



Units 3 and 4 Specialist Maths: Exam 1

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Any questions?

Check the Engage website for updated solutions, then email practiceexams@ee.org.au.

Question 1a

If A, B and C are collinear, then $\overrightarrow{OA} + t(\overrightarrow{AB}) = \overrightarrow{OC}$ for some value of t .

$$\begin{aligned}\Rightarrow \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + t\mathbf{i} - 4t\mathbf{j} + 5t\mathbf{k} &= 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} \\ \Rightarrow 1 + t = 3 \text{ and } 3 - 4t = 1 \text{ and } -2 + 5t = 2\end{aligned}$$

However, there is no value of t for which all three of these equations are true, and so A, B and C are not collinear.

Question 1b

Let $\hat{\mathbf{a}}$ be a unit vector in the direction of $\overrightarrow{OA}$, and $\hat{\mathbf{c}}$ a unit vector in the direction of $\overrightarrow{OC}$.

Thus, $\hat{\mathbf{a}} = \frac{1}{\sqrt{14}}(\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})$ and $\hat{\mathbf{c}} = \frac{1}{\sqrt{14}}(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

A vector which bisects the angle between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\mathbf{d} = \hat{\mathbf{a}} + \hat{\mathbf{c}} = \frac{1}{\sqrt{14}}(4\mathbf{i} + 4\mathbf{j})$ (or any other multiple of $\mathbf{i} + \mathbf{j}$). Hence any point D in the form $k(1, 1, 0)$, $k \in \mathbb{R}$ will satisfy the question.

Question 1c

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{\|\overrightarrow{OA}\| \|\overrightarrow{OB}\|}\right) \\ &= \cos^{-1}\left(\frac{-7}{14}\right) \\ &= \frac{2\pi}{3}\end{aligned}$$

Question 1d

$$\begin{aligned}\overrightarrow{OC}_{\text{perpendicular to } \overrightarrow{OB}} &= \overrightarrow{OC} - (\overrightarrow{OC} \cdot \widehat{\overrightarrow{OB}}) \widehat{\overrightarrow{OB}} \\ &= [3 \ 1 \ 2] - \frac{11}{14}[2 \ -1 \ 3] \\ &= \frac{10}{7}\mathbf{i} + \frac{25}{14}\mathbf{j} - \frac{5}{14}\mathbf{k}\end{aligned}$$

Question 2

Find the gradient of the tangent to the graph at the specified point:

$$\begin{aligned}\frac{d}{dx}(3x^2y + 9y^2 - 3xy) &= \frac{d}{dx}(24x) \\ 6xy + 3x^2 \frac{dy}{dx} + 18y \frac{dy}{dx} - 3y - 3x \frac{dy}{dx} &= 24 \\ \frac{dy}{dx}(18y + 3x^2 - 3x) &= 24 - 6xy + 3y \\ \frac{dy}{dx} &= \frac{24 - 6xy + 3y}{18y + 3x^2 - 3x}\end{aligned}$$

$$\begin{aligned}\text{At } (3, 2), \frac{dy}{dx} &= \frac{24 - 6(3)(2) + 3(2)}{18(2) + 3(3^2) - 3(3)} \\ &= \frac{-6}{54} = \frac{-1}{9}\end{aligned}$$

Hence the gradient of the normal = $\frac{-1}{\left(\frac{-1}{9}\right)} = 9$

Find the equation of the normal, given that it is in the form $y = 9x + c$ and passes through (3, 2):

$$\begin{aligned}2 &= 9(3) + c \\ c &= -25 \\ \Rightarrow y &= 9x - 25\end{aligned}$$

Question 3a

$$\begin{aligned}
|i\bar{z} - z| &= \left| \operatorname{cis}\left(\frac{\pi}{2}\right) \operatorname{cis}(-\theta) - \operatorname{cis}(\theta) \right| \\
&= \left| \operatorname{cis}\left(\frac{\pi}{2} - \theta\right) - \operatorname{cis}(\theta) \right| \\
&= \left| \cos\left(\frac{\pi}{2} - \theta\right) + i \sin\left(\frac{\pi}{2} - \theta\right) - \cos(\theta) - i \sin(\theta) \right| \\
&= |\sin(\theta) - \cos(\theta) + (\cos(\theta) - \sin(\theta))i| \\
&= \sqrt{(\sin(\theta) - \cos(\theta))^2 + (\cos(\theta) - \sin(\theta))^2} \\
&= \sqrt{2 - 4 \sin(\theta) \cos(\theta)} \\
&= \sqrt{2 - 2 \sin(2\theta)}
\end{aligned}$$

