

YEAR 12 Trial Exam Paper

2015

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

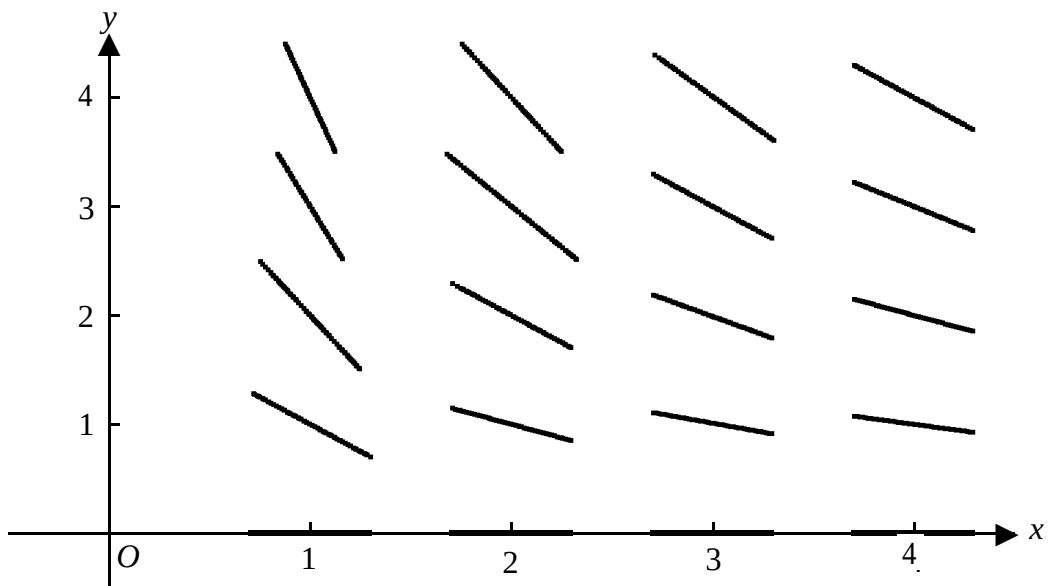
This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

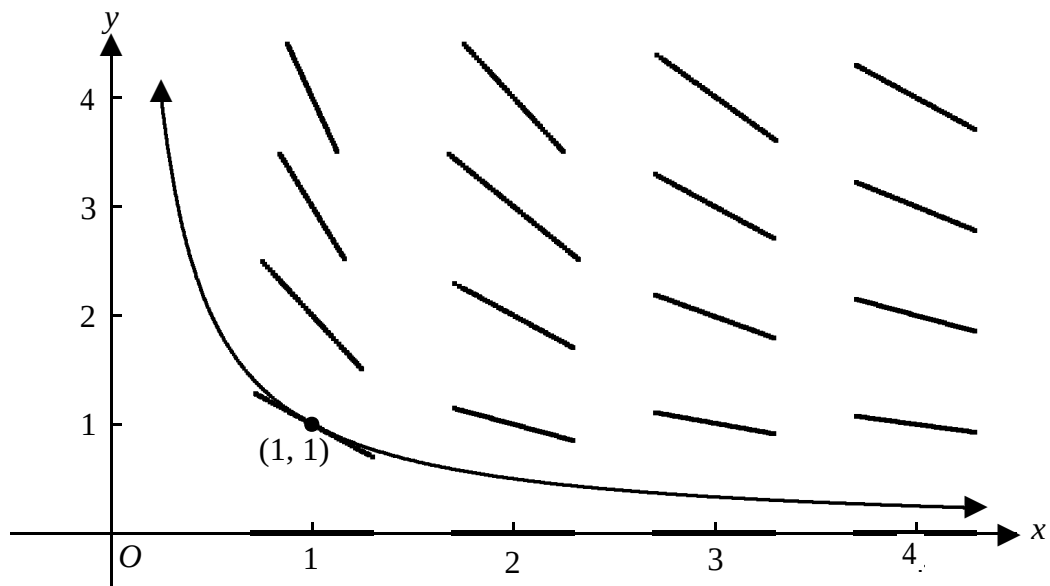
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Question 1a.**Worked solution****Mark allocation: 1 mark**

