

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2



(TSSM's 2014 trial exam updated for the current study design)

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: E

Explanation

$$y = \frac{-x}{2} + \frac{1}{2x}$$

Question 2

Answer: D

Explanation

$$\frac{d}{dx} \left(1 - 2 \cos^{-1} \left(\frac{x}{2} \right) \right) = -2 \times \frac{-1}{\sqrt{1 - \left(\frac{x}{2} \right)^2}} \times \frac{1}{2} = \frac{2}{\sqrt{4 - x^2}}$$

Question 3

Answer: C

Explanation

$$\frac{\bar{z}}{z} = \frac{3 - 2i}{3 + 2i} = \frac{(3 - 2i)^2}{9 + 4} = \frac{5 - 12i}{13}$$

Question 4

Answer: E

Explanation

$$\text{Domain: } -1 \leq \frac{x-1}{2} \leq 1 \Rightarrow -2 \leq x-1 \leq 2 \Rightarrow -1 \leq x \leq 3$$

$$\text{Range: } -\frac{\pi}{2} \leq \sin^{-1}\left(\frac{x-1}{2}\right) \leq \frac{\pi}{2}$$

Question 5

Answer: D

Explanation

$$\frac{1}{\sqrt{6}}(\vec{i} - 2\vec{j} - \vec{k}) \cdot (\vec{i} + 2\vec{j} - 3\vec{k}) = 1 - 4 + 3 = 0$$

Question 6

Answer: B

Explanation

$$\text{Let } u = 1 - x^2$$

$$\frac{du}{dx} = -2x$$

$$-\frac{1}{2} \int_1^0 (1-u)u^{1/2} du = \frac{1}{2} \int_0^1 \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du$$

Question 7

Answer: B

Explanation

$$x = -1 \Rightarrow y = 1 - \sqrt{3} \quad (\text{in third quadrant})$$

$$2x + 2(y-1)y' = 0 \Rightarrow y' = -\frac{x}{y-1}$$

$$m = -\frac{-1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

Question 8

Answer: A

Explanation

$$1 + \cot^2 t = \operatorname{cosec}^2 t$$

$$1 + \left(\frac{y+1}{3} \right)^2 = \left(\frac{x-1}{2} \right)^2$$

$$\frac{(x-1)^2}{4} - \frac{(y+1)^2}{9} = 1$$

Question 9

Answer: B

Explanation

$|z| > b$ is outside the circle.

Question 10

Answer: E

Explanation

$$z = \left(4 \operatorname{cis} \left(\frac{4\pi}{3} \right) \right)^{\frac{1}{2}} \Rightarrow z = 2 \operatorname{cis} \left(\frac{2\pi}{3} \right), 2 \operatorname{cis} \left(\frac{-\pi}{3} \right)$$

Question 11

Answer: C

Explanation

The angle is between 60° and 90° .

Question 12

Answer: C

Explanation

$$\underset{\sim}{b} \bullet (\underset{\sim}{a} - \underset{\sim}{b}) = \underset{\sim}{b} \bullet \underset{\sim}{a} - \underset{\sim}{b} \bullet \underset{\sim}{b} = \left| \underset{\sim}{b} \right| \left| \underset{\sim}{a} \right| \cos(\theta) - \left| \underset{\sim}{b} \right|^2 \neq \left| \underset{\sim}{b} \right| \left| \underset{\sim}{c} \right|$$

Question 13

Answer: B

Explanation

$$3 = 0 + \frac{1}{2}a \times 4 \Rightarrow a = \frac{3}{2}$$

$$v = 0 + 2 \times \frac{3}{2} = 3$$

$$p = mv = 12$$

Question 14

Answer: A

Explanation

$$\underset{\sim}{F} = 5\underset{\sim}{i} - \underset{\sim}{j}$$

$$2\underset{\sim}{a} = 5\underset{\sim}{i} - \underset{\sim}{j} \Rightarrow \underset{\sim}{a} = \frac{5}{2}\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j}$$

$$\left| \underset{\sim}{a} \right| = 2.5$$

Question 15

Answer: C

Explanation

Use Euler's theorem.

Question 16

Answer: B

Explanation

$$\frac{dA}{dt} = 0 - \frac{A}{25}$$

Question 17

Answer: D

Explanation

The solution to the differential equation is cubic which is represented in the slope field.

