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**SPECIALIST MATHS
TRIAL EXAMINATION 1
SOLUTIONS
2015**

Question 1 (3 marks)

$$\begin{aligned} & \int_0^{\sqrt{6}} \frac{x-5}{x^2+2} dx \\ &= \int_0^{\sqrt{6}} \frac{x}{x^2+2} dx - \int_0^{\sqrt{6}} \frac{5}{x^2+2} dx \\ &= \left[\frac{1}{2} \log_e(x^2+2) \right]_0^{\sqrt{6}} - \left[\frac{5}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) \right]_0^{\sqrt{6}} \end{aligned}$$

(1 mark)

(1 mark)

$$\begin{aligned} &= \frac{1}{2}(\log_e(8) - \log_e(2)) - \frac{5}{\sqrt{2}}(\tan^{-1}(\sqrt{3}) - \tan^{-1}(0)) \\ &= \frac{1}{2}\log_e(4) - \frac{5}{\sqrt{2}}\left(\frac{\pi}{3} - 0\right) \\ &= \log_e(2) - \frac{5\sqrt{2}\pi}{6} \end{aligned}$$

(1 mark)

Question 2 (5 marks)

a. i. $\vec{CM} = \alpha \vec{CA}$ (given in question)

$$\begin{aligned} &= \alpha \left(-\frac{1}{2}\vec{b} + \vec{a} \right) \\ &= \alpha \vec{a} - \frac{1}{2}\alpha \vec{b} \end{aligned}$$

(1 mark)

ii. $\vec{CM} = \vec{CO} + \vec{OM}$

$$\begin{aligned} &= -\frac{1}{2}\vec{b} + \beta \vec{OD} \\ &= -\frac{1}{2}\vec{b} + \beta \left(\vec{OA} + \vec{AD} \right) \\ &= -\frac{1}{2}\vec{b} + \beta \left(\vec{OA} + \frac{1}{3}\vec{AB} \right) \\ &= -\frac{1}{2}\vec{b} + \beta \left(\vec{a} + \frac{1}{3}(-\vec{a} + \vec{b}) \right) \\ &= \frac{2}{3}\beta \vec{a} + \left(\frac{1}{3}\beta - \frac{1}{2} \right) \vec{b} \end{aligned}$$

(1 mark)

b. From part a.,

$$\alpha \underline{a} - \frac{1}{2} \alpha \underline{b} = \frac{2}{3} \beta \underline{a} + \left(\frac{1}{3} \beta - \frac{1}{2} \right) \underline{b}$$

$$\text{So } \alpha = \frac{2}{3} \beta \quad (1) \quad \text{and} \quad -\frac{1}{2} \alpha = \frac{1}{3} \beta - \frac{1}{2} \quad (2)$$

Substitute (1) into (2)

$$-\frac{1}{2} \times \frac{2}{3} \beta = \frac{1}{3} \beta - \frac{1}{2}$$

$$-\frac{1}{3} \beta = \frac{1}{3} \beta - \frac{1}{2}$$

$$\frac{2}{3} \beta = \frac{1}{2}$$

$$\beta = \frac{3}{4}$$

(1 mark)

$$\text{So } \alpha = \frac{2}{3} \times \frac{3}{4}$$

$$= \frac{1}{2}$$

(1 mark)

Question 3 (3 marks)

The vector resolute of $\underline{a}$ in the direction of $\underline{b}$

$$= (\underline{a} \cdot \hat{\underline{b}}) \hat{\underline{b}}$$

$$= \left(\left(\underline{i} - 4\underline{j} + 2\underline{k} \right) \cdot \frac{1}{3} \left(2\underline{i} + \underline{j} - 2\underline{k} \right) \right) \frac{1}{3} \left(2\underline{i} + \underline{j} - 2\underline{k} \right)$$

(1 mark)

$$= \frac{1}{9} \times (2 - 4 - 4) \left(2\underline{i} + \underline{j} - 2\underline{k} \right)$$

$$= -\frac{2}{3} \left(2\underline{i} + \underline{j} - 2\underline{k} \right)$$

(1 mark)

