

NAME: _____

UNITS 3 & 4 Practice Examination

VCE® Specialist Mathematics

Written examination 2

QUESTION AND ANSWER BOOK

Structure of book

<i>Section</i>	<i>Number of questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
1	22	22	22
2	5	5	58
			80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 28 pages.
- 4 page formula booklet.
- Answer sheet for multiple-choice questions.

Instructions

- Write your name in the space provided above on this page.
- Write your name on the multiple-choice answer sheet
- All written responses must be in English.

At the end of the examination

- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Take the **acceleration due to gravity** to have magnitude $g \text{ m/s}^2$, where $g = 9.8$.

Question 1

The **domain** of the function with the rule $f(x) = \arccos(2x)$ is

A. $[-1, 1]$

B. $[-2, 2]$

C. $\left[-\frac{1}{2}, \frac{1}{2}\right]$

D. $[0, \pi]$

E. $\left[0, \frac{\pi}{2}\right]$

Question 2

The rule of the relation determined by parametric equations $x = 3 \cot(t) + 1$ and $y = 2 \operatorname{cosec}(t) - 1$ is

A. $\frac{(x+1)^2}{3} + \frac{(y-1)^2}{2} = 1$

B. $\frac{(x+1)^2}{9} - \frac{(y-1)^2}{4} = 1$

C. $\frac{(x-1)^2}{9} - \frac{(y+1)^2}{4} = 1$

D. $\frac{(y+1)^2}{4} + \frac{(x-1)^2}{9} = 1$

E. $\frac{(y+1)^2}{4} - \frac{(x-1)^2}{9} = 1$

Question 3

The graph of $y = \frac{1}{ax^2+bx+c}$ has asymptotes at $x = -2$ and $x = 4$ and $y = 0$. Given that the graph has one stationary point with a y -coordinate of $-\frac{1}{18}$, it follows that

- A. $a = 2, b = -4, c = -16$
- B. $a = 2, b = 4, c = -16$
- C. $a = 1, b = 2, c = -8$
- D. $a = 1, b = -2, c = -8$
- E. $a = 2, b = -12, c = 16$

Question 4

If $y = \cot\left(x - \frac{\pi}{2}\right)$, which of the following statements is **not** true?

- A. $y = 1/\tan\left(x - \frac{\pi}{2}\right)$
- B. $y = \tan(\pi - x)$
- C. $y = \cot\left(x + \frac{\pi}{2}\right)$
- D. $y = 1/\tan\left(\frac{\pi}{2} - x\right)$
- E. $y = -\tan(x)$

Question 5

The region of the complex plane inside the circle of radius a centred at the origin is given by the set of points z , where $z \in \mathbb{C}$, such that

- A. $z\bar{z} < a$
- B. $z\bar{z} < a^2$
- C. $(\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2 < a$
- D. $|z| < a^2$
- E. $z\bar{z} > a$

Question 6

The $z = \operatorname{cis}\left(\frac{5\pi}{6}\right)$. The imaginary part of $z + i$ is

- A. $\frac{i}{2}$
- B. $\frac{1}{2}$
- C. $\frac{\sqrt{3}}{2}$
- D. $\frac{3}{2}$
- E. $\frac{3i}{2}$

Question 7

If $z = r\text{cis}(\theta)$, then $\frac{(\bar{z})^2}{z}$ is equivalent to

- A. $r^3\text{cis}(-3\theta)$
- B. $r^3\text{cis}(\theta)$
- C. $2\text{cis}(-3\theta)$
- D. $r^3\text{cis}(-\theta)$
- E. $r\text{cis}(-3\theta)$

Question 8

The principal arguments of the solution to the equation $z^2 = 1 - i$ are

- A. $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$
- B. $-\frac{\pi}{8}$ and $\frac{7\pi}{8}$
- C. $\frac{3\pi}{8}$ and $\frac{11\pi}{8}$
- D. $-\frac{\pi}{8}$ and $-\frac{9\pi}{8}$
- E. $-\frac{\pi}{4}$ and $-\frac{5\pi}{4}$

Question 9

The definite integral $\int_{\pi/4}^{\pi/2} e^{-\cot(x)} (\cot^2(x) + 1) dx$ can be written in the form $\int_a^b \frac{1}{e^u} du$

- A. $u = \cot(x)$, $a = 1$ and $b = 0$
- B. $u = \cot(x)$, $a = 0$ and $b = 1$
- C. $u = -\cot(x)$, $a = 1$ and $b = 0$
- D. $u = -\cot(x)$, $a = 0$ and $b = 1$
- E. $u = \tan(x)$, $a = 1$ and $b = \infty$

Question 10

The region bounded by the lines $x = 0$, $y = 2$ and the graph of $y = x^{\frac{5}{3}}$, where $x \geq 0$ is rotated about the y -axis to form a solid of revolution. The volume of this solid is

- A. $\frac{5\pi}{2^{7/5}}$
- B. $\frac{20}{11} 2^{1/5} \pi$
- C. $\frac{3\pi}{2^{1/3}}$
- D. $\frac{10}{11} 2^{8/25} \pi$
- E. $\frac{12}{13} 2^{3/5} \pi$

Question 11

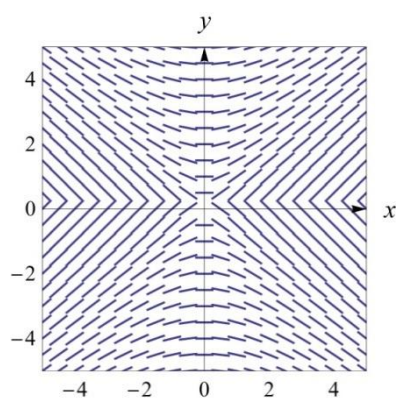
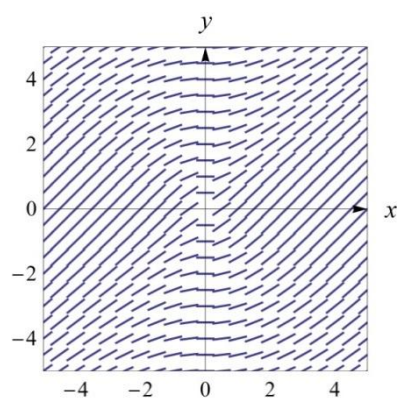
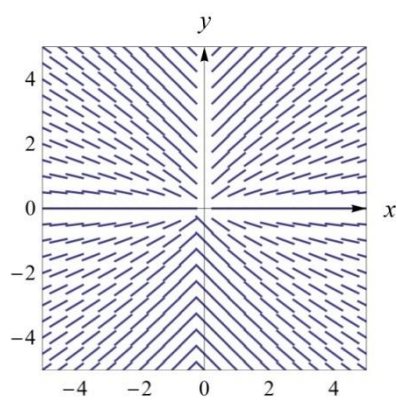
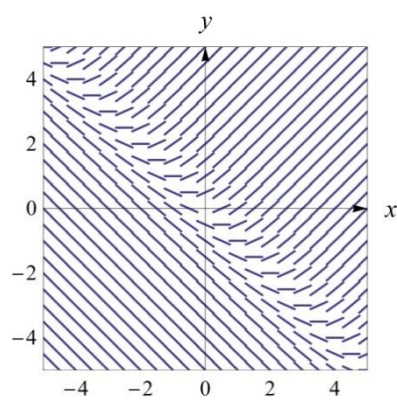
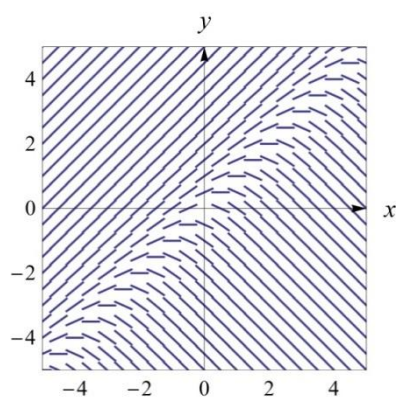
Euler's formula is used to find y_2 , where $\frac{dy}{dx} = \tan(x)$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

