - y^2) = 8y$$

$$-3 + 3y^2 = 8y$$

$$3y^2 - 8y - 3 = 0$$

Question 4**Answer is B.****Worked solution**

The domain of $y = \cos^{-1}(x)$ is $[-1, 1]$.

The domain of $y = 1 - 2\cos^{-1}(2 - x)$ can be determined by solving

$$-1 \leq (2 - x) \leq 1$$

$$-3 \leq -x \leq -1$$

$$-x \geq -3 \quad \text{and} \quad -x \leq -1$$

$$x \leq 3 \quad \text{and} \quad x \geq 1$$

Therefore, the maximal implied domain is $[1, 3]$.

When $x = 1$:

$$y = 1 - 2\cos^{-1}(1)$$

$$y = 1 - 0$$

$$y = 1$$

When $x = 3$:

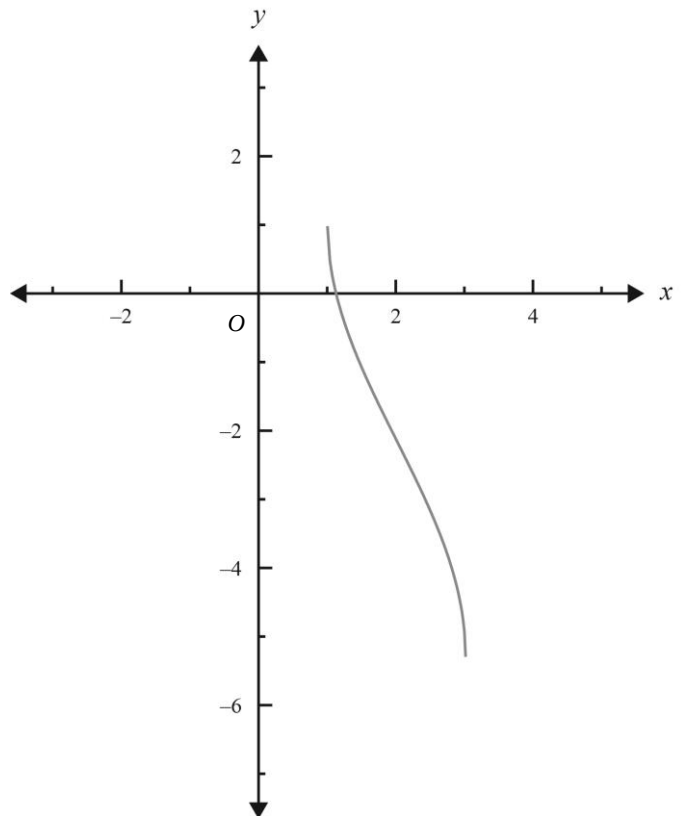
$$y = 1 - 2\cos^{-1}(-1)$$

$$y = 1 - 2\pi$$

$\therefore$ range is $[1 - 2\pi, 1]$.

**Tip**

- It would be quicker to graph the function on a CAS calculator and use the graph trace function to determine the maximal domain and range.



Question 5**Answer is A.****Worked solution**

$$\begin{aligned}
 (a-bi)(2+i) &= i^3 \\
 2a+ai-2bi+b &= -i \\
 (2a+b) + (a-2b)i &= 0-i
 \end{aligned}$$

Hence:

$$2a+b=0$$

$$a-2b=-1$$

Solving simultaneously gives:

$$a = \frac{-1}{5}, \quad b = \frac{2}{5}$$

Question 6**Answer is C.****Worked solution**

$$\begin{aligned}
 &\frac{i[\operatorname{Re}(z)-z]}{\operatorname{Im}(z)} \\
 &= \frac{i[a-(a-bi)]}{-b} \\
 &= \frac{i(a-a+bi)}{-b} \\
 &= \frac{bi^2}{-b} \\
 &= \frac{-b}{-b} \\
 &= 1
 \end{aligned}$$

Question 7**Answer is C.****Worked solution**

$$z^3 = -a$$

Let $z = r \operatorname{cis}(\theta)$, giving

$$[r \operatorname{cis}(\theta)]^3 = -a$$

$$r^3 \operatorname{cis}(3\theta) = a \operatorname{cis}(\pi)$$

$$r^3 = a$$

$$r = a^{\frac{1}{3}}$$

and

$$3\theta = \pi + 2k\pi, \quad k \in \mathbb{Z}$$

$$\theta = \frac{\pi + 2k\pi}{3}$$

When $k = 0$, $\theta = \frac{\pi}{3}$.

When $k = 1$, $\theta = \pi$.

When $k = -1$, $\theta = \frac{-\pi}{3}$.

The solutions are

$$z = a^{\frac{1}{3}} \operatorname{cis}\left(\frac{\pi}{3}\right)$$

$$z = a^{\frac{1}{3}} \operatorname{cis}(\pi) = -a^{\frac{1}{3}}$$

$$z = a^{\frac{1}{3}} \operatorname{cis}\left(\frac{-\pi}{3}\right)$$

**Tip**

- The answer can be obtained also by realising that for all equations of the form $z^3 = -a$, all three solutions will be placed equally around a circle of radius $|a|^{\frac{1}{3}}$.

Question 8**Answer is E.****Worked solution**

$$u = \frac{2}{v}$$

$$\therefore v = \frac{2}{u}$$

$$v = \frac{2 \operatorname{cis}(0)}{\operatorname{cis}\left(\frac{\pi}{4}\right)}$$

$$v = \left(\frac{2}{1}\right) \operatorname{cis}\left(0 - \frac{\pi}{4}\right)$$

$$v = 2 \operatorname{cis}\left(\frac{-\pi}{4}\right)$$

$$p = 2, \theta = \frac{-\pi}{4}$$

Question 9**Answer is D.****Worked solution**

Alternative **A** is a hyperbola and alternative **C** is a circle.

An expression of the form $|z + c| + |z - c| = 2a$, where $|a| > |c|$, represents an ellipse on a complex number plane. Only alternative **D** is of this form.

