Neap:

Trial Examination 2014

VCE Specialist Mathematics Units 3&4

Written Examination 2

Suggested Solutions

SECTION 1

Question					
1	A	B	C	D	**E**
2	A	B	C	D	**E**
3	A	**B**	C	D	E
4	A	B	**C**	D	E
5	A	B	C	D	**E**
6	**A**	B	C	D	E
7	A	B	C	**D**	E
8	A	B	**C**	D	E
9	A	**B**	C	D	E
10	**A**	B	C	D	E
11	A	B	C	**D**	E
12	A	B	**C**	D	E
13	A	**B**	C	D	E
14	A	**B**	C	D	E
15	A	B	C	D	**E**
16	A	B	C	**D**	E
17	A	B	**C**	D	E
18	A	B	C	**D**	E
19	A	**B**	C	D	E
20	**A**	B	C	D	E
21	A	B	C	**D**	E
22	**A**	B	C	D	E

SECTION 1

Question 1 **E**

The maximum and minimum values of y occur when $x = -3$, that is, when $\frac{(y-3)^2}{6} = k$.
Solving this equation for y gives $y = 3 \pm \sqrt{6k}$.
Hence the maximum value is $3 + \sqrt{6k}$.

Question 2 **E**

If $x^2 + bx - c = 0$ has two solutions then the graph of f has two vertical asymptotes.

If $x^2 + bx - c = 0$ has two solutions then $\Delta > 0$.

$b^2 - 4(1)(-c) > 0$ and so $b^2 + 4c > 0$.

So $b^2 > -4c$.

Question 3 **B**

Vertical asymptotes occur for values of x such that $\sin(2x) = 0$.

Hence $2x = n\pi$, that is, $x = \frac{n\pi}{2}$.

Question 4 **C**

$$\sin(x) = \pm\sqrt{1 - \left(\frac{1}{10}\right)^2}$$
$$= \pm\frac{\sqrt{99}}{10}$$

As $\frac{\pi}{2} < x < \pi$, $\sin(x)$ is positive.

$$\sin(x) = \frac{\sqrt{99}}{10}$$
$$= \frac{3\sqrt{11}}{10}$$

As $\operatorname{cosec}(x) = \frac{1}{\sin(x)}$, we obtain $\operatorname{cosec}(x) = \frac{10}{3\sqrt{11}}$.

Question 5 **E**

$$h'(x) = f'(g(x))g'(x)$$
$$h''(x) = f''(g(x))g'(x)g'(x) + f'(g(x))g''(x)$$

So $h''(x) = f''(g(x))(g'(x))^2 + f'(g(x))g''(x)$.

Question 6 **A**

$$\begin{aligned} z &= \frac{i}{2-i} \\ &= \frac{i}{2-i} \times \frac{2+i}{2+i} \\ &= \frac{-1+2i}{5} \end{aligned}$$

So $x = -\frac{1}{5}$ and $y = \frac{2}{5}$.

Question 7 **D**

$$\begin{aligned} (1+i)(x+yi) + (1-i)(x-yi) &= 6 \\ x + yi + xi + i^2 y + x - yi - xi + i^2 y &= 6 \\ x + yi + xi - y + x - yi - xi - y &= 6 \\ 2x - 2y &= 6 \end{aligned}$$

So $y = x - 3$.

Question 8 **C**

If $z = \cos(\theta) + i\sin(\theta)$ then $z^n = \cos(n\theta) + i\sin(n\theta)$.

If $\frac{1}{z} = \cos(\theta) - i\sin(\theta)$ then $\frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta)$.

$$\begin{aligned} z^n - \frac{1}{z^n} &= \cos(n\theta) + i\sin(n\theta) - (\cos(n\theta) - i\sin(n\theta)) \\ &= 2i\sin(n\theta) \end{aligned}$$

Question 9 **B**

Looking at the direction field, all the gradients along the diagonal with equation $y = -x$ appear to be approaching zero.

Question 10 **A**

There is a repeated linear factor in the denominator, that is, $x^2 + 6x + 9 = (x+3)^2$, so the partial fraction form is $\frac{A}{(x+3)} + \frac{B}{(x+3)^2}$.