Question 18

Answer: A

Explanation

$$g\sin(30^\circ) - g\sin(15^\circ)$$

Question 19

Answer: C

Explanation

$$V = \pi \int_0^3 y^{\frac{3}{2}} dy = \frac{18\pi\sqrt{3}}{5}$$

Question 20

Answer: A

Explanation

$$a = v \frac{dv}{dx} = \sqrt{9 - x^2} \times \frac{1}{2\sqrt{9 - x^2}} \times -2x = -x$$

$$v = \sqrt{5} \Rightarrow x = \pm 2$$

$$a = -2 \quad (\text{as } x > 0)$$

Question 21

Answer: C

Explanation

$$2 = 20 + 4a \Rightarrow a = -\frac{9}{2}$$

$$F = 2 \times \frac{9}{2} = 9 \text{ N}$$

Question 22

Answer: E

Explanation

$$R - 60g = 60 \times \frac{g}{4}$$

$$R = 75g$$

SECTION 2

Question 1

a.

$$\left(\frac{x+3}{-2}\right)^2 + \left(\frac{y-4}{3}\right)^2 = 1$$

$$\frac{(x+3)^2}{4} + \frac{(y-4)^2}{9} = 1$$

M1+A1

b.

$$y' = -\frac{9(x+3)}{4(y-4)}$$

$$\text{solve } \frac{9(x+3)}{4(y-4)} = \frac{3\sqrt{3}}{2} \quad \text{and} \quad \frac{(x+3)^2}{4} + \frac{(y-4)^2}{9} = 1$$

$$\left(-3 + \sqrt{3}, \frac{11}{2}\right) \quad \text{and} \quad \left(-3 - \sqrt{3}, \frac{5}{2}\right)$$

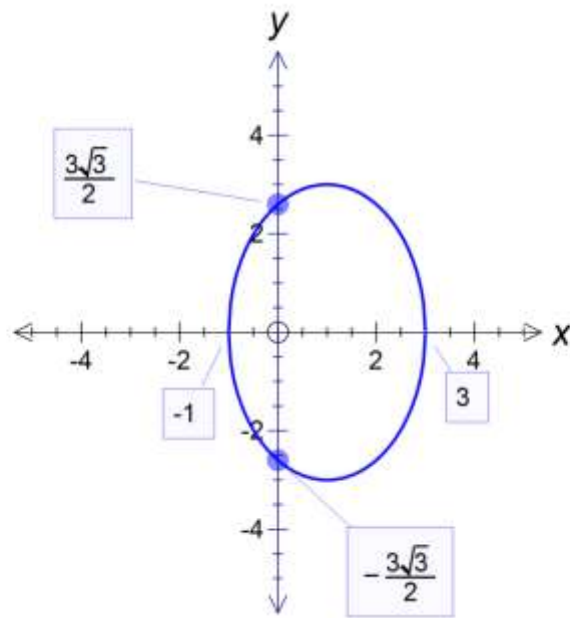
$$-3 \pm \sqrt{3} = 3 - 2\cos\left(\frac{t}{2}\right) \quad \text{for } 0 \leq t \leq 4\pi$$

$$t = \frac{\pi}{3}, \frac{11\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$$

But we require gradient to be negative, so $t = \frac{5\pi}{3}, \frac{11\pi}{3}$ only.

M2+A2

c.



2 marks for the intercepts

d.

$$V = \pi \int_1^{2.5} \left(9 - \frac{9}{4}(x-1)^2 \right) dx$$

A2

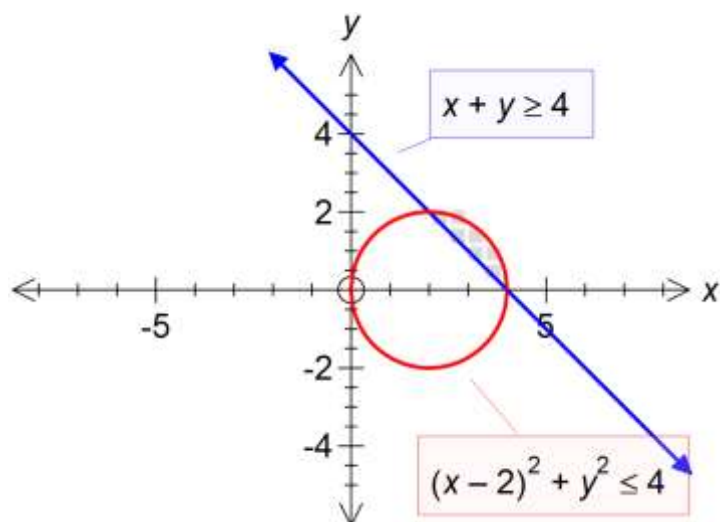
e.

$$V = \pi \int_1^{2.5} \left(9 - \frac{9}{4}(x-1)^2 \right) dx = \frac{351\pi}{32}$$

A1

Question 2

a.