- 1 mark for the four correct tangents

Question 1b.**Worked solution****Mark allocation: 1 mark**

- 1 mark for the correct curve passing through the point (1, 1)

Question 2a.**Worked solution**

$$z^4 + 3z^2 - 4 = 0$$

$$\Rightarrow (z^2 + 4)(z^2 - 1) = 0$$

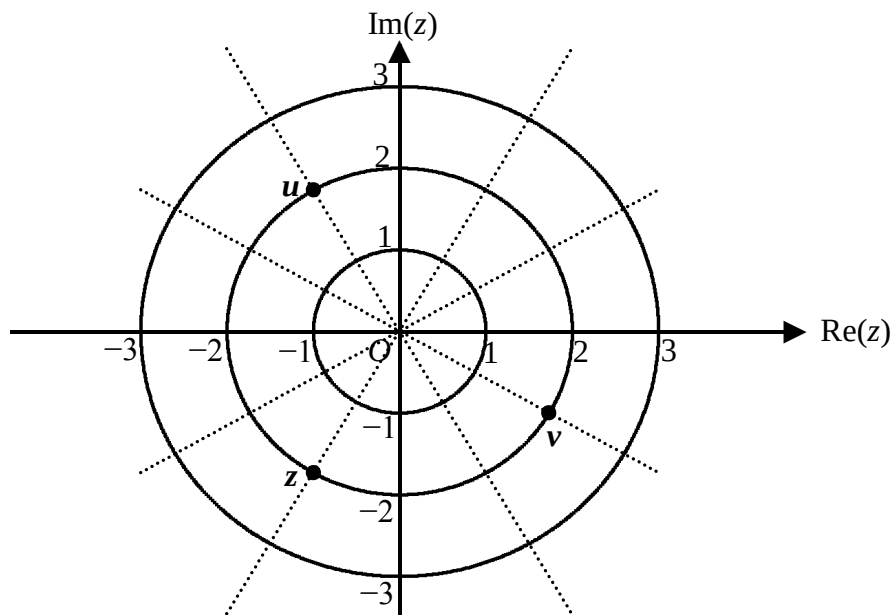
$$\Rightarrow (z^2 - 4i^2)(z^2 - 1) = 0$$

$$\Rightarrow (z - 2i)(z + 2i)(z - 1)(z + 1) = 0$$

$$\therefore z = \pm 2i, \pm 1$$

Mark allocation: 3 marks

- 1 mark for factorising as two quadratic expressions
- 1 mark for factorising as four linear expressions
- 1 mark for the correct solutions

Question 2b.**Worked solution****Mark allocation: 1 mark**

- 1 mark for the correct answer

**Tip**

- The point z is obtained by reflecting $u = \bar{z}$ through the real axis. Then, the point $v = iz$ is obtained by rotating z 90° anti-clockwise.

Question 2c.**Worked solution**

$$u = 2cis \frac{2\pi}{3}$$

$$\sqrt{u} = \left(2cis \frac{2\pi}{3} \right)^{\frac{1}{2}}$$

$$= \sqrt{2}cis \left(\frac{\pi}{3} \right) = \sqrt{2} \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$$

$$= \sqrt{2} \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right)$$

$$= \frac{\sqrt{2}}{2} + i \frac{\sqrt{6}}{2}$$

Mark allocation: 2 marks

- 1 mark for obtaining $\sqrt{u}$ correctly in polar form
- 1 mark for the correct answer

Question 3**Worked solution**

$$x^2y^2 - 4y - \log_e(x-1)^3 = 8$$

$$\frac{d}{dx}(x^2y^2) - \frac{d}{dx}(4y) - \frac{d}{dx}(\log_e(x-1)^3) = \frac{d}{dx}(8)$$

$$\text{or } \frac{d}{dx}(x^2y^2) - \frac{d}{dx}(4y) - 3 \cdot \frac{d}{dx}(\log_e(x-1)) = \frac{d}{dx}(8)$$

$$\Rightarrow 2xy^2 + x^2 2y \frac{dy}{dx} - 4 \frac{dy}{dx} - \frac{3}{x-1} = 0$$

$$\frac{dy}{dx}(2x^2y - 4) = \frac{3}{x-1} - 2xy^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{3}{x-1} - 2xy^2}{2x^2y - 4}$$

Substitute (2, 2)

$$\frac{dy}{dx} = \frac{3-16}{12} = -\frac{13}{12}, \text{ which is the gradient of the tangent at } x = 2.$$

$\therefore$ the gradient of the normal is $\frac{12}{13}$.