The value of y_2 correct to four decimal places is

- A. 1.0000 and this is an overestimate of $y(0.2)$
- B. 1.0100 and this is an underestimate of $y(0.2)$
- C. 1.0100 and this is an overestimate of $y(0.2)$
- D. 1.0303 and this is an underestimate of $y(0.2)$
- E. 1.0100 and this is an overestimate of $y(0.2)$

Question 12

Which diagram best represents the direction field of the differential equation $\frac{dy}{dx} = x/y$?

A.**B.****C.****D.****E.**

Question 13

Which of the following expressions is equivalent to $\ddot{x}dx$

A. $v dv$

B. $v \frac{dv}{dx}$

C. $\frac{d}{dx} \left(\frac{1}{2} v^2 \right)$

D. $v^2 dv$

E. $\frac{dv}{dt}$

Question 14

Consider the four vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{c} = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$, $\mathbf{d} = 3\mathbf{i} + 3\mathbf{k}$.

Which one of the following is a linearly dependent set of vectors?

A. $\{\mathbf{a}, \mathbf{b}\}$

B. $\{\mathbf{a}, \mathbf{c}\}$

C. $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$

D. $\{\mathbf{b}, \mathbf{c}, \mathbf{d}\}$

E. $\{\mathbf{a}, \mathbf{c}, \mathbf{d}\}$

Question 15

Find the distance between point $P(3,0,4)$ and point $Q(0,-2,1)$.

- A. 4
- B. 22
- C. $\sqrt{22}$
- D. 30
- E. $\sqrt{30}$

Question 16

Water containing 5 grams of salt per litre flows at a rate of 8 litres per minute into a tank that initially contained 100 litres of pure water. The concentration of salt in the tank is kept uniform by stirring and the mixture flows out of the tank at a rate of 10 litres per minute. If m grams is the amount of salt in the tank t minutes after the water begins to flow, the differential equation for m in terms of t is

- A. $\frac{dm}{dt} = 5 - \frac{m}{100}$
- B. $\frac{dm}{dt} = 5 - \frac{m}{98}$
- C. $\frac{dm}{dt} = 5 - \frac{m}{100 - 2t}$
- D. $\frac{dm}{dt} = 40 - \frac{m}{100 + 2t}$
- E. $\frac{dm}{dt} = 40 - \frac{10m}{100 - 2t}$

Question 17

Let $\mathbf{u} = 2\mathbf{i} + 3\mathbf{k}$, $\mathbf{v} = -3\mathbf{i} + 2\mathbf{k}$ and $\mathbf{w} = -2\mathbf{i} + 10\mathbf{k}$.

Which one of the following statements is **not** true?

A. $\mathbf{u} \cdot \mathbf{v} = 0$

B. $(\mathbf{u} + \mathbf{w}) \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w}$

C. $\mathbf{w} \cdot \mathbf{w} = |\mathbf{w}|^2$

D. $\mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{v}$

E. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} = |\mathbf{u}|$

Question 18

A box of mass m kg rests on the floor of a lift. Starting from rest, the lift moves downwards with a constant acceleration $a \text{ ms}^{-2}$, so that after time t , the box has traveled x m downward and is travelling with a downward velocity of $v \text{ ms}^{-1}$. Let F_N denote the normal reaction force of the lift floor on the box. Which of the following statements is **not** true?

A. $ma = mg - F_N$

B. $mv^2 = 2x(mg - F_N)$

C. $m \frac{dv}{dt} = mg - F_N$

D. $mg = -ma + F_N$

E. $x = \frac{1}{2} \left(g - \frac{F_N}{m} \right) t^2$

Question 19

A ball of mass m is projected vertically with an upwards velocity $v(x)$ m/s, where x is the distance measured upwards from ground level in metres at time t seconds. The motion of the ball is retarded by an air resistance of magnitude kv^2 newtons. The equation of motion of the ball is

A. $m \frac{dv}{dt} = mg + kv^2$

B. $m \frac{d^2x}{dt^2} = -mg + kv^2$

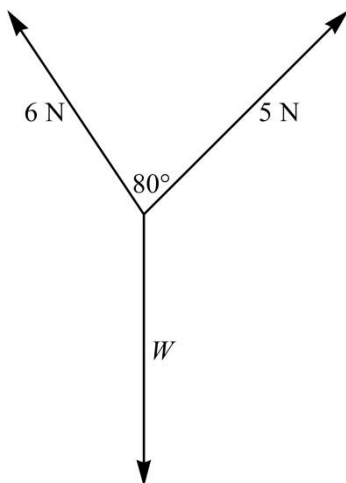
C. $\frac{dv^2}{dx} = 2g - 2 \frac{kv^2}{m}$

D. $\frac{d(v^2/2)}{dx} = mg + kv^2$

E. $mv \frac{dv}{dx} = -mg - kv^2$

Question 20

Forces of magnitude 5 N, 6 N and W N act on a particle that is in equilibrium, as shown in the diagram below.



The magnitude of W , in newtons, can be found by evaluating

- A. $\sqrt{5^2 + 6^2 - 2 \times 5 \times 6 \cos(80^\circ)}$
- B. $5^2 + 6^2 - 2 \times 5 \times 6 \cos(80^\circ)$
- C. $5^2 + 6^2 - 2 \times 5 \times 6 \cos(100^\circ)$
- D. $\sqrt{5^2 + 6^2 - 2 \times 5 \times 6 \cos(100^\circ)}$
- E. $\sin(80^\circ) \times \frac{6}{\sin(140^\circ)}$

Question 21

An ice hockey puck of mass $\frac{1}{5}$ kg, slides across a perfectly level, perfectly smooth ice rink with an initial speed of 10 ms^{-1} . A constant frictional force acts on the hockey puck, so that after travelling 9 m, the puck is travelling at 8 ms^{-1} . The magnitude of the frictional force, in newtons, is

- A. $\frac{2}{45}$
- B. $\frac{2}{9}$
- C. $\frac{2}{5}$
- D. 2
- E. 10

Question 22

The velocity $v \text{ ms}^{-1}$ of a body which is moving in a straight line, when it is x m from the origin, is given by $v = \arctan(x)$. The acceleration of the body in ms^{-2} is given by

- A. $\sec^2(x)$
- B. $\frac{\arctan(x)}{x^2 + 1}$
- C. $\tan(x)\sec^2(x)$
- D. $\frac{1}{x^2 + 1}$
- E. $\frac{1}{2} \left(\frac{2\arcsin(x)^2}{\sqrt{1-x^2}\arccos(x)^3} + \frac{2\arcsin(x)}{\sqrt{1-x^2}\arccos(x)^2} \right)$

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

Unless otherwise specified an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude $g \text{ m/s}^2$, where $g = 9.8$.

