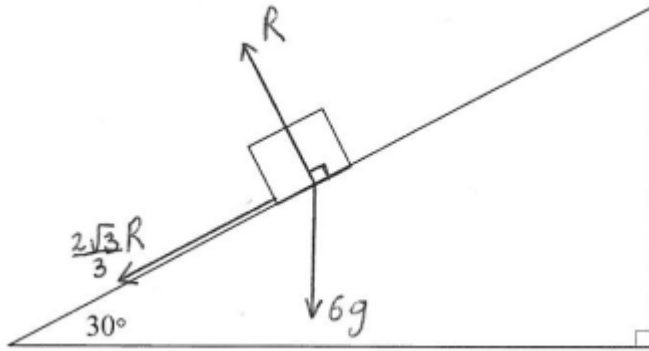


SPECIALIST MATHEMATICS 2014

Trial Written Examination 1 - SOLUTIONS

Question 1

a.



All three forces (it is acceptable for the friction force to be labelled F_{friction} etc.) [A1]

Total 1 mark

Comment:

The forces acting on the body are:

- Normal reaction force R perpendicular to the plane.
- Weight force $= mg = 6g$ down.
- Friction force of size $\mu R = \frac{2\sqrt{3}}{3}R$ (because the body is sliding) and the force is acting down the plane because the body is sliding up the plane and the friction force opposes the motion.

b.

- Resolve forces parallel to the plane (the direction of motion of the body ‘up the plane’ is taken as the positive direction):

$$F_{net} = ma = 6a.$$

$$F_{net} = -\mu R - mg \sin(30^\circ) = \frac{-2\sqrt{3}}{3}R - 3g.$$

$$\text{Therefore: } 6a = \frac{-2\sqrt{3}}{3}R - 3g. \quad \dots (1) \quad [\text{A1}]$$

- Resolve forces perpendicular to the plane (upwards is taken as the positive direction):

$$F_{net} = 0.$$

$$F_{net} = R - mg \cos(30^\circ) = R - 3\sqrt{3}g.$$

$$\begin{aligned} \text{Therefore: } 0 &= R - 3\sqrt{3}g \\ \Rightarrow R &= 3\sqrt{3}g. \quad \dots (2) \end{aligned}$$

Substitute equation (2) into equation (1):

$$6a = \frac{-2\sqrt{3}}{3}(3\sqrt{3}g) - 3g = -9g$$

$$\Rightarrow a = -\frac{3g}{2}. \quad [\text{A1}]$$

Substitute $a = -\frac{3g}{2}$, $u = 3 \text{ m/s}$ and $v = 0 \text{ m/s}$ into $2as = v^2 - u^2$ and solve for s :

$$s = \frac{3}{g} \text{ metres.} \quad [\text{A1}]$$

Total 4 marks

Question 2**a.****Method 1:** Convert the complex numbers into polar form.

$$u = -2\sqrt{3} - 2i = 4\text{cis}\left(-\frac{5\pi}{6}\right).$$

Only the argument is required.

[A1]

$$v = -1 + i\sqrt{3} = 2\text{cis}\left(\frac{2\pi}{3}\right).$$

Only the argument is required.

[A1]

$$-\frac{5\pi}{6} - \frac{2\pi}{3} = -\frac{9\pi}{6} = -\frac{3\pi}{2}.$$

Therefore:

$$\text{Arg}\left(\frac{u}{v}\right) = -\frac{3\pi}{2} + 2\pi = \frac{\pi}{2}.$$

[A1]**Total 3 marks****Method 2:**

$$\frac{-2\sqrt{3} - 2i}{i\sqrt{3} - 1} = \frac{(-2\sqrt{3} - 2i)}{(-1 + i\sqrt{3})} \times \frac{(-1 - i\sqrt{3})}{(-1 - i\sqrt{3})}$$

$$= \frac{2\sqrt{3} + 6i + 2i - 2\sqrt{3}}{4}$$

[A1]

$$= \frac{8i}{4} = 2i.$$

[A1]

$$\text{Arg}(2i) = \frac{\pi}{2}.$$

[A1]**Total 3 marks**

b.**Method 1:**

Since all the coefficients are real, it follows from the conjugate root theorem that $-\frac{3}{2} - i\sqrt{2}$ is also a root of $p(z)$.