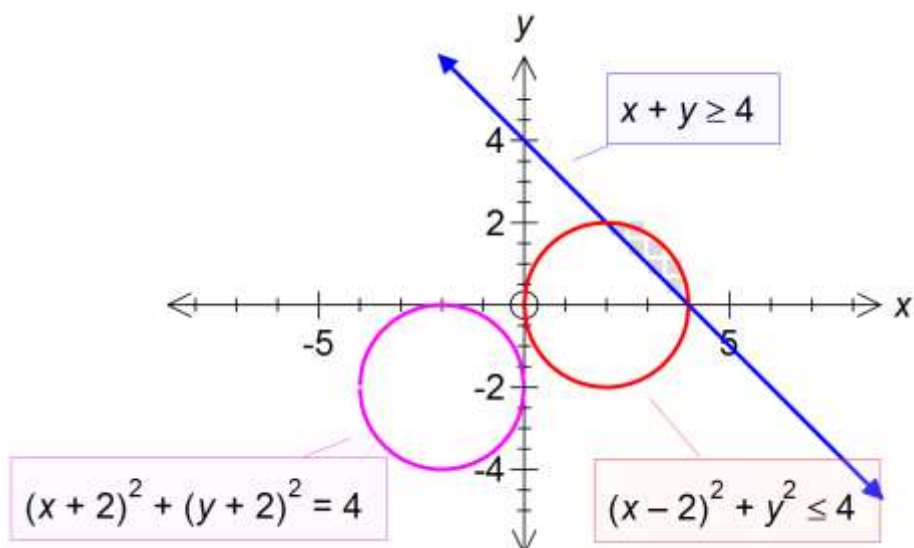


3 marks for line and circle, 1 mark for shading the intersection area

b.
$$Area = \int_2^4 \left(\sqrt{4 - (x - 2)^2} - (4 - x) \right) dx = \pi - 2$$

M2

c.



1 mark for centre and 1 mark for correct radius

d.

$$C_1(-2, -2) \text{ and } C_2(2, 0)$$

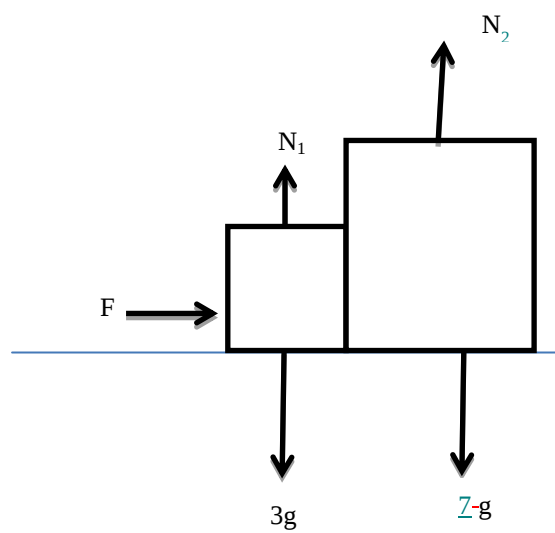
$$C_1C_2 = \sqrt{(2+2)^2 + (2-0)^2} = \sqrt{20} = 2\sqrt{5}$$

$$\text{Min distance} = 2\sqrt{5} - 2 - 2 = 2\sqrt{5} - 4$$

M1+A1

Question 3

a.



4 marks

b.

$$F = 10a \Rightarrow a = \frac{F}{10}$$

M1 + A1

c.