$|z + 4| + |z - 4| = 10$ can be converted to Cartesian form $\frac{x^2}{25} + \frac{y^2}{9} = 1$, but this is a lengthy process

and not recommended for a multiple-choice question. In fact, $|z + 4| + |z - 4| = 10$ can be described as a locus of points whereby the distance to the point (4, 0) plus the distance to the point (−4, 0) is a constant. These points represent the foci of the ellipse.

**Tip**

- Students should be familiar with general equations to represent lines, rays, circles, ellipses and hyperbolas on a complex number plane.

Question 10**Answer is E.****Worked solution**Let $u = \cos(2x)$

$$\therefore \frac{du}{dx} = -2 \sin(2x)$$

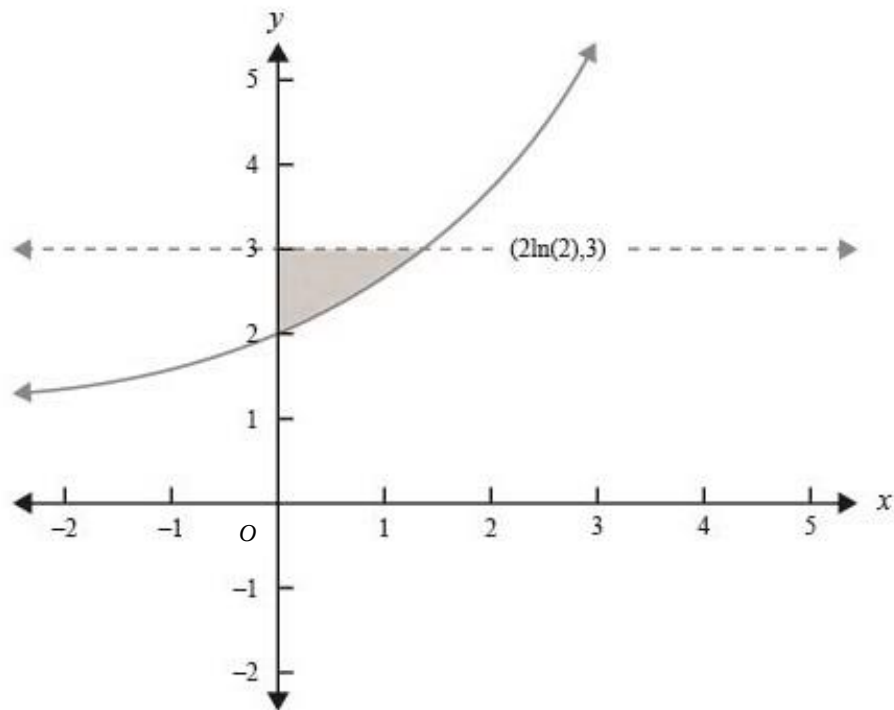
$$\begin{aligned} & \int_0^{\frac{\pi}{6}} \sin(4x) \cos(2x) dx \\ &= \int_0^{\frac{\pi}{6}} 2 \sin(2x) \cos(2x) \cos(2x) dx \\ &= \int_0^{\frac{\pi}{6}} 2 \sin(2x) \cos^2(2x) dx \\ &= -\int_1^{\frac{1}{2}} \cos^2(2x) \frac{du}{dx} dx \\ &= -\int_1^{\frac{1}{2}} u^2 du \\ &= \int_{\frac{1}{2}}^1 u^2 du \end{aligned}$$

Question 11**Answer is C.****Worked solution**

$$\begin{aligned} & \frac{1-x^2}{(x-1)^2} \\ &= \frac{-x^2 + 0x + 1}{x^2 - 2x + 1} \end{aligned}$$

When dividing, this becomes $-1 + \frac{2-2x}{(x-1)^2}$ whereby the expression $\frac{2-2x}{(x-1)^2}$, with the repeated linear factor in the denominator, can be further divided as $\frac{A}{(x-1)^2} + \frac{B}{(x-1)}$.

Therefore, the expression $\frac{1-x^2}{(x-1)^2}$ can be written as $\frac{A}{(x-1)^2} + \frac{B}{x-1} - 1$.

Question 12**Answer is A.****Worked solution**

$$y = 1 + e^{\frac{1}{2}x}$$

$$e^{\frac{1}{2}x} = y - 1$$

$$\frac{1}{2}x = \log_e(y - 1)$$

$$x = 2 \log_e(y - 1)$$

$$A = \int_2^3 2 \log_e(y - 1) dy$$

Question 13**Answer is D.****Worked solution**

$$y = 4e^{\frac{1}{2}x} - 4e^{-\frac{1}{2}x}$$

$$y' = 2e^{\frac{1}{2}x} + 2e^{-\frac{1}{2}x}$$

$$y'' = e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}$$

$$\Rightarrow y'' = \frac{1}{4}y$$

Question 14**Answer is E.****Worked solution**

If $\underline{a}$ is parallel to $\underline{b}$,

then, $\underline{a} = \lambda \underline{b}$.