The vector resolute of $\underline{a}$ perpendicular to $\underline{b}$

$$= \underline{a} - (\underline{a} \cdot \hat{\underline{b}}) \hat{\underline{b}}$$

$$= \underline{i} - 4\underline{j} + 2\underline{k} - \left(-\frac{2}{3} \right) \left(2\underline{i} + \underline{j} - 2\underline{k} \right)$$

$$= \frac{7}{3} \underline{i} - \frac{10}{3} \underline{j} + \frac{2}{3} \underline{k}$$

(1 mark)

If you have time, add the two resolutes together to make sure they give $\underline{a}$.

$$-\frac{4}{3} \underline{i} - \frac{2}{3} \underline{j} + \frac{4}{3} \underline{k} + \frac{7}{3} \underline{i} - \frac{10}{3} \underline{j} + \frac{2}{3} \underline{k}$$

$$= \underline{i} - 4\underline{j} + 2\underline{k}$$

$$= \underline{a}$$

Question 4 (4 marks)

Since $z = 2i$ is a solution then $z = -2i$ is also a solution because the coefficients of the terms in the equation are all real (conjugate root theorem).

So $(z - 2i)(z + 2i)$

$$= z^2 + 4 \text{ is a factor}$$

(1 mark)

Method 1 – long division

$$\begin{array}{r} z^2 - 2z + 4 \\ z^2 + 4 \overline{) z^4 - 2z^3 + 8z^2 - 8z + 16} \\ \underline{z^4 + 4z^2} \\ -2z^3 + 4z^2 - 8z \\ \underline{-2z^3 - 8z} \\ 4z^2 + 16 \\ \underline{4z^2 + 16} \\ 0 \end{array}$$

Method 2 – inspection

$$\begin{aligned} & z^4 - 2z^3 + 8z^2 - 8z + 16 \\ &= (z^2 + 4)(z^2 - 2z + 4) \\ &= (z^2 + 4)(1z^2 - 2z + 4) \\ &= (z^2 + 4)(z^2 - 2z + 4) \\ &= (z^2 + 4)(z^2 - 2z + 4) \end{aligned}$$

$$\text{So } f(z) = (z^2 + 4)(z^2 - 2z + 4)$$

(1 mark)

$$= (z^2 + 4)((z^2 - 2z + 1) - 1 + 4)$$

$$= (z^2 + 4)((z - 1)^2 + 3)$$

$$= (z^2 + 4)((z - 1)^2 - 3i^2)$$

$$= (z^2 + 4)(z - 1 - \sqrt{3}i)(z - 1 + \sqrt{3}i)$$

(1 mark)

The other solutions to $f(z) = 0$ are $z = -2i, 1 \pm \sqrt{3}i$.

(1 mark)

Question 5 (5 marks)

a. $u = 10$ $v = u + at$ (1 mark)

$v = 0$ $0 = 10 + 4a$

$t = 4$ $a = -2.5$

$a = ?$

The acceleration is -2.5ms^{-2} . (1 mark)

b.

Method 1

$$\begin{aligned} u &= 10 & s &= ut + \frac{1}{2}at^2 \\ v &= 0 & &= 10 \times 4 + \frac{1}{2} \times -2.5 \times 16 \\ t &= 4 & &= 40 - 20 \\ a &= -2.5 & &= 20 \\ s &= ? \end{aligned}$$

The box travels 20m.

Method 2

$$\begin{aligned} v^2 &= u^2 + 2as \\ 0 &= 100 + 2 \times -2.5s \\ s &= \frac{100}{5} \\ &= 20 \end{aligned}$$

(1 mark)

c. Draw the forces on the diagram.

$$\underline{R} = m \underline{a}$$

$$-F \underline{i} + (N - 2g) \underline{j} = 2a \underline{i} \quad (1 \text{ mark})$$

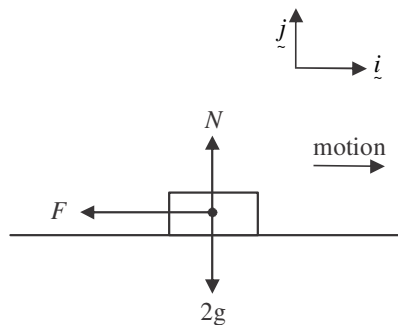
$$-F = 2a \quad \text{and} \quad N = 2g$$

$$-\mu N = 2 \times -2.5$$

$$-\mu \times 2g = -5$$

$$\mu = \frac{5}{2g}$$

So $r = 5$ and $s = 2$.