Question 1 (11 marks)

A curve is defined by parametric equations

$$x = 2 \sec(t) + 1$$

$$y = 3 \tan(t) - 2$$

for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

a. Find the Cartesian equation of the curve.

2 marks

b. Find the values of t for which the gradient of the curve is 3.

2 marks

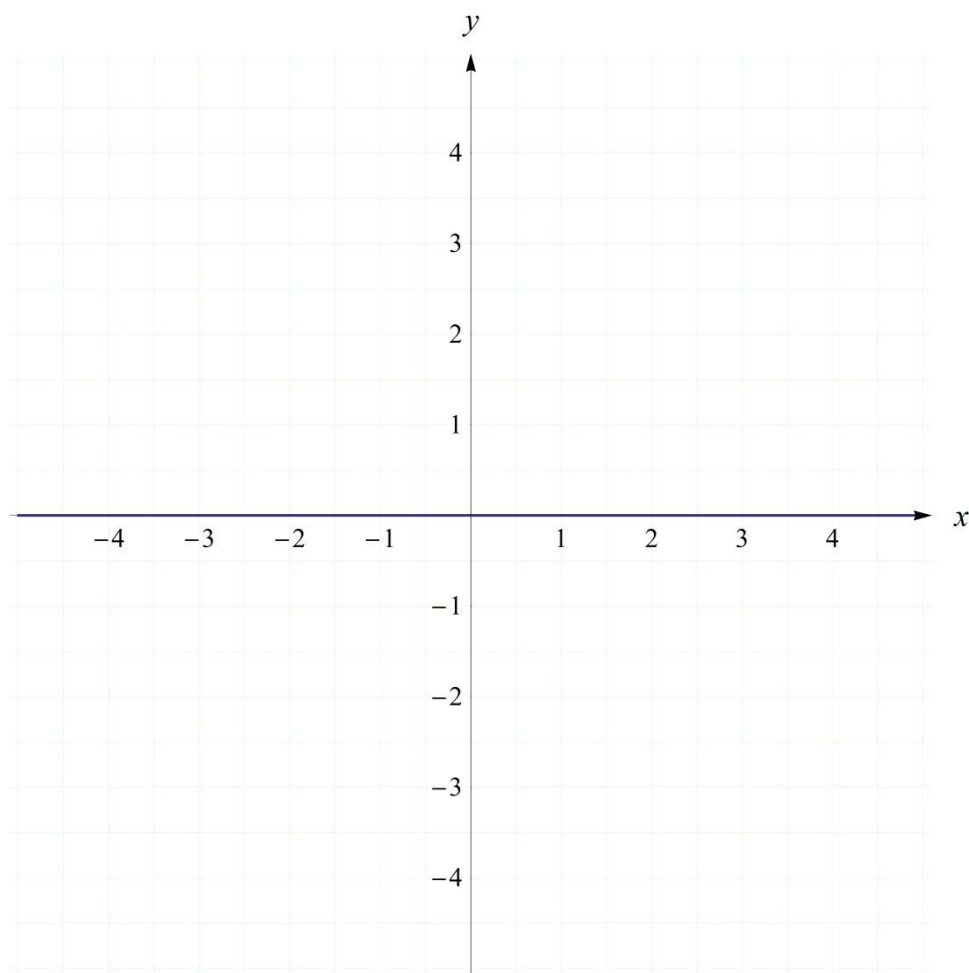
Consider the **different** relation $\frac{x^2}{9} + \frac{(y-2)^2}{4} = 1$.

- c. Find the value(s) of y if the tangent line is vertical

2 marks

- d. Sketch the graph of $\frac{x^2}{9} + \frac{(y-2)^2}{4} = 1$, showing the coordinates of any relevant vertices.

2 marks



The region in the first quadrant enclosed by the graph of $\frac{x^2}{9} + \frac{(y-2)^2}{4} = 1$, the y -axis, and the lines $y = 1$ and $y = 3$ is rotated about the y -axis to form a solid of revolution.

Consider a **different** relation $\frac{x^2}{9} + \frac{(y-2)^2}{4} = 1$.

- e. i. Write down a definite integral, in terms of y , that gives the volume of this solid of revolution.

2 marks

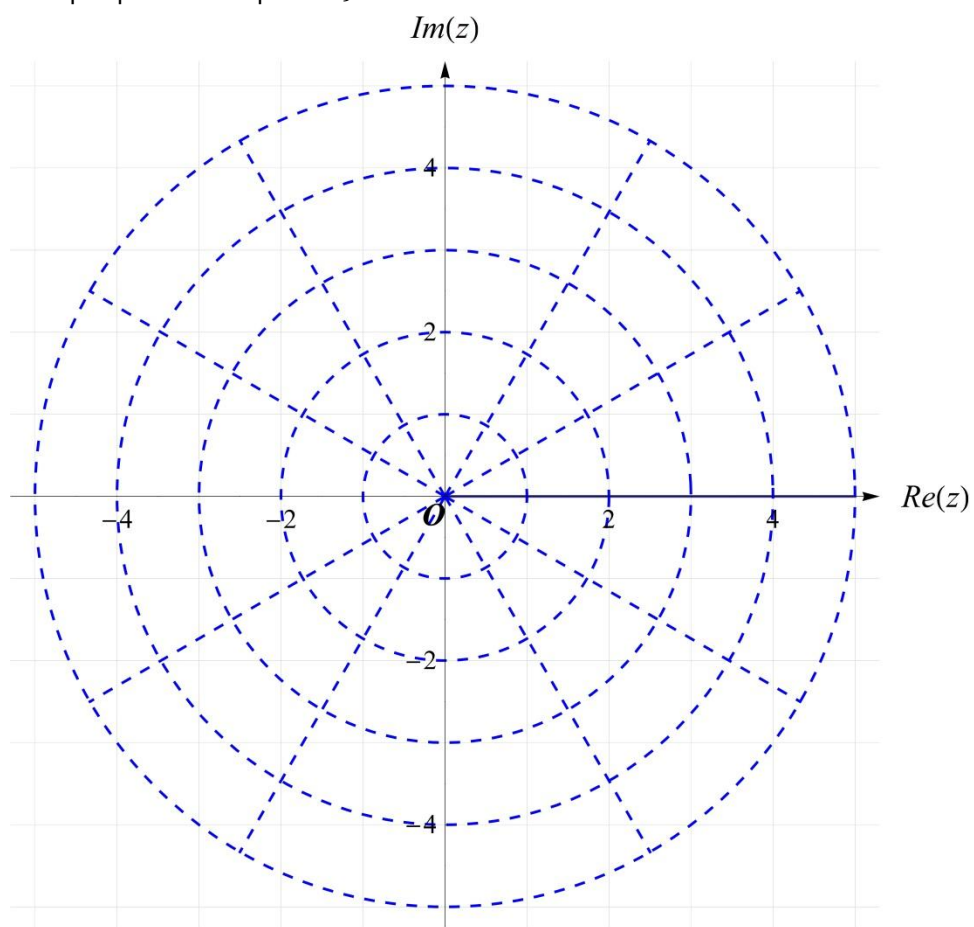
- ii. Find the volume of this solid of revolution.