$$5\underline{i} - \underline{j} + 2\underline{k} = \lambda(3m\underline{i} - 6\underline{j} + 6n\underline{k})$$

$$5\underline{i} - \underline{j} + 2\underline{k} = 3\lambda m\underline{i} - 6\lambda \underline{j} + 6\lambda n\underline{k}$$

$$\Rightarrow -6\lambda = -1$$

$$\therefore \lambda = \frac{1}{6}$$

$$3\lambda m = 5$$

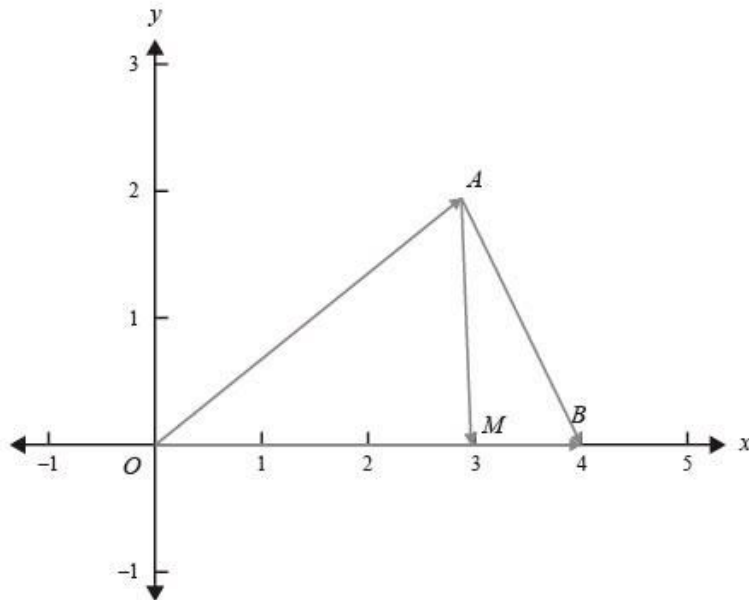
$$\frac{1}{2}m = 5$$

$$m = 10$$

and

$$6\lambda n = 2$$

$$n = 2$$

Question 15**Answer is A.****Worked solution**

$$\vec{AM} = -2\vec{j}$$

$$\vec{AB} = \vec{i} - 2\vec{j}$$

$$\vec{AM} \cdot \vec{AB} = |\vec{AM}| |\vec{AB}| \cos(\theta)$$

$$4 = 2\sqrt{5} \cos(\theta)$$

$$\cos(\theta) = \frac{2}{\sqrt{5}}$$

$$\theta = 27^\circ$$

Question 16**Answer is C.****Worked solution**

$$x = 35 + 20t - 5t^2$$

$$\therefore \frac{dx}{dt} = 20 - 10t$$

$$\therefore v = 20 - 10t$$

At the maximum height, $v = 0$.

$$20 - 10t = 0$$

$$t = 2$$

When $t = 2$:

$$x = 35 + 20(2) - 5(2)^2$$

$$x = 55$$

Therefore, the object is 20 m above the building when it reaches its maximum height of 55 m.

Question 17**Answer is D.****Worked solution**

$$\vec{r}(t) = 2\cos(3t)\vec{i} - 3\sin(3t)\vec{j}$$

$$\dot{\vec{r}}(t) = -6\sin(3t)\vec{i} - 9\cos(3t)\vec{j}$$

$$\text{When } t = 0, \dot{\vec{r}}(t) = -9\vec{j}.$$

$$\text{Speed} = |\dot{\vec{r}}(t)|$$

$$\text{Speed} = 9$$

Question 18*Answer is E.***Worked solution**

$$a = v \frac{dv}{dx}$$

$$v = 5 - x$$

$$\frac{dv}{dx} = -1$$

$$a = -(5 - x)$$

$$a = x - 5$$

Question 19*Answer is E.***Worked solution**

$$\frac{dV}{dt} = -\sqrt{h}$$

$$\therefore V = \pi a^2 h$$

$$\frac{dV}{dh} = \pi a^2$$

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$\frac{dh}{dt} = -\sqrt{h} \times \frac{1}{\pi a^2}$$

$$\frac{dh}{dt} = \frac{-\sqrt{h}}{\pi a^2}$$

Question 20**Answer is A.****Worked solution**

$$y(0) = 2$$

$$\Rightarrow x_0 = 0, y_0 = 2$$

$$y_{n+1} = y_n + hf(x_n, y_n) \text{ and } x_{n+1} = x_n + h$$

When $x = 0.5$:

$$y_1 = y_0 + 0.5f(x_0, y_0)$$

$$y_1 = 2 + 0.5\left(\frac{0}{2}\right)$$

$$y_1 = 2$$

When $x = 1$:

$$y_2 = y_1 + 0.5f(x_1, y_1) \text{ and } x_1 = x_0 + h = 0.5$$

$$y_2 = 2 + 0.5\left(\frac{0.5}{2}\right)$$

$$y_2 = 2.125$$

Question 21**Answer is D.****Worked solution**

The weight force is Mg newton and acts in a direction vertically down.

The normal reaction is N newton and acts in a direction that is perpendicular to the surface of the plane.

The frictional force is μN newton and acts in a direction that is parallel to the plane opposing the direction of motion.

Question 22**Answer is C.****Worked solution**Alternative **A** is correct.

$$F_1 \cos(\alpha) = F_2 \cos(90 - \alpha)$$

$$F_1 \cos(\alpha) = F_2 \sin(\alpha)$$

Alternative **B** is correct.

If the particle is in equilibrium, then

$$\sum \vec{F} = \vec{0}$$

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0}$$

Alternative **C** is incorrect.From **A**,

$$F_1 \cos(\alpha) = F_2 \sin(\alpha)$$

$$\cot(\alpha) = \frac{F_2}{F_1}$$

Alternative **D** is correct.

The net horizontal force is zero, so

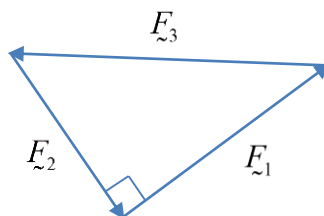
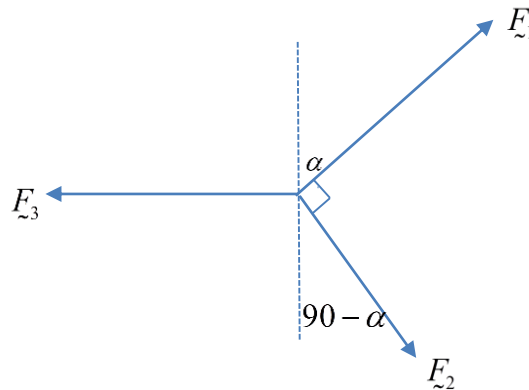
$$F_3 = F_1 \sin(\alpha) + F_2 \sin(90 - \alpha)$$

$$F_3 = F_1 \sin(\alpha) + F_2 \cos(\alpha)$$

Alternative **E** is correct.