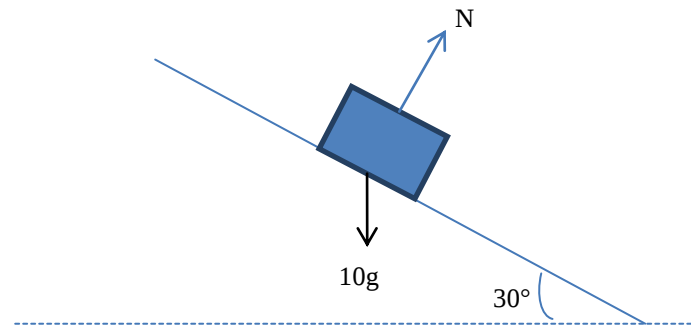
$$7 \times \frac{F}{10} = 0.7F$$

M1+A1

d. $a = \frac{120}{10} = 12 \text{ m/s}^2$

A1

e.



A2

- f. Acceleration perpendicular to the block is 0
 Let acceleration parallel to the incline be a , then
 $mg \sin(30^\circ) = ma$
 $a = g \sin(30^\circ) = 4.9 \text{ m/s}^2$
 Thus resultant acceleration $= 4.9 \text{ m/s}^2$

M2+A1

g.

$$N = mg \cos(30^\circ)$$

$$N = 10 \times 9.8 \times \frac{\sqrt{3}}{2} = 84.87 \text{ N}$$

M1+A1

Question 4**a.**

$$(t^3 - 9t + 8)\hat{i} + t^2\hat{j} = (2 - t^2)\hat{i} + (3t - 2)\hat{j}$$

$$t^3 - 9t + 8 = 2 - t^2 \Rightarrow t^3 + t^2 - 9t + 6 = 0$$

$$\text{and } t^2 = 3t - 2 \Rightarrow t^2 - 3t + 2 = 0$$

$$t = 2, 1$$

$$\text{At } t = 1, \quad t^3 + t^2 - 9t + 6 = -1 \neq 0$$

$$\text{Thus } t = 2$$

The particles collide after 2 seconds.

M3+A1

b.

$$\dot{\vec{r}}_A(t) = (3t^2 - 9)\hat{i} + 2t\hat{j}$$

$$\dot{\vec{r}}_A(2) = 3\hat{i} + 4\hat{j}$$

$$\text{Speed}_A = \sqrt{9 + 16} = 5$$

$$\dot{\vec{r}}_B(t) = -2t\hat{i} + 3\hat{j}$$

$$\dot{\vec{r}}_B(2) = -4\hat{i} + 3\hat{j}$$

$$\text{Speed}_B = \sqrt{16 + 9} = 5$$

The particles collide when their speed is 5m/s.

M3+A1

$$\text{c. } \dot{\vec{r}}_A(t) \cdot \dot{\vec{r}}_B(t) = -12 + 12 = 0$$

A1

d. The particles are travelling at right angles at the time of collision.

A2

e.

$$\ddot{r}_{\sim B}(t) = -2i_{\sim}$$

$$\ddot{r}_{\sim B}(2) = -2 \text{ m/s}^2$$

M1+A1

Question 5

a.

$$\log_e N = 6 - 3e^{-0.4t}$$

$$\frac{1}{N} \frac{dN}{dt} = -3e^{-0.4t} \times -0.4 \Rightarrow \frac{1}{N} \frac{dN}{dt} = 1.2e^{-0.4t}$$

$$LHS = 1.2e^{-0.4t} + 0.4(6 - 3e^{-0.4t}) - 2.4 = 0$$

M2

b.

$$\log_e N = 6 - 3e^0$$

$$N = e^3$$

$$N = 20$$

A1

c.

$$\log_e N = 6 - 3e^{-0.4t}$$

$$\text{As } t \rightarrow \infty, \log_e N = 6 \Rightarrow N = e^6 \Rightarrow N = 403$$

M1+A1

d.

$$\log_e N = 6 - 3e^{-0.4t}$$

$$\frac{dN}{dt} = 1.2N \left(\frac{6 - \log_e N}{3} \right) = 0.4N(6 - \log_e N)$$

$$\frac{d^2 N}{dt^2} = 0.4N \times \frac{-1}{N} \frac{dN}{dt} + (6 - \log_e N) \times 0.4 \frac{dN}{dt}$$

$$\frac{d^2 N}{dt^2} = -1.6N(6 - \log_e N) + 1.6N(6 - \log_e N)^2$$

M1+A1

e.

$$\frac{d^2N}{dt^2} = 0 \Rightarrow N = 148$$

 $(2.7, 148)$ or $(3, 148)$ to the nearest integers.

M1+A1