(1 mark)

Question 6 (4 marks)

$$2xy - \arctan\left(\frac{x}{2}\right) + y^2 = 5 - \frac{\pi}{4}$$

$$2y + 2x \frac{dy}{dx} - \frac{2}{4+x^2} + 2y \frac{dy}{dx} = 0$$

(1 mark) - $\frac{2}{4+x^2}$ term

$$(2x + 2y) \frac{dy}{dx} = -2y + \frac{2}{4+x^2}$$

(1 mark) - other terms

$$= \frac{-2y(4+x^2) + 2}{4+x^2}$$

$$\frac{dy}{dx} = \frac{-8y - 2yx^2 + 2}{(4+x^2) \times 2(x+y)}$$

$$= \frac{-4y - yx^2 + 1}{(4+x^2)(x+y)}$$

(1 mark)

$$\text{At } (2,1) \quad \frac{dy}{dx} = \frac{-4 - 4 + 1}{8 \times 3}$$

$$= \frac{-7}{24}$$

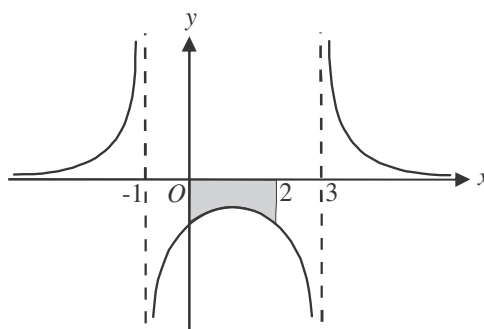
(1 mark)

Question 7 (4 marks)

Do a quick sketch.

$$y = \frac{1}{x^2 - 2x - 3}$$

$$= \frac{1}{(x-3)(x+1)}$$



$$\text{area} = -\int_0^2 \frac{1}{(x-3)(x+1)} dx$$

(1 mark)

$$\text{Let } \frac{1}{(x-3)(x+1)} \equiv \frac{A}{(x-3)} + \frac{B}{(x+1)}$$

$$\equiv \frac{A(x+1) + B(x-3)}{(x-3)(x+1)}$$

$$\text{True iff } 1 \equiv A(x+1) + B(x-3)$$

$$\text{Put } x = -1, \text{ true iff } 1 = -4B, \quad B = -\frac{1}{4}$$

$$\text{Put } x = 3, \text{ true iff } 1 = 4A, \quad A = \frac{1}{4}$$

$$\text{So area} = -\int_0^2 \left(\frac{1}{4(x-3)} - \frac{1}{4(x+1)} \right) dx$$

(1 mark)

$$= -\frac{1}{4} [\log_e |x-3| - \log_e |x+1|]_0^2$$

(1 mark)

$$= -\frac{1}{4} \left[\log_e \frac{|x-3|}{|x+1|} \right]_0^2$$

$$= -\frac{1}{4} \left(\log_e \frac{|-1|}{|3|} - \log_e \frac{|-3|}{|1|} \right)$$

$$= -\frac{1}{4} \left(\log_e \left(\frac{1}{3} \right) - \log_e \left(\frac{3}{1} \right) \right)$$

$$= -\frac{1}{4} \log_e \left(\frac{1}{9} \right)$$

$$= -\frac{1}{4} \log_e (3^{-2})$$

$$= \frac{1}{2} \log_e (3) \text{ square units}$$

(1 mark)

Question 8 (5 marks)**a.** Method 1

$$v = \sqrt{2x+4}$$

$$a = v \frac{dv}{dx} \quad (\text{from formula sheet})$$

(1 mark)

$$\begin{aligned} \text{Now } \frac{dv}{dx} &= \frac{1}{2}(2x+4)^{-\frac{1}{2}} \times 2 \\ &= \frac{1}{\sqrt{2x+4}} \end{aligned}$$

$$\text{So } a = \sqrt{2x+4} \times \frac{1}{\sqrt{2x+4}}$$

$$a = 1$$

Hence acceleration is constant.