1 mark

Question 2 (12 marks)

- a. On the Argand diagram below, sketch $\{z: |z| = 3, z \in \mathbb{C}\}$ **and** sketch $\{z: |z + \sqrt{3} - i| = |z - \sqrt{3} + i|, z \in \mathbb{C}\}$.

3 marks



- b.** Find all the elements of $\{z: |z| = 3, z \in \mathbb{C}\} \cap \{z: |z + \sqrt{3} - i| = |z - \sqrt{3} + i|, z \in \mathbb{C}\}$, expressing your answer(s) in the form $a + ib$. 2 marks

One of the roots of $z^3 + 27 = 0$ is $z = \frac{3}{2} + i \frac{3\sqrt{3}}{2}$

- c.** Write down the other roots in cartesian form. 2 marks

- d.** Plot and label all these roots on the Argand diagram provided in **part a.** 1 mark

- e.** Express $z^3 + 27$ as a product of linear factors in terms of z . 1 mark

- f.** On the diagram provided in **part a.**, shade the region defined by $\{z: |z| \leq 3, z \in \mathbb{C}\} \cap \{z: \text{Arg}(z) > \frac{\pi}{3}, z \in \mathbb{C}\}$ 1 mark

- g.** Find the area of the region shaded in **part e.** 2 marks

Question 3 (11 marks)

The number of individuals n (**measured in thousands**) in a large city that have been infected by a flu virus, t months since the beginning of the year is modelled by $\log_e(n) = 5 - 4e^{-6t/5}$, $t \geq 0$.

- a. Verify that $\log_e(n) = 5 - 4e^{-6t/5}$ satisfies the differential equation

$$\frac{1}{n} \frac{dn}{dt} = \frac{6}{5} (5 - \log_e(n))$$
 2 marks

- b. Find the initial number of infected individuals in the city. Express your answer correct to the nearest individual. 1 mark

- c. Find the limiting number of individuals that would eventually become infected with the flu virus. Express your answer correct to the nearest individual. 2 marks

- d. i. Show that $\frac{d^2n}{dt^2} = \frac{36}{25} n(4 - \log_e(n))(5 - \log_e(n))$.

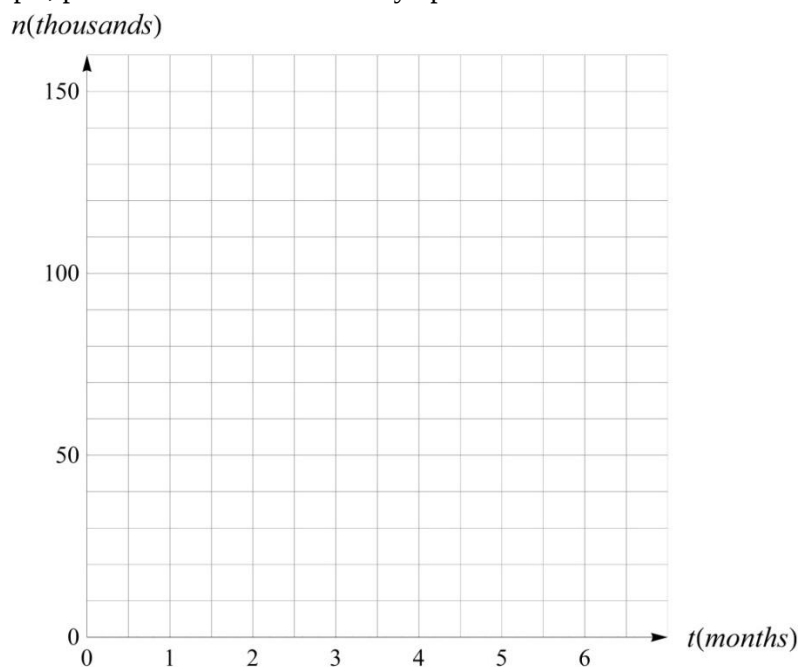
2 marks

- ii. The graph of n as a function t has a point of inflection. Find the exact coordinates of this point.

2 marks

- e. Plot the graph of n as a function of t on the axes below for $0 \leq t \leq 6$ showing any relevant axis intercepts, points of inflexion and/or asymptotes.

2 marks



Question 4 (12 marks)

Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{m} = 18\mathbf{i} + 18\mathbf{j} + 0\mathbf{k}$.

- a.** Resolve $\mathbf{a}$ into two vector components, one is parallel to $\mathbf{m}$ and one is perpendicular to $\mathbf{m}$. 2 marks

- b.** Find the acute angle θ between vectors $\mathbf{a}$ and $\mathbf{m}$. 2 marks

- c. Find the value of real number β such that $\mathbf{b} = 4\mathbf{i} + \beta\mathbf{j} - 4\mathbf{k}$ makes an angle $\frac{\pi}{4}$ with vector $\mathbf{m}$ where $\mathbf{b} \neq \mathbf{a}$. 2 marks

- d. Find α if $\mathbf{m} = 6\mathbf{a} + \alpha\mathbf{b}$. 2 marks

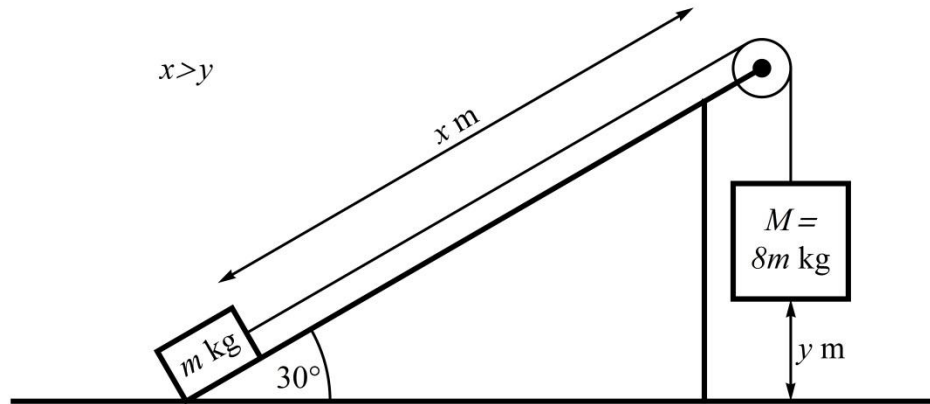
Consider three new vectors $\mathbf{p}$, $\mathbf{r}$ and $\mathbf{q} = |\mathbf{r}|\mathbf{p} + |\mathbf{p}|\mathbf{r}$.