Mark allocation: 3 marks

- 1 mark for implicitly differentiating the relation correctly
- 1 mark for the correct gradient function
- 1 mark for the correct answer

**Tip**

- The relation cannot be explicitly expressed as a function of x , so implicit differentiation is required to obtain $\frac{dy}{dx}$.

Question 4**Worked solution**

$$\tan(x) = \sin(2x)$$

$$\Rightarrow \frac{\sin(x)}{\cos(x)} = \sin(2x)$$

$$\Rightarrow \frac{\sin(x)}{\cos(x)} = 2 \sin(x) \cos(x)$$

$$\Rightarrow \sin(x) = 2 \sin(x) \cos^2(x)$$

$$\Rightarrow 2 \sin(x) \cos^2(x) - \sin(x) = 0$$

$$\sin(x)(2\cos^2(x) - 1) = 0$$

$$\Rightarrow \sin(x) = 0 \text{ or } \cos^2(x) = \frac{1}{2}$$

$$\Rightarrow \sin(x) = 0 \text{ or } \cos(x) = \pm \frac{1}{\sqrt{2}}$$

$$\therefore x = n\pi \text{ or } x = \frac{(2n+1)\pi}{4}, n \in \mathbb{Z}$$

Mark allocation: 3 marks

- 1 mark for obtaining a correct equation in terms of $\sin(x)$ and $\cos(x)$
- 1 mark for obtaining correct equations, one in terms of $\sin(x)$ only and the other in terms of $\cos(x)$ only
- 1 mark for the correct solutions

Question 5**Worked solution**

$$f(x) = \arcsin\left(\frac{x}{2}\right)$$

$$f'(x) = \frac{1}{\sqrt{4-x^2}}$$

$$= (4-x^2)^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{2}(-2x)(4-x^2)^{-\frac{3}{2}}$$

$$= \frac{x}{(4-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow f''(\sqrt{3}) = \frac{\sqrt{3}}{\frac{3}{1^2}} = \sqrt{3}$$

Mark allocation: 3 marks

- 1 mark for the correct first derivative, $f'(x)$
- 1 mark for the correct second derivative, $f''(x)$
- 1 mark for the correct answer

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Question 6a.**Worked solution**

$$\begin{aligned}
 \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\
 &= 2\hat{i} - \hat{j} + \hat{k} - (\hat{i} - \hat{k}) \\
 &= \hat{i} - \hat{j} + 2\hat{k}
 \end{aligned}$$

Mark allocation

- 1 mark for the correct answer

Question 6b.**Worked solution**

$$\begin{aligned}
 \overrightarrow{AB} &= \hat{i} - \hat{j} + 2\hat{k} \\
 \overrightarrow{OB} &= 2\hat{i} - \hat{j} + \hat{k} \\
 \overrightarrow{AB} \cdot \overrightarrow{OB} &= 2 + 1 + 2 = 5 \\
 |\overrightarrow{AB}| &= \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6} \\
 |\overrightarrow{OB}| &= \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6} \\
 \cos \theta &= \frac{\overrightarrow{AB} \cdot \overrightarrow{OB}}{|\overrightarrow{AB}| \cdot |\overrightarrow{OB}|} = \frac{5}{\sqrt{6} \cdot \sqrt{6}} = \frac{5}{6}
 \end{aligned}$$

Mark allocation: 2 marks

- 1 mark for the correct values of $\overrightarrow{AB} \cdot \overrightarrow{OB}$, $|\overrightarrow{AB}|$ and $|\overrightarrow{OB}|$
- 1 mark for correct evaluation of $\cos \theta$