Therefore $z + \frac{3}{2} - i\sqrt{2}$ and $z + \frac{3}{2} + i\sqrt{2}$ are linear factors of $p(z)$.

Therefore a quadratic factor of $p(z)$ is

$$\begin{aligned} \left(z + \frac{3}{2} - i\sqrt{2}\right)\left(z + \frac{3}{2} + i\sqrt{2}\right) &= \left(\left[z + \frac{3}{2}\right] - i\sqrt{2}\right)\left(\left[z + \frac{3}{2}\right] + i\sqrt{2}\right) \\ &= \left(z + \frac{3}{2}\right)^2 + 2 \\ &= z^2 + 3z + \frac{17}{4}. \end{aligned} \quad [\text{A1}]$$

By equating the coefficients of $\left(z^2 + 3z + \frac{17}{4}\right)(\alpha z - \beta)$ with $p(z)$ (the coefficient of z^3 is 4 and the constant term is -34) it follows that $(4z - 8)$ is also linear factor of $p(z)$. Therefore:

$$\begin{aligned} p(z) &= \left(z^2 + 3z + \frac{17}{4}\right)(4z - 8) \\ &= (4z^2 + 12z + 17)(z - 2) \\ &= 4z^3 + 4z^2 - 7z - 34. \end{aligned} \quad [\text{M1}]$$

Therefore $a = 4$ and $b = -7$.

Both values.

[A1]

Total 3 marks

Method 2:

Let two of the roots be α and β .

Then a quadratic factor is $(z - \alpha)(z - \beta) = z^2 - (\alpha + \beta)z + \alpha\beta$.

$\alpha = -\frac{3}{2} + i\sqrt{2}$ (given) and $\beta = \bar{\alpha} = -\frac{3}{2} - i\sqrt{2}$ (from the conjugate root theorem).

Therefore:

$$\alpha + \beta = -3.$$

$$\alpha\beta = \left(-\frac{3}{2}\right)^2 + (\sqrt{2})^2 = \frac{17}{4}.$$

So a quadratic factor is $z^2 + 3z + \frac{17}{4}$.

Question 3Substitute $u = 3 - 2x$:**[M1]**

$$\bullet \quad \frac{du}{dx} = -2 \Rightarrow dx = \frac{du}{-2}.$$

$$\bullet \quad x = \frac{3-u}{2}.$$

$$\bullet \quad x = \frac{1}{2} \Rightarrow u = 2 \quad \text{and} \quad x = 1 \Rightarrow u = 1.$$

$$\bullet \quad \int_{1/2}^1 \frac{x-1}{\sqrt{3-2x}} dx$$

$$= \int_2^1 \frac{\frac{3-u}{2} - 1}{\sqrt{u}} \left(\frac{du}{-2} \right)$$

[M1]

$$= -\frac{1}{4} \int_2^1 \frac{1-u}{\sqrt{u}} du$$

$$= -\frac{1}{4} \int_2^1 u^{-1/2} - u^{1/2} du$$

[M1]

$$= -\frac{1}{4} \left[2u^{1/2} - \frac{2}{3}u^{3/2} \right]_2^1$$

$$= \frac{\sqrt{2} - 2}{6}.$$

[A1]**Round the final total DOWN to the nearest integer****Total 4 marks**

Question 4**a.**

$$F = ma = 3a.$$

$$v = 1 - x^2$$

$$\Rightarrow a = v \frac{dv}{dx} = (1 - x^2)(-2x).$$

Therefore:

$$F = 3(1 - x^2)(-2x) = -6x(1 - x^2).$$

[A1]Substitute $x = \frac{2}{3}$:

$$F = -\frac{60}{27} = -\frac{20}{9}$$

$$\textbf{Answer: } F = -\frac{60}{27} = -\frac{20}{9}.$$

[A1]**Total 2 marks****b.**

$$v = 1 - x^2$$

$$\Rightarrow \frac{dx}{dt} = 1 - x^2$$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{1 - x^2}$$