Question 11 **D**

The amount of dissolved chemical at t minutes is y grams.
So the amount of undissolved chemical at t minutes is $(8 - y)$ grams.
As the chemical dissolves at a rate equal to 10% of $(8 - y)$ per minute, then $\frac{dy}{dt} = \frac{8-y}{10}$.

Question 12 C

Let V be the volume.

Using $V = \pi\int_a^b y^2\,dx$ we obtain:

$$V = \pi\int_0^{\frac{\pi}{4}} \sec^2(x)\,dx$$

$$= \pi[\tan(x)]_0^{\frac{\pi}{4}}$$

$$= \pi\left[\tan\left(\frac{\pi}{4}\right) - \tan(0)\right]$$

$$= \pi$$

Question 13 B

Let $u = \sin(x)$ and so $\frac{du}{dx} = \cos(x)$.

When $x = 0,\ u = 0$ and when $x = \frac{\pi}{3},\ u = \frac{\sqrt{3}}{2}$.

$$\int_0^{\frac{\pi}{3}} \frac{\cos(x)}{1+\sin^2(x)}dx = \int_0^{\frac{\pi}{3}} \frac{\frac{du}{dx}}{1+u^2}dx$$

$$= \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{1+u^2}du$$

Question 14 B

$$\left|\overrightarrow{AB}\right|\cos(\theta) = \frac{\overrightarrow{AB}\cdot \underset{\sim}{v}}{|\underset{\sim}{v}|}$$

$$= \frac{(-2\underset{\sim}{i} - 11\underset{\sim}{j} + 9\underset{\sim}{k})\cdot(\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k})}{\sqrt{1^2 + (-2)^2 + (-2)^2}}$$

$$= \frac{-2 + 22 - 18}{3}$$

$$= \frac{2}{3}$$

Question 15 E

Two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, are linearly dependent if they are parallel.

If $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$, then $\underset{\sim}{a} = k\underset{\sim}{b}$, $k \neq 0$.

This is the case in **E**, where $\underset{\sim}{a} = -\frac{1}{3}\underset{\sim}{b}$.

Question 16 D

$$\overrightarrow{OP} = 300\underset{\sim}{j} + (200\cos(30°)\underset{\sim}{i} + 200\sin(30°)\underset{\sim}{j})$$
$$= 100\sqrt{3}\underset{\sim}{i} + (300 + 100)\underset{\sim}{j}$$
$$= 100\sqrt{3}\underset{\sim}{i} + 400\underset{\sim}{j}$$

Question 17 C

The parametric equations are:

$x = t - 1$ (1)

$y = 4(t - 1)^2$ (2)

Substituting (2) into (1) we obtain $y = 4x^2$.

If $t \geq 0$ then from (1) we obtain $x \geq -1$.

Question 18 D

The parametric equations are $x = t^3 - 2t^2 - 5$ and $y = t^4 + 2t^2 - 8t$.

So $\frac{dx}{dt} = 3t^2 - 4t$ and $\frac{dy}{dt} = 4t^3 + 4t - 8$.

Using $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ we obtain $\frac{dy}{dx} = \frac{4t^3 + 4t - 8}{t(3t - 4)}$.

Vertical tangents occur when $\frac{dy}{dx}$ is undefined.

This occurs for $t = 0$ and $t = \frac{4}{3}$ only.

Question 19 B

$v^2 = 36x - 4x^2$ and so $\frac{1}{2}v^2 = 18x - 2x^2$.

Using $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ we obtain $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 18 - 4x$.

So $a = 18 - 4x$.

When $x = 9$:

$a = 18 - (4)(9)$

$= -18$

So the acceleration is –18 m/s^2.

Question 20 A

Let the normal reaction force exerted by the lift floor on the man be R newtons.

The equation of motion is $85g - R = 85a$.

So $R = 85(g - a)$.

As the downward acceleration is 3 m/s^2, we obtain $R = 85(g - 3)$.

Question 21 D

The initial momentum (p_i) is 0 (kg m/s).

To calculate the final momentum (p_f) we need to find the particle's final velocity.

Taking downwards as a positive, we have $u = 0$, $a = 9.8$ and $t = 2$.

Using $v = u + at$ we obtain:

$v = 0 + (9.8)(2)$

$= 19.6$

The final momentum (p_f) is 0.25 × 19.6, that is, 4.9 (kg m/s).