Question 6c.**Worked solution**

$$\text{Area} = \frac{1}{2} \times |\overline{AB}| \times |\overline{OB}| \times \sin \theta$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta}$$

$$= \sqrt{1 - \left(\frac{5}{6}\right)^2} = \sqrt{1 - \frac{25}{36}} = \sqrt{\frac{11}{36}} = \frac{\sqrt{11}}{6}$$

$$\begin{aligned} \therefore \text{Area} &= \frac{1}{2} \times \sqrt{6} \times \sqrt{6} \times \frac{\sqrt{11}}{6} \\ &= \frac{\sqrt{11}}{2} \text{ square units} \end{aligned}$$

Mark allocation: 2 marks

- 1 mark for the correct value of $\sin \theta$
- 1 mark for the correct answer

Question 7a.**Worked solution**

$$\int_{e^{-\frac{\pi}{2}}}^{e^{\frac{\pi}{2}}} \frac{1}{x \sec(\log_e x)} dx$$

$$= \int_{e^{-\frac{\pi}{2}}}^{e^{\frac{\pi}{2}}} \frac{\cos(\log_e x)}{x} dx$$

Let $u = \log_e x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$x = e^{\frac{\pi}{2}} \Rightarrow u = \log_e e^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$x = e^{-\frac{\pi}{2}} \Rightarrow u = \log_e e^{-\frac{\pi}{2}} = -\frac{\pi}{2}$$

$$\begin{aligned} \therefore \int_{e^{-\frac{\pi}{2}}}^{e^{\frac{\pi}{2}}} \frac{\cos(\log_e x)}{x} dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(u) \frac{du}{dx} dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(u) du \\ &= [\sin u]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) \\ &= 1 - (-1) = 2 \end{aligned}$$

Mark allocation: 3 marks

- 1 mark for the correct substitution
- 1 mark for the correct integrand in terms of u
- 1 mark for the correct answer

**Tip**

- To anti-differentiate, use the substitution $u = \log_e x$ because its derivative $\frac{1}{x}$ is a factor of the integrand.