If the particle is in equilibrium, then the vector sum forms a right-angled triangle, so

$$F_3^2 = F_1^2 + F_2^2$$



SECTION 2

Question 1a.

Worked solution

$$x = 2 \sec(\theta)$$

$$x = 2[\cos(\theta)]^{-1}$$

$$\frac{dx}{d\theta} = -2[\cos(\theta)]^{-2}[-\sin(\theta)]$$

$$\frac{dx}{d\theta} = \frac{2 \sin(\theta)}{\cos^2(\theta)}$$

$$y = \tan(\theta)$$

$$\frac{dy}{d\theta} = \sec^2(\theta)$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos^2(\theta)} \cdot \frac{\cos^2(\theta)}{2 \sin(\theta)}$$

$$\frac{dy}{dx} = \frac{1}{2 \sin(\theta)}$$

$$\text{When } \theta = \frac{\pi}{6}, x = \frac{4}{\sqrt{3}}, y = \frac{1}{\sqrt{3}}.$$

$$\text{When } \theta = \frac{\pi}{6}, \frac{dy}{dx} = 1.$$

The equation of the tangent is

$$y - \frac{1}{\sqrt{3}} = 1 \left(x - \frac{4}{\sqrt{3}} \right)$$

$$y = x - \frac{3}{\sqrt{3}}$$

$$y = x - \sqrt{3}$$

Mark allocation: 4 marks

- 1 method mark for $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$
- 1 mark for correctly obtaining $\frac{dy}{dx} = \frac{1}{2 \sin(\theta)}$
- 1 mark for $m_T = 1$ and point $\left(\frac{4}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$
- 1 mark for $y = x - \sqrt{3}$

Question 1b.**Worked solution**

$$x = 2 \sec(\theta)$$

$$\Rightarrow \sec(\theta) = \frac{x}{2}$$

$$y = \tan(\theta)$$

$$\text{Now, } \tan^2(\theta) + 1 = \sec^2(\theta)$$

$$y^2 + 1 = \left(\frac{x}{2}\right)^2$$

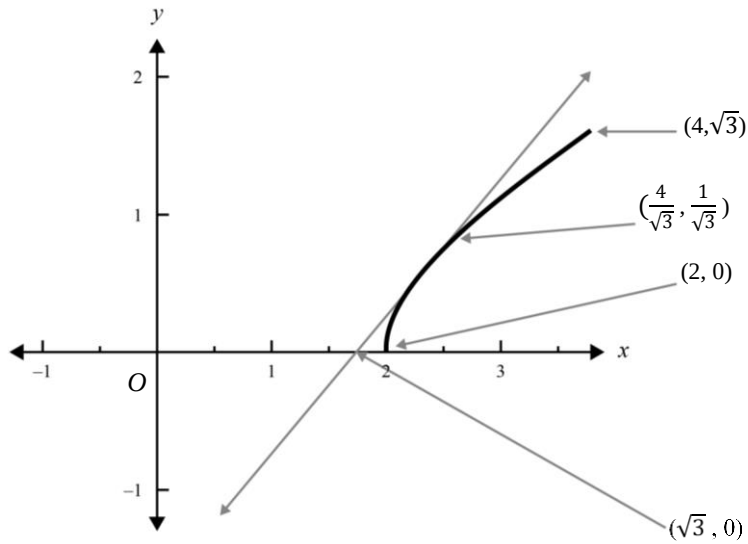
$$\frac{x^2}{4} - y^2 = 1$$

$$\text{When } \theta = 0, x = 2 \Rightarrow a = 2.$$

$$\text{When } \theta = \frac{\pi}{3}, x = 4 \Rightarrow b = 4.$$

Mark allocation: 3 marks

- 1 method mark for using the identity $\tan^2(\theta) + 1 = \sec^2(\theta)$
- 1 mark for showing the Cartesian equation is $\frac{x^2}{4} - y^2 = 1$
- 1 mark for finding $a = 2$ and $b = 4$

Question 1c.**Worked solution****Mark allocation: 2 marks**

- 1 mark for the hyperbola with the endpoints (2, 0) and (4, \sqrt{3}) labelled clearly
- 1 mark for the tangent $y = x - \sqrt{3}$ with the points $(\frac{4}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $(\sqrt{3}, 0)$ labelled clearly

Question 1d.**Worked solution**

$$\begin{aligned}
 V &= \pi \int_{\sqrt{3}}^{\frac{4}{\sqrt{3}}} (x - \sqrt{3})^2 dx - \pi \int_2^{\frac{\sqrt{3}}{2}} \left(\frac{x^2}{4} - 1 \right) dx \\
 &= \frac{\pi(7\sqrt{3} - 12)}{9} \text{ cubic units}
 \end{aligned}$$

Mark allocation: 2 marks

- 1 mark for $V = \pi \int_{\sqrt{3}}^{\frac{4}{\sqrt{3}}} (x - \sqrt{3})^2 dx - \pi \int_2^{\frac{\sqrt{3}}{2}} \left(\frac{x^2}{4} - 1 \right) dx$
- 1 mark for $\frac{\pi(7\sqrt{3} - 12)}{9}$ cubic units