(1 mark)Method 2

$$v = \sqrt{2x+4}$$

$$a = \frac{d}{dx} \left(\frac{1}{2} v^2 \right) \quad (\text{from formula sheet})$$

(1 mark)

$$= \frac{1}{2} \frac{d}{dx} (2x+4)$$

$$= \frac{1}{2} \times 2$$

$$\text{So } a = 1$$

Hence acceleration is constant.

(1 mark)**b.**

$$v = \sqrt{2x+4}$$

$$\frac{dx}{dt} = \sqrt{2x+4}$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{2x+4}}$$

$$t = \int \frac{1}{\sqrt{2x+4}} dx \quad \textbf{(1 mark)}$$

$$\text{let } u = 2x+4$$

$$= \int u^{-\frac{1}{2}} \times \frac{1}{2} \frac{du}{dx} dx$$

$$\frac{du}{dx} = 2$$

$$= \frac{1}{2} \int u^{-\frac{1}{2}} du$$

$$t = \frac{1}{2} u^{\frac{1}{2}} \times 2 + c$$

$$t = \sqrt{2x+4} + c$$

(1 mark)When $t = 0$, $x = 0$, so $0 = \sqrt{4} + c$ and $c = -2$.

$$\text{So } t = \sqrt{2x+4} - 2$$

$$\text{When } t = 3, \quad 3 = \sqrt{2x+4} - 2$$

$$5 = \sqrt{2x+4}$$

$$25 = 2x+4$$

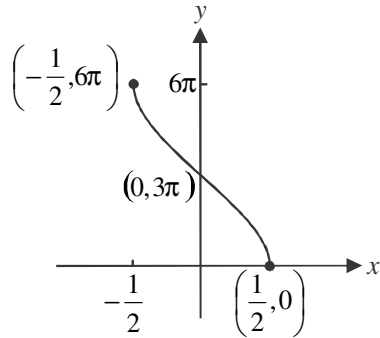
$$x = \frac{21}{2}$$

(1 mark)

Question 9 (7 marks)

a. $d_f = \left[-\frac{1}{2}, \frac{1}{2}\right]$ (1 mark)
 $r_f = [0, 6\pi]$ (1 mark)

b.



(1 mark) – correct endpoints and y-intercept
 (1 mark) – correct shape

c.

$$\text{volume} = \pi \int_0^{3\pi} x^2 dy$$

(1 mark)

$$\text{since } y = 6\arccos(2x)$$

$$\frac{y}{6} = \arccos(2x)$$

$$\cos\left(\frac{y}{6}\right) = 2x$$

$$x = \frac{1}{2}\cos\left(\frac{y}{6}\right)$$

$$x^2 = \frac{1}{4}\cos^2\left(\frac{y}{6}\right)$$

$$\text{Since } \cos(2\theta) = 2\cos^2(\theta) - 1$$

$$2\cos^2(\theta) = \cos(2\theta) + 1$$

$$\cos^2(\theta) = \frac{1}{2}(\cos(2\theta) + 1)$$

$$x^2 = \frac{1}{8}\left(\cos\left(\frac{y}{3}\right) + 1\right)$$

$$\text{volume} = \frac{\pi}{8} \int_0^{3\pi} \left(\cos\left(\frac{y}{3}\right) + 1\right) dy$$

(1 mark)

$$= \frac{\pi}{8} \left[3\sin\left(\frac{y}{3}\right) + y \right]_0^{3\pi}$$

$$= \frac{\pi}{8} \{ (3\sin(\pi) + 3\pi) - (3\sin(0) + 0) \}$$

$$= \frac{3\pi^2}{8} \text{ units}^3$$

1 mark