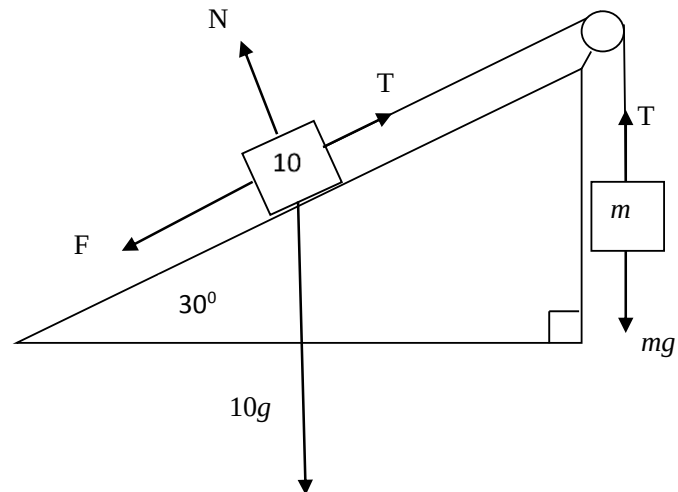
Question 7b.**Worked solution**

$$y_1 = y_0 + h \times f(x_0, y_0)$$

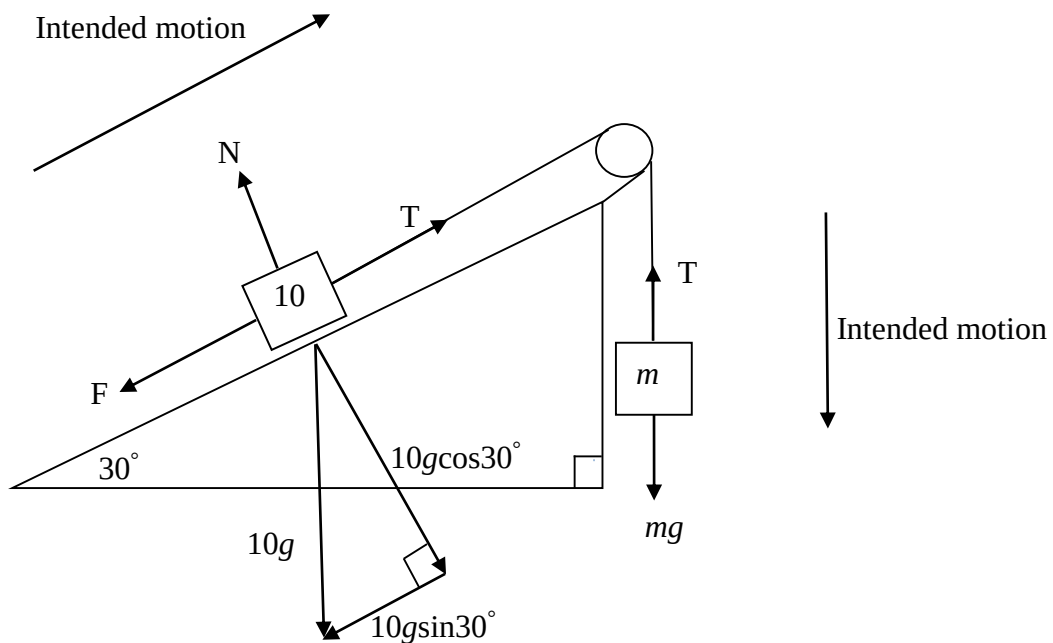
$$y_1 = 1 + 0.2 \times \frac{-2}{2^2} = 1 - 0.1 = 0.9$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 8a.**Worked solution****Mark allocation: 1 mark**

- 1 mark for showing the four forces acting on the 10 kg mass and the two forces acting on mass m

Question 8b.**Worked solution**

Forces acting on mass m :

$$R = mg - T = 0$$

$$\Rightarrow T = mg$$

Forces acting on mass of 10 kg:

$$\underline{R} = (T - F - 10g \sin 30^\circ) \underline{i} + (N - 10g \cos 30^\circ) \underline{j} = \underline{0}$$

$$\Rightarrow (T - F - 5g) \underline{i} + (N - 5\sqrt{3}g) \underline{j} = \underline{0}$$

$$N - 5\sqrt{3}g = 0$$

$$\Rightarrow N = 5\sqrt{3}g$$

$$\therefore \text{Friction, } F = N\mu = 5\sqrt{3}g \times \frac{1}{\sqrt{3}} = 5g$$

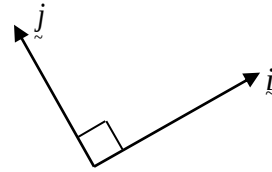
$$T - F - 5g = 0$$

$$\Rightarrow mg - 5g - 5g = mg - 10g = 0$$

$$\Rightarrow (m - 10)g = 0$$

$$\Rightarrow m = 10$$

$\therefore$ 10 kg is the maximum value of m for which the 10 kg mass will not move up the incline.



Mark allocation: 3 marks

- 1 mark for correctly resolving the forces acting on the mass m and the 10 kg mass
- 1 mark for correctly calculating the friction force
- 1 mark for the correct answer