Question 2a.**Worked solution**

$$\frac{dN}{dt} = kN(1500 - N)$$

$$\frac{dt}{dN} = \frac{1}{kN(1500 - N)}$$

$$t = \frac{1}{k} \int \frac{1}{N(1500 - N)} dN$$

Resolving $\frac{1}{N(1500 - N)}$ into partial fractions:

$$\frac{1}{N(1500 - N)} = \frac{A}{N} + \frac{B}{1500 - N}$$

$$\frac{1}{N(1500 - N)} = \frac{A(1500 - N) + BN}{N(1500 - N)}$$

$$\frac{1}{N(1500 - N)} = \frac{1500A - AN + BN}{N(1500 - N)}$$

$$\text{So } 1500A = 1 \Rightarrow A = \frac{1}{1500}$$

$$-A + B = 0, B = \frac{1}{1500}$$

$$t = \frac{1}{k} \int \frac{\frac{1}{1500}}{N} + \frac{\frac{1}{1500}}{1500 - N} dN$$

$$t = \frac{1}{1500k} \int \frac{1}{N} + \frac{1}{1500 - N} dN$$

$$t = \frac{1}{1500k} (\log_e |N| - \log_e |1500 - N|) + c$$

$$t = \frac{1}{1500k} \log_e \left| \frac{N}{1500 - N} \right| + c$$

When $t = 0$, $N = 1$.

$$0 = \frac{1}{1500k} \log_e \left(\frac{1}{1499} \right) + c$$

$$\Rightarrow c = \frac{-1}{1500k} \log_e \left(\frac{1}{1499} \right)$$

$$t = \frac{1}{1500k} \log_e \left| \frac{N}{1500 - N} \right| - \frac{1}{1500k} \log_e \left(\frac{1}{1499} \right)$$

$$t = \frac{1}{1500k} \log_e \left| \frac{1499N}{1500 - N} \right|$$

$$1500kt = \log_e \left| \frac{1499N}{1500 - N} \right|$$

$$\frac{1499N}{1500 - N} = e^{1500kt}, \text{ as } t=0, N=1 \text{ is satisfied.}$$

$$1499N = e^{1500kt} (1500 - N)$$

$$1499N + Ne^{1500kt} = 1500e^{1500kt}$$

$$N(1499 + e^{1500kt}) = 1500e^{1500kt}$$

$$N = \frac{1500e^{1500kt}}{1499 + e^{1500kt}}$$

Therefore, $A = 1500$ and $B = 1499$.

Mark allocation: 5 marks

- 1 mark for inverting $\frac{dN}{dt} = kN(1500 - N)$
- 1 mark for finding $A = \frac{1}{1500}$ and $B = \frac{1}{1500}$ by resolving $\frac{1}{N(1500 - N)}$ into partial fractions
- 1 mark for correctly integrating to find $t = \frac{1}{1500k} \log_e \left| \frac{N}{1500 - N} \right|$
- 1 mark for finding $c = \frac{-1}{1500k} \log_e \left(\frac{1}{1499} \right)$
- 1 mark for transposing to $N = \frac{1500e^{1500kt}}{1499 + e^{1500kt}}$ and stating that $A = 1500$ and $B = 1499$

Question 2b.**Worked solution**

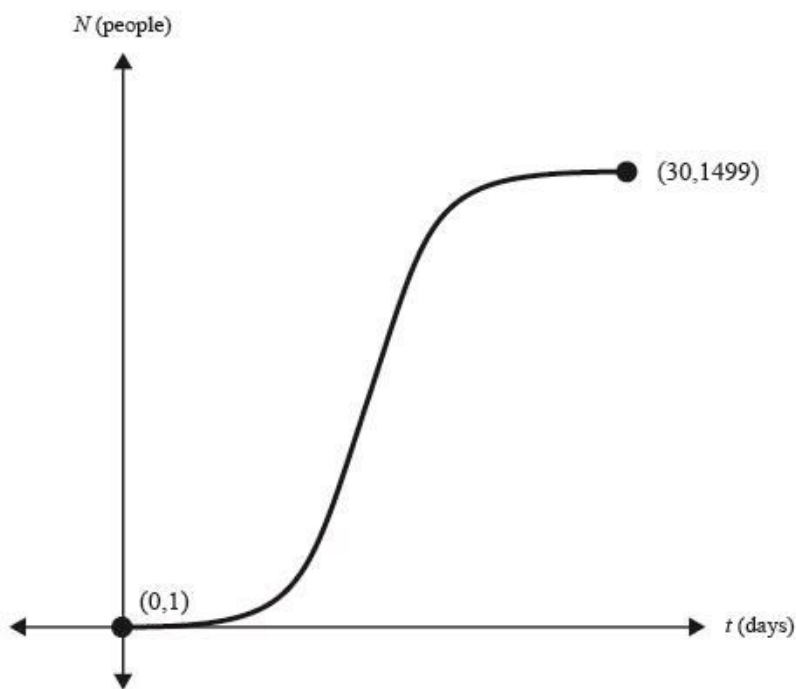
When $k = \frac{1}{3000}$, $t = 10$.