Question 3b

$$\text{Note: } \theta \in (-\pi, \pi] \Rightarrow 2\theta \in (-2\pi, 2\pi]$$

$\operatorname{Arg}(i\bar{z} - z)$ is undefined when $|i\bar{z} - z| = 0$

$$\Rightarrow 2 - 2 \sin(2\theta) = 0$$

$$\sin(2\theta) = 1$$

$$\Rightarrow 2\theta = -\frac{3\pi}{2}, \frac{\pi}{2}$$

$$\theta = -\frac{3\pi}{4}, \frac{\pi}{4}$$

Since domain is in the form $\theta \in (-\alpha, \alpha)$,

$$\alpha = \frac{\pi}{4}$$

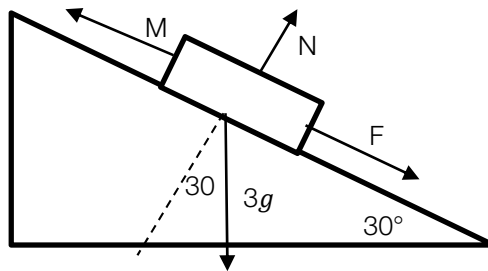
Question 3c

$$\begin{aligned}
\operatorname{Arg}(i\bar{z} - z) &= \operatorname{Arg}(\sin(\theta) - \cos(\theta) + (\cos(\theta) - \sin(\theta))i) \\
&= \tan^{-1}\left(\frac{\cos(\theta) - \sin(\theta)}{\sin(\theta) - \cos(\theta)}\right) \\
&= \tan^{-1}(-1) \\
&= \frac{3\pi}{4}
\end{aligned}$$

Note: It is possible to ascertain by graphical means that for $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, $(i\bar{z} - z)$ lies in the 2nd quadrant, not the 4th quadrant.

Question 4

$$\begin{aligned}
 \int_3^7 \frac{3x^2 + 2x^3 - 12x}{\sqrt{x-3}} dx &= \int_0^4 \frac{3(u+3)^2 + 2(u+3)^3 - 12(u+3)}{\sqrt{u}} du \quad \text{where } u = x - 3 \\
 &= \int_0^4 \frac{3(u+3)^2 + 2(u+3)^3 - 12(u+3)}{\sqrt{u}} du \\
 &= \int_0^4 \frac{(3u^2 + 18u + 27) + (2u^3 + 18u^2 + 54u + 54) - (12u + 36)}{\sqrt{u}} du \\
 &= \int_0^4 \frac{2u^3 + 21u^2 + 60u + 45}{\sqrt{u}} du \\
 &= \int_0^4 2u^{\frac{5}{2}} + 21u^{\frac{3}{2}} + 60u^{\frac{1}{2}} + 45u^{-\frac{1}{2}} du \\
 &= \left[2\left(\frac{2}{7}\right)u^{\frac{7}{2}} + 21\left(\frac{2}{5}\right)u^{\frac{5}{2}} + 60\left(\frac{2}{3}\right)u^{\frac{3}{2}} + 45(2)u^{\frac{1}{2}} \right]_0^4 \\
 &= \left[\frac{4u^{\frac{7}{2}}}{7} + \frac{42u^{\frac{5}{2}}}{5} + 40u^{\frac{3}{2}} + 90u^{\frac{1}{2}} \right]_0^4 \\
 &= \frac{4}{7}(2^7) + \frac{42}{5}(2^5) + 40(2^3) + 90(2) \\
 &= \frac{512}{7} + \frac{21 \cdot 64}{5} + 320 + 180 \\
 &= 500 + \frac{512}{7} + \frac{1344}{5}
 \end{aligned}$$