- e. Using properties of the dot product, show that the cosine of the angle between vectors $\mathbf{p}$ and $\mathbf{q}$ is $\frac{|\mathbf{r}||\mathbf{p}| + \mathbf{r} \cdot \mathbf{p}}{|\mathbf{q}|}$ 2 marks

- f. Hence, show that vector $\mathbf{q} = |\mathbf{r}|\mathbf{p} + |\mathbf{p}|\mathbf{r}$ is the **angle bisector** of vectors $\mathbf{p}$ and $\mathbf{r}$. (Hint: Find the cosine of the angle between vectors $\mathbf{r}$ and $\mathbf{q}$). 2 marks

Question 5 (12 marks)

Two blocks of stone of mass m kg and $M = 8m$ kg are connected by a light, strong, inextensible rope. The rope passes over a smooth pulley at the top of a rough ramp elevated at 30° to the horizontal. The smaller mass m is x metres from the pulley, and the heavier mass M , is initially held in place with its bottom y metres above the ground, as shown in the figure below. You may assume that $x > y$.



The coefficient of friction between the ramp and the smaller block is $\frac{1}{\sqrt{3}}$. At time $t = 0$ seconds, the heavier block is released, and the system begins to move.

- a. Find** a pair of equations for the initial acceleration a ms^{-2} and rope tension τ newton, over the time interval before M reaches ground level. 3 marks

- b. Hence**, find the acceleration a up the plane (leave answer in terms of g if necessary). 1 mark

- c.** Find the speed $V \text{ ms}^{-1}$ of the heavier block M , just before it hits the ground (Express your answer in terms of y). 2 marks

- d.** Find the speed $v \text{ ms}^{-1}$ of the lighter block m , just before it reaches the pulley at the top of the ramp. 4 marks

- e. If block m comes to rest instantaneously as it reaches the pulley, show that $x = \frac{16}{9}y$ 2 marks

End of Examination

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.

This formula sheet is provided for your reference.

Specialist Mathematics Formulas

Mensuration

area of a trapezium: $\frac{1}{2}(a + b)h$

curved surface area of a cylinder: $2\pi rh$

volume of a cylinder: $\pi r^2 h$

volume of a cone: $\frac{1}{3}\pi r^2 h$

volume of a pyramid: $\frac{1}{3}Ah$

volume of a sphere: $\frac{4}{3}\pi r^3$

area of a triangle: $\frac{1}{2}bc \sin A$

sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

cosine rule: $c^2 = a^2 + b^2 - 2ab \sin C$

Coordinate geometry

ellipse: $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$

hyperbola: $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$

Circular (trigonometric) functions

$\cos^2(x) + \sin^2(x) = 1$

$\sin(2x) = 2 \sin(x) \cos(x)$

$1 + \tan^2(x) = \sec^2(x)$

$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$

$\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y)$

$\sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y)$

$\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$

$\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y)$

$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \tan(y)}$

$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x) \tan(y)}$

$$\begin{aligned}
 \cos(2x) &= \cos^2(x) - \sin^2(x) \\
 &= 2\cos^2(x) - 1 \\
 &= 1 - 2\sin^2(x)
 \end{aligned}$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

function	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1, 1]$	$[-1, 1]$	$\mathbb{R}$
range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (complex numbers)

$$z = x + iy = r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta$$

$$|z| = \sqrt{x^2 + y^2} = r$$

$$z_1 z_2 = r_1 r_2 \operatorname{cis} (\theta_1 + \theta_2)$$

$$z^n = r^n \operatorname{cis} (n\theta) \text{ (de Moivre's theorem)}$$

$$-\pi < \operatorname{Arg}(z) \leq \pi$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis} (\theta_1 - \theta_2)$$

Calculus

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

$$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin(ax)) = a \cos(ax)$$

$$\frac{d}{dx}(\cos(ax)) = -a \sin(ax)$$

$$\frac{d}{dx}(\tan(ax)) = a \sec^2(ax)$$

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$$

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$$

$$\int \frac{1}{x} dx = \log_e |x| + c$$

$$\int \sin(ax) dx = -\frac{1}{a} \cos(ax) + c$$

$$\int \cos(ax) dx = \frac{1}{a} \sin(ax) + c$$

$$\int \sec^2(ax) dx = \frac{1}{a} \tan(ax) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c, a > 0$$

$$\frac{d}{dx}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}$$

$$\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1}(ax) + c, a > 0$$

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$$

$$\int \frac{a}{a^2+x^2} dx = \sin^{-1}(ax) + c$$

product rule:

$$\frac{d}{dx}(uv) = u \frac{d}{dx}(v) + v \frac{d}{dx}(u)$$

quotient rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{d}{dx}(u) - u \frac{d}{dx}(v)}{v^2}$$

chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Euler's method:

$$\begin{aligned} \frac{dy}{dx} &= f(x), x_0 = a \text{ and } y_0 = b \\ \Rightarrow x_{n+1} &= x_n + h, \quad y_{n+1} = y_n + hf(x_n) \end{aligned}$$

acceleration:

$$a = \frac{d^2x}{dx^2} = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

constant (uniform) acceleration: $v = u + at, s = ut + \frac{1}{2}at^2, v^2 = u^2 + 2as, s = \frac{1}{2}(u + v)t$

Vectors in two and three dimensions

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} = r$$

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos(\theta) = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

Mechanics

momentum:

$$\mathbf{p} = m\mathbf{v}$$

equation of motion:

$$\mathbf{R} = m\mathbf{a}$$

friction:

$$F \leq \mu N$$

Multiple Choice Answer Sheet

Student Name:

Shade the letter that corresponds to each correct answer.

Question	A	B	C	D	E
Question 1	☐	☐	☐	☐	☐
Question 2	☐	☐	☐	☐	☐
Question 3	☐	☐	☐	☐	☐
Question 4	☐	☐	☐	☐	☐
Question 5	☐	☐	☐	☐	☐
Question 6	☐	☐	☐	☐	☐
Question 7	☐	☐	☐	☐	☐
Question 8	☐	☐	☐	☐	☐
Question 9	☐	☐	☐	☐	☐
Question 10	☐	☐	☐	☐	☐
Question 11	☐	☐	☐	☐	☐
Question 12	☐	☐	☐	☐	☐
Question 13	☐	☐	☐	☐	☐
Question 14	☐	☐	☐	☐	☐
Question 15	☐	☐	☐	☐	☐
Question 16	☐	☐	☐	☐	☐
Question 17	☐	☐	☐	☐	☐
Question 18	☐	☐	☐	☐	☐
Question 19	☐	☐	☐	☐	☐
Question 20	☐	☐	☐	☐	☐
Question 21	☐	☐	☐	☐	☐
Question 22	☐	☐	☐	☐	☐

Solution Pathway

Below are sample answers. Please consider the merit of alternative responses.