Change in momentum $(\Delta p) = p_f - p_i$

$= 4.9$ (kg m/s)

Question 22 A

Let the tension in the string be T newtons.

Resolving forces vertically:

$mg = T\sin(\alpha) + T\sin(\alpha)$

$mg = 2T\sin(\alpha)$

So $T = \frac{mg}{2\sin(\alpha)}$.

SECTION 2

Question 1 (9 marks)

a. $3.8 - 9.8t = 0 \Rightarrow t = 0.3877\ldots$ (s) A1

$y(t) = 11.6 + 3.8t - 4.9t^2$ M1

$y(0.3877\ldots) = 12.34$ (m) (correct to two decimal places) A1

b. Solving $11.6 + 3.8T - 4.9T^2 = 0$ for T gives $T = 1.97$ (s) (correct to two decimal places). M1 A1

c. $d = \int_0^{1.9744\ldots} \sqrt{(0.9)^2 + (3.8 - 9.8t)^2}\,dt$ M1

$d = 13.3$ (m) (correct to one decimal place) A1

d. When $t = 1.9744\ldots$, $\alpha = \tan^{-1}\left(\frac{3.8 - 9.8(1.9744\ldots)}{0.9}\right)$. M1

So $\alpha = 86.7°$ (correct to the nearest tenth of a degree). A1

Question 2 (12 marks)

a. As $a, b, c \in R$ and $k \neq 0$, the complex linear factors of $P(z)$ occur in conjugate pairs, that is, $(z - ki)$ and $(z + ki)$ are both complex linear factors of $P(z)$. A1

b. **Method 1**

$P(ki) = 0 \Rightarrow k^4 - ak^3 i + bki + c = 0$ M1

Equating imaginary parts we obtain $-ak^3 + bk = 0 \Rightarrow bk = ak^3$.

As $k \neq 0$, $b = ak^2$. A1

Award A1 only if $-ak^3 + bk$ or $bk = ak^3$ are seen.

Award as above for $P(-ki) = 0 \Rightarrow k^4 + ak^3 i - bki + c = 0$, leading to $ak^3 - bk = 0 \Rightarrow bk = ak^3$.

OR

Method 2

$P(z) = (z^2 + k^2)(z^2 + mz + n)$ M1

Equating coefficients of z^3 we obtain $a = m$.
Equating coefficients of z we obtain $b = k^2 m$.
So $b = ak^2$. A1

Award A1 only if $a = m$ and $b = ak^2 m$ are seen.

c. $P(ki) = 0 \Rightarrow k^4 - ak^3 i + bki + c = 0$

Equating real parts we obtain $k^4 + c = 0$. M1

$k^2 = \frac{b}{a} \Rightarrow \frac{b^2}{a^2} + c = 0$

So $b^2 + a^2 c = 0$. A1

d. Solving $P(2) = 0$ for b with $c = -\frac{b^2}{a^2}$. M1 A1

So $b = 2a(a + 2)$ or $b = -4a$, that is, b is an even number. A1

e. If $W(u) = 0$, then $mu^3 + nu^2 + pu + q = 0$.

$\overline{mu^3 + nu^2 + pu + q} = \bar{0}$ (taking the conjugate of both sides) M1

$\overline{mu^3} + \overline{nu^2} + \overline{pu} + \bar{q} = \bar{0}$ ($\overline{z_1 + z_2 + z_3 + \ldots} = \bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \ldots$) A1

$\bar{a} = a$ when $a \in R$ and $(\bar{z})^n = \overline{z^n}$. A1

So $m(\bar{u})^3 + n(\bar{u})^2 + p(\bar{u}) + q = 0$ and $P(\bar{u}) = 0$. A1

Question 3 (13 marks)

a. The equations of motion are $60g\cos(30°) - F_r = 60a$ and $N - 60g\sin(30°) = 0$. A1

Attempting to solve for a with $F_r = 15g\sin(30°)$ (or equivalent). M1

$a = \frac{g(4\sqrt{3} - 1)}{8}$ (m/s^2) A1

b. Use of $v^2 = u^2 + 2as$ with $u = 0$, $a = \frac{g(4\sqrt{3} - 1)}{8}$ and $s = 30$. M1