$$N = \frac{1500e^5}{1499 + e^5}$$

$N = 135$ people

Mark allocation: 1 mark

- 1 mark for $N = 135$ people

Question 2c.**Worked solution****Mark allocation: 2 marks**

- 1 mark for graph
- 1 mark for correctly labelled endpoints

Question 3a.**Worked solution**

$$(z - i)(\bar{z} + i) = 4$$

$$z\bar{z} - i\bar{z} + iz - i^2 = 4$$

Let $z = x + yi$, giving

$$(x + yi)(x - yi) - i(x - yi) + i(x + yi) + 1 = 4$$

$$x^2 + y^2 - xi - y + xi - y + 1 = 4$$

$$x^2 + y^2 - 2y + 1 = 4$$

$$x^2 + (y - 1)^2 = 4$$

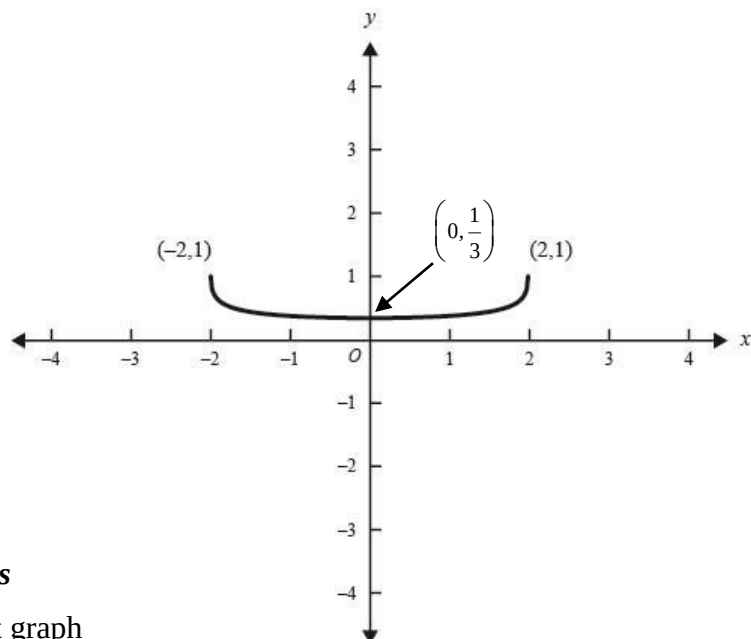
Mark allocation: 2 marks

- 1 mark for correctly expanding
 $(z - i)(\bar{z} + i) = 4$
 $z\bar{z} - i\bar{z} + iz - i^2 = 4$
- 1 mark for substituting $z = x + yi$ and simplifying to $x^2 + (y - 1)^2 = 4$

Question 3b.**Worked solution**

The domain is $[-2, 2]$.

The range is $\left[\frac{1}{3}, 1\right]$.

**Mark allocation: 3 marks**

- 1 mark for correct graph
- 1 mark for labelled endpoints and intercept
- 1 mark for domain and range

Question 3c.**Worked solution**

$$z + 3 = k(iz + i)$$

Let $z = x + yi$, giving

$$x + yi + 3 = k[i(x + yi) + i]$$

$$x + yi + 3 = k(xi - y + i)$$

$$(x + 3) + yi = kxi - ky + ki$$

$$(x + 3) + yi = -ky + ik(x + 1)$$

Equating real and imaginary coefficients:

$$x + 3 = -ky \quad (1) \text{ and}$$

$$y = k(x + 1) \quad (2)$$

From (1): $k = \frac{x+3}{-y}$

Substitute into (2):

$$y = \frac{x+3}{-y}(x+1)$$

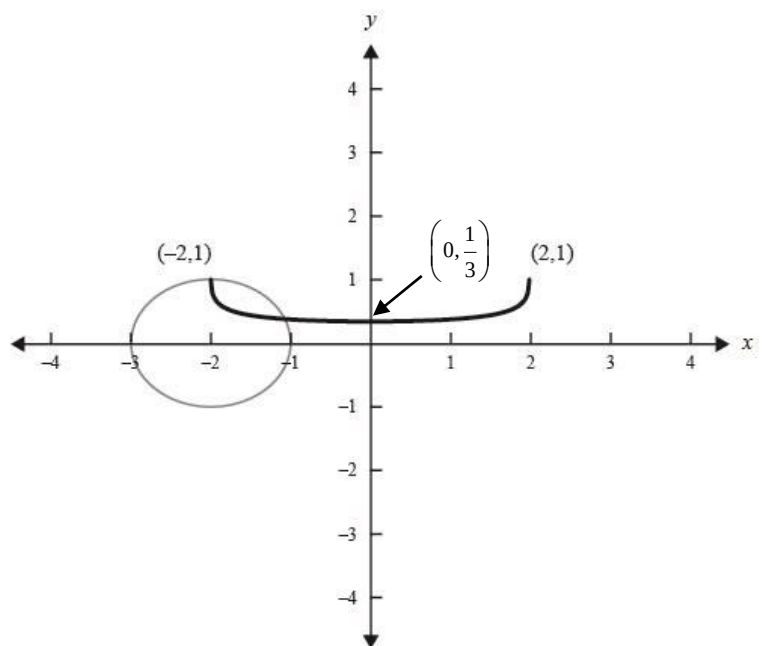
$$-y^2 = (x+3)(x+1)$$

$$-y^2 = x^2 + 4x + 3$$

$$1 - y^2 = x^2 + 4x + 4$$

$$1 - y^2 = (x+2)^2$$

$$(x+2)^2 + y^2 = 1$$

**Mark allocation: 4 marks**

- 1 mark for obtaining $(x + 3) + yi = -ky + ik(x + 1)$
- 1 mark for equating real and imaginary coefficients $x + 3 = -ky$ and $y = k(x + 1)$
- 1 mark for eliminating k to obtain $(x+2)^2 + y^2 = 1$
- 1 mark for sketching graph of circle with centre $(-2, 0)$ and radius of 1

Question 3d.**Worked solution**

Solving $y = \frac{1}{1+\sqrt{4-x^2}}$ and $(x+2)^2 + y^2 = 1$ using CAS gives $x = -2$, $x = -1.07174$

$$\text{Area} = \int_{-2}^{-1.07174} \left(\sqrt{1-(x+2)^2} - \frac{1}{1+\sqrt{4-x^2}} \right) dx$$

$$\text{Area} = 0.32$$

Mark allocation: 3 marks

- 1 mark for solving equations simultaneously to obtain terminals $x = -2$, $x = -1.07174$
- 1 mark for Area = $\int_{-2}^{-1.07174} \left(\sqrt{1-(x+2)^2} - \frac{1}{1+\sqrt{4-x^2}} \right) dx$
- 1 mark for Area = 0.32

Question 4a.**Worked solution**

$$\underline{v}_A = 4\hat{i} + \hat{j}$$

The speed of A is:

$$|\underline{v}_A| = \sqrt{4^2 + 1^2}$$

$$|\underline{v}_A| = \sqrt{17}$$

$$\underline{v}_B = -8\alpha \cos(\alpha t)\hat{i} - 8\alpha \sin(\alpha t)\hat{j}$$