[M1]

$$\Rightarrow t = \int_0^{2/3} \left| \frac{1}{1 - x^2} \right| dx \quad \left(\text{because the total time is the area of the region bounded by } y = \frac{1}{1 - x^2} \right.$$

and the x -axis between $x = 0$ and $x = \frac{2}{3}$)

$$= \int_0^{2/3} \frac{1}{1 - x^2} dx \quad \left(\text{since } \frac{1}{1 - x^2} > 0 \text{ for } 0 < x < \frac{2}{3} \right)$$

$$= \int_0^{2/3} \frac{A}{1 - x} + \frac{B}{1 + x} dx.$$

[M1]

Partial fraction calculation:

$$\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$= \frac{A(1+x) + B(1-x)}{1-x^2}$$

$$\Rightarrow 1 = A(1+x) + B(1-x) \quad \text{for all values of } x.$$

There are two options for finding the values of A and B :

Option 1: Substitute convenient values of x into $1 = A(1+x) + B(1-x)$.

$$\text{Substitute } x = 1: 1 = 2A \Rightarrow A = \frac{1}{2}.$$

$$\text{Substitute } x = -1: 1 = 2B \Rightarrow B = \frac{1}{2}.$$

Option 2: Use simultaneous equations.

Expand and group like terms:

$$1 = (A+B)x + A+B.$$

Equate coefficients of powers of x :

$$0 = A - B. \quad \dots (1)$$

Equate constant terms:

$$1 = A + B. \quad \dots (2)$$

Solve equations (1) and (2) simultaneously:

$$A = \frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}.$$

$$A = \frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}.$$

[A1]

Therefore:

$$t = \frac{1}{2} \int_0^{2/3} \frac{1}{1-x} + \frac{1}{1+x} dx$$

$$= \frac{1}{2} \left[\log_e \left| \frac{1+x}{1-x} \right| \right]_0^{2/3}$$

$$= \frac{1}{2} \log_e \left(\frac{1+\frac{2}{3}}{1-\frac{2}{3}} \right) = \frac{1}{2} \log_e (5) \quad \text{seconds.}$$

Unit not essential.

[A1]

Total 4 marks

Question 5

Use implicit differentiation with respect to x :

$$3(y+1)^2 - 2xy - x^2 = 7$$

$$\Rightarrow 6(y+1)\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} - 2x = 0. \quad [\text{M1}]$$

Substitute $\frac{dy}{dx} = 0$ and simplify:

$$y = -x. \quad [\text{M1}]$$

Solve the pair of equations

$$3(y+1)^2 - 2xy - x^2 = 7 \quad \dots (1)$$

$$y = -x \quad \dots (2)$$

simultaneously for y . Substitute equation (2) into equation (1):

$$3(y+1)^2 - 2(-y)y - (-y)^2 = 7$$

Expand, re-arrange and simplify:

$$\Rightarrow 2y^2 + 3y - 2 = 0 \quad [\text{A1}]$$

$$\Rightarrow (y+2)(2y-1) = 0$$

$$\Rightarrow y = -2, \frac{1}{2}.$$

Therefore the equations of the tangents are

$$y = -2 \text{ and } y = \frac{1}{2}. \quad [\text{A1}]$$

Total 4 marks

Alternate method (very inefficient and time consuming):

$$3(y+1)^2 - 2xy - x^2 = 7$$

Expand, re-arrange and simplify:

$$\Rightarrow 3y^2 + (6-2x)y - (4+x^2) = 0$$

Use the quadratic formula to solve for y :

$$y = \frac{2x-6 \pm 2\sqrt{(3-x)^2 + 3(4+x^2)}}{6}$$

Now calculate $\frac{dy}{dx}$ and solve $\frac{dy}{dx} = 0$ etc.