$v = \frac{\sqrt{30g(4\sqrt{3} - 1)}}{2}$ (m/s) A1

c. The equations of motion are $-F_r = 60a$ and $N - 60g = 0$. A1

$a = -\frac{g}{4}$ (m/s^2) M1

Use of $V^2 = u^2 + 2as$ with $u = \frac{\sqrt{30g(4\sqrt{3} - 1)}}{2}$, $a = -\frac{g}{4}$ and $s = 15$. M1

$V = \sqrt{15g(2\sqrt{3} - 1)}$ (m/s) A1

d. Use of $y = ut + \frac{1}{2}at^2$ with $y = 1.5$, $u = 0$ and $a = g$ to obtain $1.5 = \frac{1}{2}gt^2$. M1

$t = \sqrt{\frac{3}{g}}$ (s) A1

Use of $x = Vt$ with $V = \sqrt{15g(2\sqrt{3} - 1)}$ and $t = \sqrt{\frac{3}{g}}$. M1

$x = 10.5$ (m) (correct to one decimal place) A1

Question 4 (11 marks)

a. $V = \pi\int_0^{\pi}(3\cos(2y) + 4)^2 dy$ A1

$= \frac{41\pi^2}{2}$ (m^3) A1

b. attempting to use $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$ with $\frac{dV}{dt} = 2$ M1

$\frac{dV}{dh} = \frac{d}{dh}\left(\pi\int_0^h (3\cos(2y) + 4)^2 dy\right)$ M1

So $\frac{dV}{dh} = \pi(3\cos(2h) + 4)^2$. A1

$\frac{dh}{dt} = \frac{2}{\pi(3\cos(2h) + 4)^2}$ A1

When $h = \frac{\pi}{4}$, $\frac{dh}{dt} = \frac{1}{8\pi}$ (m/min) A1

c. $\frac{dh}{dt} = \frac{2}{\pi(3\cos(2h) + 4)^2}$ A1

using either integration or a differential equation solver with $t = 0$ when $h = 0$ M1

$t = \frac{\pi}{16}(9\sin(4h) + 4(24\sin(2h) + 41h))$ (or equivalent) A1

When $h = \frac{\pi}{4}$, $t = 44.1$ (min). A1

Question 5 (13 marks)

a. $\theta = \cos^{-1}\left(\frac{(\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}) \cdot (3\underset{\sim}{i} - 6\underset{\sim}{j} + 6\underset{\sim}{k})}{|\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}||3\underset{\sim}{i} - 6\underset{\sim}{j} + 6\underset{\sim}{k}|}\right)$ M1 A1

$= \cos^{-1}\left(-\frac{9}{9\sqrt{11}}\right)$

$= \cos^{-1}\left(-\frac{1}{\sqrt{11}}\right)$ A1

b. $A = \frac{1}{2}|\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}||3\underset{\sim}{i} - 6\underset{\sim}{j} + 6\underset{\sim}{k}|\sin\left(\cos^{-1}\left(-\frac{1}{\sqrt{11}}\right)\right)$ M1

$= \frac{9\sqrt{10}}{2}$ A1

c. $\overrightarrow{RM} = (-1 - 2\lambda)\underset{\sim}{i} + (7 + 4\lambda)\underset{\sim}{j} + (-3 - 4\lambda)\underset{\sim}{k}$ and $\overrightarrow{RQ} = -2\underset{\sim}{i} + 9\underset{\sim}{j} - 5\underset{\sim}{k}$. A1

$\left|\frac{((-1 - 2\lambda)\underset{\sim}{i} + (7 + 4\lambda)\underset{\sim}{j} + (-3 - 4\lambda)\underset{\sim}{k}) \cdot (-2\underset{\sim}{i} + 9\underset{\sim}{j} - 5\underset{\sim}{k})}{|-2\underset{\sim}{i} + 9\underset{\sim}{j} - 5\underset{\sim}{k}|}\right| = \sqrt{110}$ M1 A1

attempting to solve the above equation for λ M1

$\lambda = -\frac{19}{6}$ or $\lambda = \frac{1}{2}$ A1

d. $\overrightarrow{MP} = -2\lambda(-\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$ and $\overrightarrow{PQ} = -\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$. A1 A1

$\overrightarrow{MP} = -2\lambda(\overrightarrow{PQ})$ and so M, P and Q are collinear for $\lambda \in R$. A1