Tip

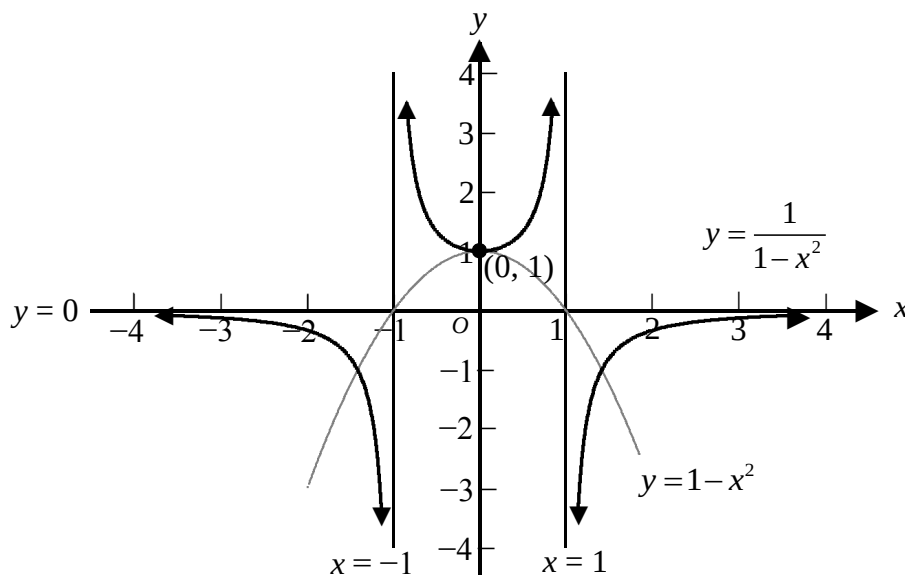
- The maximum value of m occurs when the 10 kg mass is on the verge of moving up the plane and the friction is acting down the plane.

Question 9a.**Worked solution**

$$\begin{aligned}\frac{1}{2(1-x)} + \frac{1}{2(1+x)} &= \frac{2(1+x) + 2(1-x)}{4(1-x^2)} \\ &= \frac{4}{4(1-x^2)} = \frac{1}{1-x^2}\end{aligned}$$

Mark allocation: 1 mark

- 1 mark for correctly expressing the two rational expressions with a common denominator

Question 9b.**Worked solution****Mark allocation: 2 marks**

- 1 mark for the correct asymptotes $y = 0$, $x = -1$ and $x = 1$ and the y -intercept $(0, 1)$
- 1 mark for the correct curve $y = \frac{1}{1-x^2}$

**Tip**

- Sketch the graph of $y = 1 - x^2$, and then obtain the graph of its reciprocal $y = \frac{1}{1-x^2}$.

Question 9c.**Worked solution**

$$\text{Volume} = \pi \int_{-3}^{-1} x^2 dy$$

$$y = \frac{1}{1-x^2}$$

$$\Rightarrow 1-x^2 = \frac{1}{y}$$

$$\therefore x^2 = 1 - \frac{1}{y}$$

$$\text{Volume} = \pi \int_{-3}^{-1} \left(1 - \frac{1}{y}\right) dy$$

$$= \pi \left[y - \log_e |y| \right]_{-3}^{-1}$$

$$= \pi \left[-1 - \log_e 1 - (-3 - \log_e 3) \right]$$

$$= \pi(2 + \log_e 3) \text{ cubic units}$$

Mark allocation: 3 marks

- 1 mark for the correct integrand representing the volume required
- 1 mark for correctly anti-differentiating
- 1 mark for the correct answer

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Question 10a.**Worked solution**

$$\underline{r}(t) = e^t \underline{i} + \log_e(t+1) \underline{j}$$

$$\Rightarrow \underline{v}(t) = e^t \underline{i} + \frac{1}{t+1} \underline{j}$$

$$\Rightarrow \underline{v}(0) = \underline{i} + \underline{j}$$

$$\therefore v(0) = \sqrt{2}$$

Mark allocation: 2 marks

- 1 mark for the correct velocity vector
- 1 mark for the correct answer

Question 10b.**Worked solution**

$$\underline{v}(t) = e^t \underline{i} + \frac{1}{t+1} \underline{j}$$

$$\Rightarrow \underline{a}(t) = e^t \underline{i} - \frac{1}{(t+1)^2} \underline{j}$$

$$\Rightarrow \underline{a}(0) = \underline{i} - \underline{j}$$

$$\Rightarrow \underline{v}(0) \cdot \underline{a}(0) = (\underline{i} + \underline{j}) \cdot (\underline{i} - \underline{j}) = 1 - 1 = 0$$

$\therefore$ the initial acceleration of the particle is perpendicular to its initial velocity.

Mark allocation: 2 marks

- 1 mark for the correct acceleration vector
- 1 mark for showing that the dot product of the initial acceleration vector and the initial velocity vector equals zero

END OF SOLUTIONS BOOK