The speed of B is:

$$|\underline{v}_B| = \sqrt{64\alpha^2 \cos^2(\alpha t) + 64\alpha^2 \sin^2(\alpha t)}$$

$$|\underline{v}_B| = \sqrt{64\alpha^2 (\cos^2(\alpha t) + \sin^2(\alpha t))}$$

$$|\underline{v}_B| = \sqrt{64\alpha^2}$$

$$|\underline{v}_B| = 8\alpha \text{ since } \alpha > 0.$$

Mark allocation: 3 marks

- 1 mark for differentiating $\underline{r}_A$ and $\underline{r}_B$ correctly
- 1 mark for $|\underline{v}_A| = \sqrt{17}$
- 1 mark for $|\underline{v}_B| = 8\alpha$

Question 4b.**Worked solution**

Ant (A)

$$x = 4t, y = t, t \geq 0$$

$$y = \frac{x}{4}, x \geq 0$$

Beetle (B)

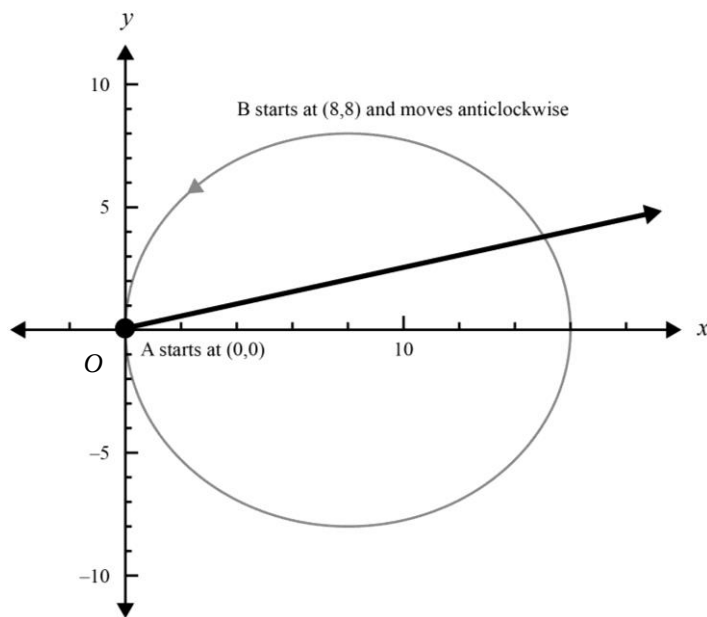
$$x = 8 - 8\sin(\alpha t), y = 8\cos(\alpha t), t \geq 0$$

$$\sin(\alpha t) = \frac{8-x}{8}, \cos(\alpha t) = \frac{y}{8}$$

$$\sin^2(\alpha t) + \cos^2(\alpha t) = 1$$

$$\frac{(8-x)^2}{64} + \frac{y^2}{64} = 1$$

$$(x-8)^2 + y^2 = 64, 0 \leq x \leq 16$$

**Mark allocation: 5 marks**

- 1 mark for the Cartesian equation of the ant's path: $y = \frac{x}{4}, x \geq 0$
- 1 mark for the Cartesian equation of the beetle's path: $(x-8)^2 + y^2 = 64, 0 \leq x \leq 16$
- 1 mark for the graph of the ant's path: $y = \frac{x}{4}, x \geq 0$
- 1 mark for the graph of the beetle's path: $(x-8)^2 + y^2 = 64, 0 \leq x \leq 16$
- 1 mark for indicating the starting position and direction of motion for both insects

Question 4c.**Worked solution**

$$y = \frac{x}{4}, x \geq 0 \text{ intersecting } (x-8)^2 + y^2 = 64, 0 \leq x \leq 16$$

$$(x-8)^2 + \left(\frac{x}{4}\right)^2 = 64$$

$$x^2 - 16x + 64 + \frac{x^2}{16} = 64$$

$$16x^2 - 256x + x^2 = 0$$

$$17x^2 - 256x = 0$$

$$x(17x - 256) = 0$$

$$x = 0, x = \frac{256}{17}$$

When $x = 0$, $y = 0$.

$$\text{When } x = \frac{256}{17}, y = \frac{64}{17}.$$

The points of intersection are $(0, 0)$ and $(\frac{256}{17}, \frac{64}{17})$.

Mark allocation: 2 marks

- 1 mark for solving equations simultaneously to obtain $x = 0, x = \frac{256}{17}$
- 1 mark for finding coordinates of intersecting points $(0, 0)$ and $(\frac{256}{17}, \frac{64}{17})$

Question 4d.**Worked solution**

The insects cannot collide at (0, 0) because the ant starts at (0, 0) and the beetle starts at (8, 8).

If the insects collide at $\left(\frac{256}{17}, \frac{64}{17}\right)$, they must be there at the same time.

The ant (A) is at $\left(\frac{256}{17}, \frac{64}{17}\right)$ when $t = \frac{64}{17}$.

For the beetle (B):

$$\frac{256}{17} = 8 - 8\sin(\alpha t), \quad \frac{64}{17} = 8\cos(\alpha t), \quad t \geq 0$$

$$\sin(\alpha t) = \frac{\left(8 - \frac{256}{17}\right)}{8}, \quad \cos(\alpha t) = \frac{\left(\frac{64}{17}\right)}{8}$$

When $t = \frac{64}{17}$:

$$\sin\left(\frac{64\alpha}{17}\right) = \frac{\left(8 - \frac{256}{17}\right)}{8}$$

$$\sin\left(\frac{64\alpha}{17}\right) = \frac{-15}{17} \quad (1)$$

$$\cos\left(\frac{64\alpha}{17}\right) = \frac{\left(\frac{64}{17}\right)}{8}$$

$$\cos\left(\frac{64\alpha}{17}\right) = \frac{8}{17} \quad (2)$$

Using CAS to solve equation (1) for positive values of α :

$\alpha = 1.12, 1.38, 2.79, 3.05 \dots$

Using CAS to solve equation (2) for positive values of α :

$\alpha = 0.29, 1.38, 1.96, 3.05 \dots$

The smallest positive value of α that satisfies both equations (1) and (2) is 1.38.