Question 6

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = -4 - t + 2 = -2 - t. \quad \dots (1)$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left| \underset{\sim}{a} \right| \left| \underset{\sim}{b} \right| \cos(\theta) = 3\sqrt{5+t^2} \cos(\theta). \quad \dots (2) \quad [\text{M1}]$$

Equate equations (1) and (2):

$$-2 - t = 3\sqrt{5+t^2} \cos(\theta)$$

$$\Rightarrow (-2 - t)^2 = 9(5+t^2) \cos^2(\theta) \quad [\text{M1}]$$

$$\Rightarrow (2+t)^2 = 9(5+t^2) \cos^2(\theta). \quad \dots (3)$$

$$\sin(\theta) = \frac{4\sqrt{5}}{9}$$

$$\Rightarrow \sin^2(\theta) = \frac{80}{81}$$

$$\Rightarrow \cos^2(\theta) = \frac{1}{81}.$$

Substitute into equation (3):

$$\Rightarrow (2+t)^2 = \frac{1}{9}(5+t^2) \quad [\text{A1}]$$

$$\Rightarrow 9(t+2)^2 = 5+t^2$$

$$\Rightarrow 8t^2 + 36t + 31 = 0.$$

Option 1: Complete the square.

$$8t^2 + 36t + 31 = 8\left(t^2 + \frac{9}{2}t + \frac{31}{8}\right) = 8\left[\left(t + \frac{9}{4}\right)^2 - \frac{81}{16} + \frac{31}{8}\right] = 8\left[t + \frac{9}{4}\right]^2 - \frac{81}{2} + 31 = 8\left[t + \frac{9}{4}\right]^2 - \frac{19}{2}:$$

$$8\left[t + \frac{9}{4}\right]^2 - \frac{19}{2} = 0$$

$$\Rightarrow \left[t + \frac{9}{4}\right]^2 = \frac{19}{16}$$

$$\Rightarrow t + \frac{9}{4} = \pm \frac{\sqrt{19}}{4}$$

$$\Rightarrow t = \frac{-9 \pm \sqrt{19}}{4}.$$

[A1]

Option 2: Use the quadratic formula.

$$\Delta = 36^2 - 32 \times 31$$

$$= 4^2(9^2 - 2 \times 31) = 16 \times 19.$$

Therefore:

$$t = \frac{-36 \pm \sqrt{16 \times 19}}{16} = \frac{-36 \pm 4\sqrt{19}}{16} = \frac{-9 \pm \sqrt{19}}{4}.$$

[A1]

Total 4 marks

Question 7

$$\underset{\sim}{v} = \frac{d \underset{\sim}{r}}{dt} = \left(2t \cos(t) - t^2 \sin(t) \right) \underset{\sim}{i} - \left(\sin(t) + t \cos(t) \right) \underset{\sim}{j}.$$

[A1]

Substitute $t = \pi$:

$$\underset{\sim}{v} = \left(2\pi \cos(\pi) - \pi^2 \sin(\pi) \right) \underset{\sim}{i} - \left(\sin(\pi) + \pi \cos(\pi) \right) \underset{\sim}{j}$$

$$= -2\pi \underset{\sim}{i} + \pi \underset{\sim}{j}.$$

[A1]

$$\text{Speed} = \left| \underset{\sim}{v} \right| = \sqrt{(-2\pi)^2 + \pi^2}$$

$$= \sqrt{5}\pi.$$

[A1]

Total 3 marks

Question 8

$$m_{normal} = \frac{-1}{m_{tangent}} \text{ and } m_{tangent} = \frac{dy}{dx}.$$

Use the chain rule to get $\frac{dy}{dx}$.

$$\text{Let } u = \frac{3}{x}.$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \text{ where:}$$

- $\frac{du}{dx} = -\frac{3}{x^2}$
- $y = \cos^{-1}(u).$
- $\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}.$

Therefore:

$$\begin{aligned} \frac{dy}{dx} &= \frac{-1}{\sqrt{1-u^2}} \times \left(-\frac{3}{x^2}\right) \\ &= \frac{-1}{\sqrt{1-\left(\frac{3}{x}\right)^2}} \times \left(-\frac{3}{x^2}\right). \end{aligned}$$

[M1]

Substitute $x = -2\sqrt{3}$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{-1}{\sqrt{1-\left(\frac{3}{-2\sqrt{3}}\right)^2}} \times \left(-\frac{3}{12}\right) \\ &= \frac{-1}{\sqrt{1-\frac{3}{4}}} \times \left(-\frac{3}{12}\right) \end{aligned}$$

$$= \frac{1}{2}$$

[A1]

$$\Rightarrow m_{normal} = -2.$$

Substitute $x = -2\sqrt{3}$ into $y = \cos^{-1}\left(\frac{3}{x}\right)$:

$$y = \cos^{-1}\left(\frac{3}{-2\sqrt{3}}\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

Equation of normal:

$$y - \frac{5\pi}{6} = -2(x + 2\sqrt{3}) \quad \Rightarrow y = -2x - 4\sqrt{3} + \frac{5\pi}{6}.$$

Any correct form.