Question 5a**Question 5b**

In this question, redefine $\mathbf{i}$ and $\mathbf{j}$ in line with the hill. Splitting equations of movement we have:

$$N - 3g \cos(30^\circ) = 0$$

$$N = 3g \left(\frac{\sqrt{3}}{2} \right) = \frac{3\sqrt{3}g}{2}$$

Similarly, in the other direction we have:

$$M - \mu N - 3g \sin(30^\circ) = 0$$

$$M = \mu N + 3g \sin(30^\circ)$$

$$M = \frac{1}{\sqrt{3}} \cdot \left(\frac{3\sqrt{3}g}{2} \right) + 3g \left(\frac{1}{2} \right)$$

$$M = 3g = 29.4\text{N}$$

Question 5c

Movement in $\mathbf{i}$ direction would then be:

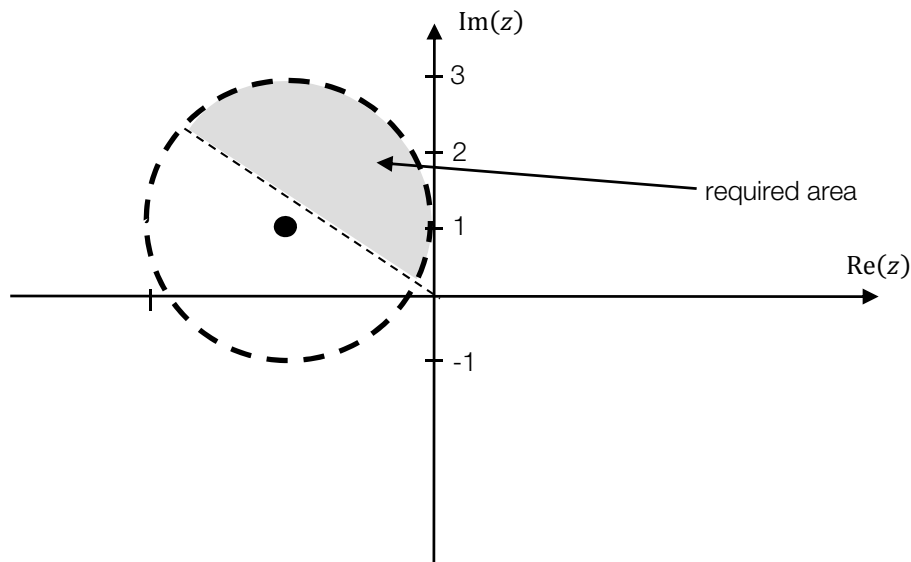
$$3g - 3g \sin(30^\circ) = ma$$

$$3g - \frac{3g}{2} = 3a$$

$$a = \frac{g}{2} = 4.9\text{m/s}^2$$

Question 6

$$\begin{aligned}
 V &= \pi \int_0^2 y^2 dx \\
 &= \pi \int_0^2 \frac{(x+3)^2}{x^2+1} dx \\
 &= \pi \int_0^2 \left(\frac{x^2+1}{x^2+1} + \frac{6x}{x^2+1} + \frac{8}{x^2+1} \right) dx \\
 &= \pi \left[\int_0^2 1 dx + 3 \int_0^2 \frac{2x}{x^2+1} dx + 8 \int_0^2 \frac{1}{x^2+1} dx \right] \\
 &= \pi \left[[x]_0^2 + 3 \int_1^5 \frac{1}{u} du + 8 [\tan^{-1}(x)]_0^2 \right] \text{ where } u = x^2 + 1 \\
 &= \pi [2 + 3 [\ln|u|]_1^5 + 8 \tan^{-1}(2)] \\
 &= \pi (2 + 3 \ln(5) + 8 \tan^{-1}(2))
 \end{aligned}$$

Question 7**Question 8**

The two particles will collide when they are in the same position at the same time.

Equating **i** components:

$$\begin{aligned}
 12 - 2t &= 5t - t^2 \\
 t^2 - 7t + 12 &= 0 \\
 (t - 3)(t - 4) &= 0 \\
 \Rightarrow t &= 3 \text{ or } t = 4
 \end{aligned}$$

Equating **j** components:

$$\begin{aligned}
 6t - 2 &= t^2 + 6 \\
 t^2 - 6t + 8 &= 0 \\
 (t - 2)(t - 4) &= 0 \\
 \Rightarrow t &= 2 \text{ or } t = 4
 \end{aligned}$$

Hence the particles collide at $t = 4$ seconds.