Mark allocation: 4 marks

- 1 mark for finding that when A is at $\left(\frac{256}{17}, \frac{64}{17}\right)$, $t = \frac{64}{17}$
- 1 mark for substituting into $\sin(\alpha t) = \frac{8-x}{8}$, $\cos(\alpha t) = \frac{y}{8}$ to obtain

$$\sin\left(\frac{64\alpha}{17}\right) = \frac{-15}{17} \quad (1)$$

$$\cos\left(\frac{64\alpha}{17}\right) = \frac{8}{17} \quad (2)$$

- 1 mark for solving equations (1) and (2) to obtain positive solutions for α :
- 1 mark for finding that the smallest positive value of α is 1.38

Question 4e.**Worked solution**

The insects collide when $t = \frac{64}{17}$.

Distance travelled = speed of insect $\times$ time taken

For the ant (A)

$$\text{Distance} = \sqrt{17} \times \frac{64}{17} = 15.5$$

For the beetle (B):

$$\text{Distance} = 8\alpha \times \frac{64}{17}$$

$$\text{Distance} = 41.6$$

Mark allocation: 2 marks

- 1 mark for 15.5
- 1 mark for 41.6

Question 5a.**Worked solution**

For the M kg object:

$$T - Mg = Ma \quad (1)$$

For the $2M$ kg object:

$$2Mg - T = 2Ma \quad (2)$$

Adding equations (1) and (2) gives:

$$Mg = 3Ma$$

$$a = \frac{g}{3}$$

Mark allocation: 2 marks

- 1 mark for $T - Mg = Ma$ and $2Mg - T = 2Ma$
- 1 mark for showing that $a = \frac{g}{3} \text{ ms}^{-2}$

Question 5b.**Worked solution**

Consider the motion of the smaller object before the string breaks (upwards is positive)

$$u = 0, t = 3, a = \frac{g}{3}, s = ?$$

$$s = ut + \frac{1}{2}at^2$$

$$s = 0 + \left(\frac{1}{2} \times \frac{g}{3} \times 3^2 \right)$$

$$s = 14.7$$

When the string breaks, the velocity of the smaller object is

$$v = u + at$$

$$v = 0 + \left(\frac{g}{3} \times 3 \right)$$

$$v = 9.8$$

Consider the motion of the smaller object when the string breaks to now determine how long it takes to return to its original position.

$$u = 9.8, t = ?, s = -14.7, a = -g(-9.8)$$

$$s = ut + \frac{1}{2}at^2$$

$$-14.7 = 9.8t - \left(\frac{1}{2} \times 9.8 \times t^2 \right)$$

$$-14.7 = 9.8t - 4.9t^2$$

$$-147 = 98t - 49t^2$$

$$-3 = 2t - t^2$$

$$t^2 - 2t - 3 = 0$$

$$(t - 3)(t + 1) = 0$$

$$t = 3 \text{ or } t = -1$$

Because $t \geq 0$, it takes 3 seconds for the smaller object to return to its original position.

Mark allocation: 3 marks

- 1 mark for finding the distance travelled by the smaller object before the string breaks
- 1 mark for finding the velocity of the smaller object when the string breaks
- 1 mark for determining the time taken for the smaller object to return to its original position

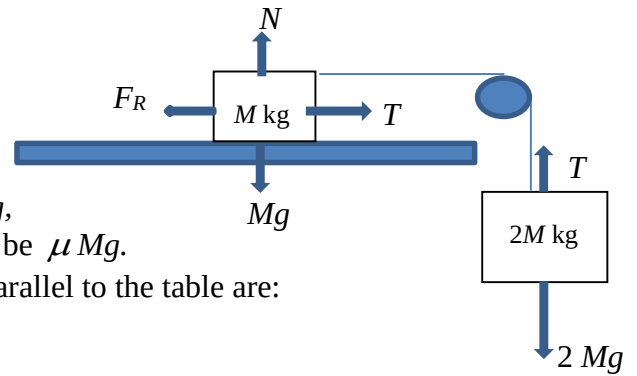
Question 5c.**Worked solution**

Consider the motion of the smaller object.

Resolving the forces perpendicular to the table $N = Mg$,
the frictional force F_R on the smaller mass would then be μMg .

If the tension in the string is T , then resolving forces parallel to the table are:

$$T - \mu Mg = Ma \quad (1)$$



Consider the motion of the larger object.

$$2Mg - T = 2Ma \quad (2)$$

Adding equations (1) and (2) gives:

$$2Mg - \mu Mg = 3Ma$$

$$Mg(2 - \mu) = 3Ma$$

$$a = \frac{g(2 - \mu)}{3}$$

Substituting into equation (1) gives:

$$T - \mu Mg = \frac{Mg(2 - \mu)}{3}$$

$$3T - 3\mu Mg = Mg(2 - \mu)$$

$$3T = 2Mg - \mu Mg + 3\mu Mg$$

$$3T = 2Mg(1 + \mu)$$

$$T = \frac{2Mg(1 + \mu)}{3}$$

Mark allocation: 4 marks

- 1 mark for resolving all forces on the smaller object to obtain $T - \mu Mg = Ma$
- 1 mark for resolving forces on the larger object to obtain $2Mg - T = 2Ma$
- 1 mark for solving equations simultaneously to obtain $a = \frac{g(2 - \mu)}{3}$
- 1 mark for finding $T = \frac{2Mg(1 + \mu)}{3}$

Question 5d.**Worked solution**

The system will not move if $a \leq 0$.

$$\frac{g(2-\mu)}{3} \leq 0$$

$$2-\mu \leq 0$$

$$\mu \geq 2$$

$$\therefore k = 2$$

Mark allocation: 2 marks

- 1 method mark for determining $a \leq 0$
- 1 mark for showing $k = 2$

END OF SOLUTIONS BOOK