Specialist Mathematics Exam 2: SOLUTIONS**Section 1: Multiple-choice Answers**

1. C	2. E	3. A	4. D	5. B
6. D	7. E	8. B	9. B	10. B
11. B	12. A	13. A	14. C	15. C
16. E	17. E	18. D	19. E	20. D
21. C	22. B			

Section 1: Multiple-choice Solutions**Question 1**

$$\text{Domain arccos} = [-1, 1] \Rightarrow -1 \leq 2x \leq 1 \Rightarrow x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \text{ Answer: C}$$

Question 2

$$\operatorname{cosec}^2(t) - \cot^2(t) = 1 \Rightarrow \frac{(y+1)^2}{4} - \frac{(x-1)^2}{9} = 1 \text{ Answer: E}$$

Question 3

The denominator $ax^2 + bx + c$ has intercepts at $x = -2$ and $x = 4$ and a t. p. of -18 at $x = 1$
 $\Rightarrow a(x+2)(x-4)|_{x=1} = -18 \Rightarrow a = 2$

Finally, expanding gives $ax^2 + bx + c = 2x^2 - 4x - 16 \Rightarrow a = 2, b = -4 \text{ and } c = -16 \text{ Answer: A}$

Question 4

$$\frac{1}{\tan\left(\frac{\pi}{2} - x\right)} = \frac{1}{\cot(x)} \neq \cot(x) \text{ Answer: D}$$

Question 5

A region of the complex plane inside the circle of radius a centred at the origin is

$$|z| < a \Rightarrow \sqrt{z\bar{z}} < a \Rightarrow z\bar{z} < a^2 \text{ Answer: B}$$

Question 6

$$\operatorname{Im}\left(\operatorname{cis}\left(\frac{5\pi}{6}\right) + i\right) = \operatorname{Im}\left(\cos\left(\frac{5\pi}{6}\right) + i\left(\sin\left(\frac{5\pi}{6}\right) + 1\right)\right) = \frac{3}{2} \text{ Answer: D}$$

Question 7

$$\frac{(\bar{z})^2}{z} = \frac{(\overline{r\operatorname{cis}(\theta)})^2}{r\operatorname{cis}(\theta)} = \frac{(r\operatorname{cis}(-\theta))^2}{r\operatorname{cis}(\theta)} = \frac{r^2\operatorname{cis}(-2\theta)}{r\operatorname{cis}(\theta)} = r\operatorname{cis}(-3\theta) \text{ Answer: E}$$

Question 8

$$z^2 = 1 - i \Rightarrow r^2\operatorname{cis}(2\theta) = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4} + 2k\pi\right), k = 0, 1 \text{ Solving for } \theta$$

$$\theta = \frac{1}{2}\left(-\frac{\pi}{4} + 2k\pi\right), k = 0, 1 \Rightarrow \theta \in \left\{-\frac{\pi}{8}, \frac{7\pi}{8}\right\} \text{ Answer: B}$$

Question 9

$$e^{-\cot(x)} = \frac{1}{e^{\cot(x)}} \Rightarrow u = \cot(x), \frac{du}{dx} = -\operatorname{cosec}^2(x) = -(\cot^2(x) + 1)$$

$$\text{when } x = \frac{\pi}{4}, u = 1 \text{ and when } x = \frac{\pi}{2}, u = 0$$

$$\therefore \int_{\pi/4}^{\pi/2} e^{-\cot(x)}(\cot^2(x) + 1) dx = -\int_1^0 \frac{1}{e^u} du = \int_0^1 \frac{1}{e^u} du \text{ Answer: B}$$

Question 10

$$y = x^{\frac{5}{3}} \Rightarrow x = y^{\frac{3}{5}}$$

$$V = \pi \int_0^2 x^2(y) dy = \pi \int_0^2 y^{\frac{6}{5}} dy = \pi \frac{20}{11} 2^{1/5} \text{ Answer: B}$$

Question 11

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Exact value $y(0.2) = 1 + \int_0^{0.2} \tan(x) \, dx = 1 - \log_e(\cos(0.2)) \approx 1.020$

Euler's estimate $y(0.2) = 1 + 0.1 \tan(0) + 0.1 \tan(0.1) \approx 1.0100$ **Answer: B**

Question 12

If $f(x, y) = x/y$ then $f(-x, y) = -f(x, y)$ and $f(x, -y) = -f(x, y) \Rightarrow$ **Answer: A**

Question 13

$$\ddot{x}dx = \frac{dv}{dt} \frac{dx}{dt} dt = \frac{dx}{dt} \frac{dv}{dt} dt = v dv \quad \text{Answer: A}$$

Question 14

By inspection $\mathbf{a} + \mathbf{b} = \mathbf{c} \Rightarrow$ **Answer: C**

Question 15

If $(3, 0, 4)$, $Q(0, -2, 1)$ then $\overrightarrow{PQ} = -3\mathbf{i} - 2\mathbf{j} - 3\mathbf{k} \Rightarrow |PQ| = \sqrt{22}$ **Answer: C**

Question 16

$$\frac{dm}{dt} = 5 \times 8 - \frac{m}{100 + (8 - 10) \times t} \quad \text{Answer: E}$$

Question 17

Clearly, $\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$ **Answer: E**

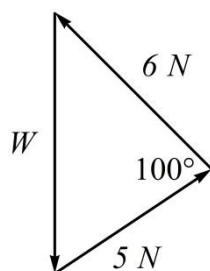
Question 18

Weight is in same direction as acceleration, and normal reaction force is in the opposite direction to acceleration. **Answer: D**

Question 19

Up is positive, weight and air resistance are down, so they are negative in Newton's Law.

$$\Rightarrow ma = -mg - kv^2, \text{ but } a = v \frac{dv}{dx} \Rightarrow \text{Answer: E}$$

Question 20

Application of the cosine rule to the triangle above gives

$$W = \sqrt{5^2 + 6^2 - 2 \times 5 \times 6 \cos(100^\circ)} \quad \textbf{Answer: D}$$

Question 21

$$a = \frac{v^2 - u^2}{2 \times s} = \frac{8^2 - 10^2}{18} = -2\text{ms}^{-2}$$

$$\text{magnitude of the frictional force} = m|a| = \frac{2}{5} \quad \textbf{Answer: C}$$

Question 22

$$\text{acceleration} = v \frac{dv}{dx} = \frac{\arctan(x)}{x^2 + 1} \quad \textbf{Answer: B}$$

Section 2: Extended Answer Solutions

Question 1 (11 marks)

a. (2 marks)

$$\sec(t) = \frac{x-1}{2} \text{ and } \tan(t) = \frac{y+2}{3} \quad (\text{M1})$$

$$\sec^2(t) - \tan^2(t) = 1 \Rightarrow \frac{(x-1)^2}{4} - \frac{(y+2)^2}{9} = 1 \quad (\text{A1})$$

b. (2 marks)

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3}{2 \sin(t)} \quad (\text{M1})$$

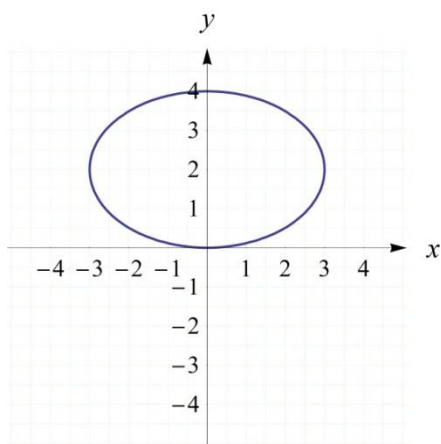
$$\frac{dy}{dx} = \frac{3}{2 \sin(t)} = 3 \Rightarrow t = \frac{\pi}{6} \quad (\text{A1})$$

c. (2 marks)

$$\frac{dy}{dx} = \frac{4x}{9(y-2)} \quad (\text{A1})$$

$$\frac{dy}{dx} = \infty \Rightarrow y = 2 \quad (\text{A1})$$

d.



Shape (A1)

Correct Vertices (A1)

e. i. (2 marks)

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$$V = \pi \int_1^3 9(1 - (y - 2)^2/4) dy$$

Integrand(A1)

Limits (A1)

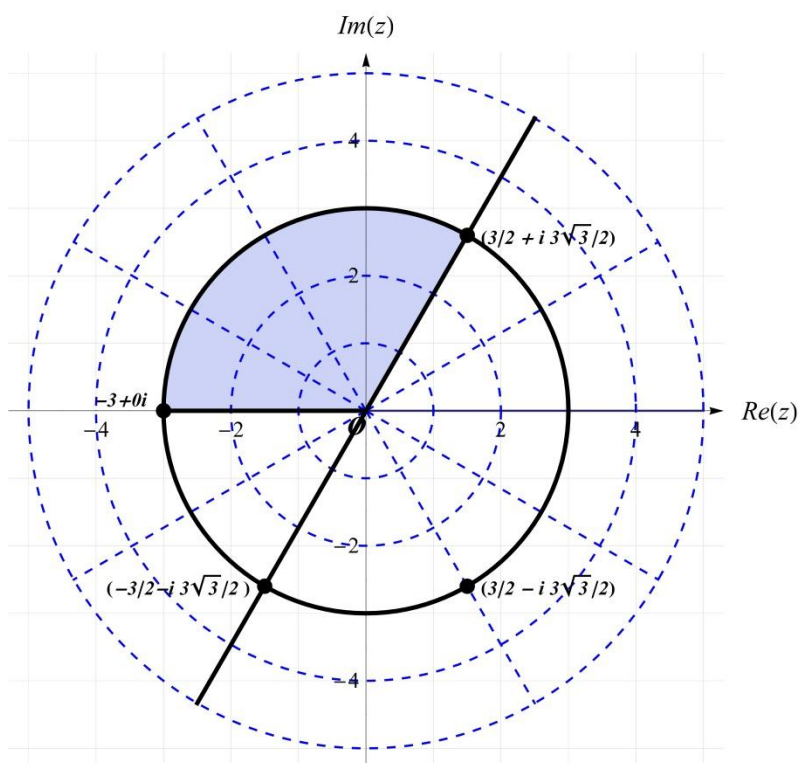
ii. (1 mark)

$$\frac{33\pi}{2}$$

(A1)

Question 2 (12 marks)

a. (3 marks)



Circle – correct centre and radius

(A1)

Straight Line

Shape(A1)

Slope (A1)

b. (2 marks)

$$\frac{3}{2} + i \frac{3\sqrt{3}}{2} \text{ and } -\frac{3}{2} - i \frac{3\sqrt{3}}{2}$$

(A2)

c. (2 marks)

$$-3 \text{ and } \frac{3}{2} - i \frac{3\sqrt{3}}{2}$$

(A2)

d. (1 mark)

Totally correct (A1)

e. (1 mark)

$$(z + 3) \left(z - \frac{3}{2} - i \frac{3\sqrt{3}}{2} \right) \left(z - \frac{3}{2} + i \frac{3\sqrt{3}}{2} \right) \quad (\text{A1})$$

f. (1 mark)

Totally correct (A1)

h. (2 marks)

$$\text{Area} = \frac{(\frac{2\pi}{3})}{2\pi} \times \pi \times 3^2 \quad (\text{M1})$$

$$\text{Area} = 3\pi \text{ square units} \quad (\text{A1})$$

Question 3 (11 marks)**a.** (2 marks)

Differentiating both sides with respect to time and using the chain rule

$$\frac{d}{dt} \log_e(n) = \frac{d}{dt} (5 - 4e^{-6t/5}) \Rightarrow \frac{1}{n} \frac{dn}{dt} = \frac{24}{5} e^{-6t/5} \quad (\text{M1})$$

$$\text{but } \frac{24}{5} e^{-6t/5} = \frac{6}{5} (5 - \log_e(n)) \Rightarrow \frac{1}{n} \frac{dn}{dt} = \frac{6}{5} (5 - \log_e(n)) \quad (\text{M1})$$

b. (1 mark)

$$t = 0 \Rightarrow \log_e(n(0)) = 5 - 4e^{-6 \times 0/5} \Rightarrow n(0) = e \Rightarrow 1000 \times e \approx 2718 \text{ individuals} \quad (\text{A1})$$

c. (2 marks)

$$\lim_{t \rightarrow \infty} \log_e(n) = \lim_{t \rightarrow \infty} (5 - 4e^{-6t/5}) = 5 \quad (\text{M1})$$

$$\text{Number of individuals eventually infected } 1000 \times e^5 \approx 148413 \quad (\text{A1})$$

d. i. (2 marks)Differentiating both sides of $\frac{dn}{dt} = \frac{6}{5} n (5 - \log_e(n))$ with respect to t and using the chain rule gives

$$\frac{d^2n}{dt^2} = \frac{d}{dn} \left(\frac{6}{5} n (5 - \log_e(n)) \right) \frac{dn}{dt} = \frac{6}{5} (4 - \log_e(n)) \frac{dn}{dt} \quad (\text{M1})$$

$$\text{Substituting } \frac{dn}{dt} = \frac{6}{5} n (5 - \log_e(n)) \text{ in the preceding equation gives the required result} \quad (\text{M1})$$

ii. (2 marks)At a point of inflection, $\frac{d^2n}{dt^2} = \frac{6}{5} n (4 - \log_e(n)) (5 - \log_e(n))$ changes sign

$$\begin{array}{ccccc} 0 & < n < & e^4 & < n < & e^5 \\ \frac{6}{5}n & & (4 - \log_e(n)) & & (5 - \log_e(n)) \end{array}$$

$$\frac{d^2n}{dt^2} = 0 \quad \frac{d^2n}{dt^2} > 0 \quad \frac{d^2n}{dt^2} = 0 \quad \frac{d^2n}{dt^2} < 0 \quad \frac{d^2n}{dt^2} = 0$$

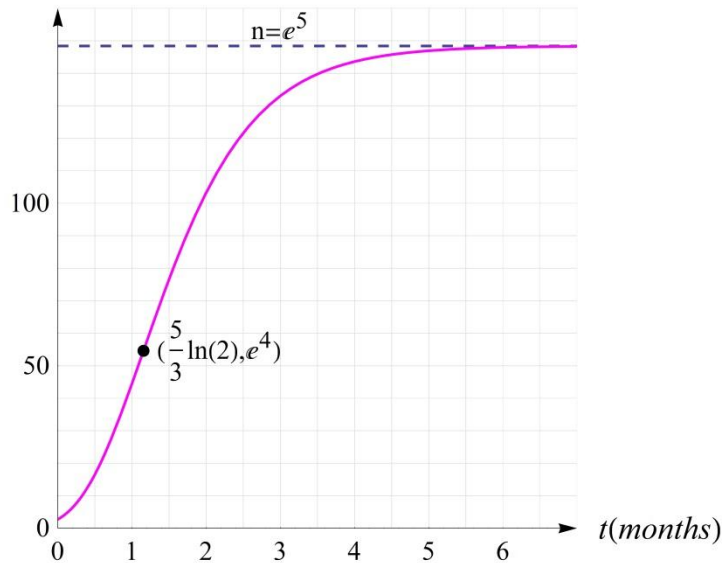
Or using a sign diagram shows that $\frac{d^2n}{dt^2}$ changes sign at $n = e^4$ (M1)

$$\text{Solving } \log_e(n) = 5 - 4e^{-6t/5} \text{ for } t \text{ when } n = e^4 \text{ gives } t = \frac{5}{3} \log_e(2)$$

$$\therefore (t, n) = \left(\frac{5}{3} \log_e(2), e^4\right) \quad (\text{A1})$$

e. (2 marks)

$n(\text{thousands})$



asymptote

(A1)

shape

(A1)

Question 4 (12 marks)

a. (2 marks)

Vector component parallel to $\mathbf{m}$ is $\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}$ (A1)

Vector component perpendicular to $\mathbf{m}$ is $-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$ (A1)

b. (2 marks)

$$\theta = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{m}}{\sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{m} \cdot \mathbf{m}}}\right) \quad (\text{M1})$$

$$\theta = \frac{\pi}{4} \quad (\text{A1})$$

c. (2 marks)

$$\frac{\mathbf{b} \cdot \mathbf{m}}{\sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{m} \cdot \mathbf{m}}} = \frac{4+\beta}{\sqrt{2} \sqrt{32+\beta^2}} = \cos\left(\frac{\pi}{4}\right) \quad (\text{M1})$$

Solving for β gives $\beta = 2$ (A1)

d. (2 marks)

$$18\mathbf{i} + 18\mathbf{j} + 0\mathbf{k} = 6(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) + \alpha(4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) \quad (\text{M1})$$

Equating components on both sides gives $\alpha = 3$.

e. (2 marks)

cosine of the angle between vectors $\mathbf{p}$ and $\mathbf{q}$ is

$$\frac{\mathbf{p} \cdot (|\mathbf{r}| \mathbf{p} + |\mathbf{p}| \mathbf{r})}{|\mathbf{p}| (|\mathbf{r}| \mathbf{p} + |\mathbf{p}| \mathbf{r})} \quad (\text{M1})$$

$$= \frac{(|\mathbf{r}| \mathbf{p} \cdot \mathbf{p} + |\mathbf{p}| \mathbf{p} \cdot \mathbf{r})}{|\mathbf{p}| (|\mathbf{r}| \mathbf{p} + |\mathbf{p}| \mathbf{r})} = \frac{(|\mathbf{r}| |\mathbf{p}| + \mathbf{p} \cdot \mathbf{r})}{|\mathbf{r}| \mathbf{p} + |\mathbf{p}| \mathbf{r}} \quad (\text{A1})$$

f. (2 marks)

cosine of the angle between vectors $\mathbf{r}$ and $\mathbf{q}$ is

$$\frac{\mathbf{r} \cdot (|\mathbf{r}| \mathbf{p} + |\mathbf{p}| \mathbf{r})}{|\mathbf{r}| (|\mathbf{r}| \mathbf{p} + |\mathbf{p}| \mathbf{r})} = \frac{(|\mathbf{r}| \mathbf{r} \cdot \mathbf{p} + |\mathbf{p}| \mathbf{r} \cdot \mathbf{r})}{|\mathbf{r}| (|\mathbf{r}| \mathbf{p} + |\mathbf{p}| \mathbf{r})} = \frac{(|\mathbf{r}| |\mathbf{p}| + \mathbf{p} \cdot \mathbf{r})}{|\mathbf{r}| \mathbf{p} + |\mathbf{p}| \mathbf{r}} \quad (\text{M1})$$

cosine of the angle between vectors $\mathbf{p}$ and $\mathbf{q}$ = cosine of the angle between vectors $\mathbf{r}$ and $\mathbf{q}$

$\therefore \mathbf{q}$ bisects the angle between vectors $\mathbf{p}$ and $\mathbf{r}$ (A1)

Question 5 (12 marks)

a. (3 marks)

Applying Newtons 2nd Law to the smaller mass, taking positive in the direction up the plane gives

$$\tau - mg \sin(30^\circ) - \frac{mg}{\sqrt{3}} \cos(30^\circ) = m a \quad (\text{M1})$$

Simplifying

$$\tau - mg = m a \quad (1) \quad (\text{M1})$$

Applying Newtons 2nd Law to the heavier mass, taking positive in the downward direction gives

$$8mg - \tau = 8ma \quad (2) \quad (\text{A1})$$

b. (1 mark)

Solving equations (1) and (2) gives

$$a = \frac{7}{9} g \text{ ms}^{-2} \quad (\text{A1})$$

c. (2 marks)

Attempt to use the constant acceleration formula $V^2 = u^2 + 2 a y$ (M1)

$$u = 0, a = \frac{7}{9} g \Rightarrow V = \frac{\sqrt{14gy}}{3} \text{ ms}^{-1} \text{ downwards} \quad (\text{A1})$$

d. Find the speed $v \text{ ms}^{-1}$ of the lighter block m , just before it reaches the pulley at the top of the ramp (4 marks)

From equation (1), $a = -g$ after the heavier block reaches ground level (A1)

Attempt to use the constant acceleration formula $v^2 = V^2 + 2a(x - y)$ (M1)

Attempt to substitute $a = -g$ and $V = \frac{\sqrt{14gy}}{3}$ in preceding equation (M1)

Simplifying

$$v = \sqrt{\frac{14gy}{9} - 2g(x - y)} \quad (\text{A1})$$

e. (2 marks)

If block comes to rest at top of ramp, $v = 0$ in preceding equation (M1)

Solving

$$\frac{14gy}{9} - 2g(x - y) = 0 \Rightarrow x = \frac{16y}{9} \quad (\text{M1})$$