[A1]

Total 3 marks

Comment:

The following is NOT required but is included as a teaching point (see line *).

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - \left(\frac{3}{x}\right)^2}} \times \left(-\frac{3}{x^2}\right)$$

$$= \frac{3}{x^2 \sqrt{1 - \frac{9}{x^2}}}$$

$$= \frac{3}{x^2 \sqrt{\frac{x^2 - 9}{x^2}}}$$

$$= \frac{3\sqrt{x^2}}{x^2 \sqrt{x^2 - 9}}$$

$$= \frac{3|x|}{x^2 \sqrt{x^2 - 9}} \quad \textbf{not} \quad \frac{3x}{x^2 \sqrt{x^2 - 9}} \quad *$$

$$= \frac{3}{|x| \sqrt{x^2 - 9}}.$$

Substitute $x = -2\sqrt{3}$:

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3}\sqrt{12-9}} = \frac{1}{2}.$$

Question 9**a.**

$$f(x) = 1 - 2\operatorname{cosec}\left(\frac{\pi x}{3}\right) = 1 - \frac{2}{\sin\left(\frac{\pi x}{3}\right)}$$

has vertical asymptotes when

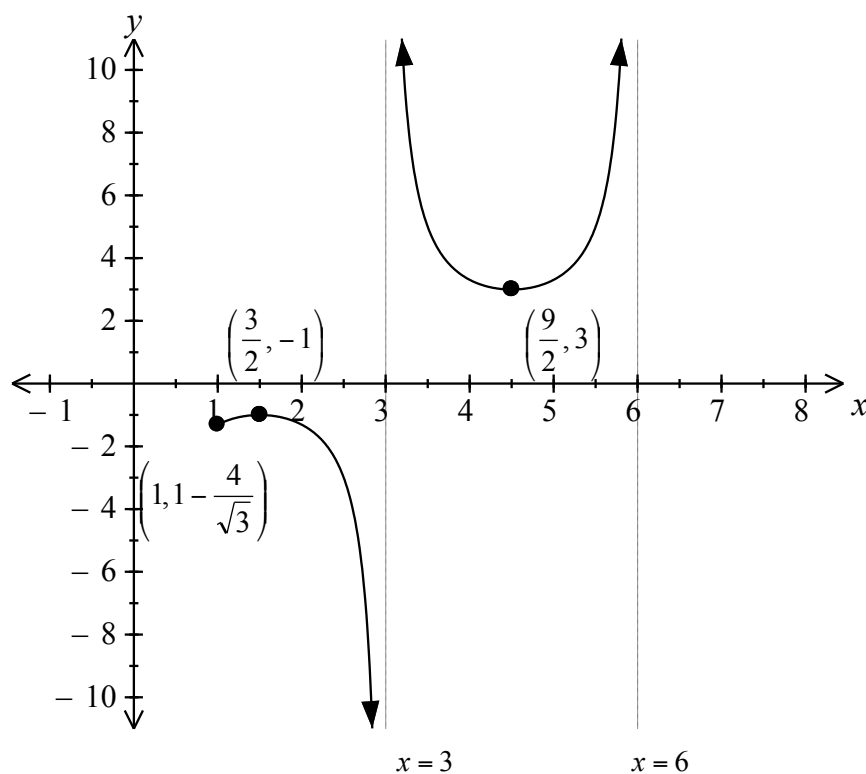
$$\sin\left(\frac{\pi x}{3}\right) = 0$$

$$\Rightarrow \frac{\pi x}{3} = n\pi, \quad n \in \mathbb{Z}$$

$$\Rightarrow x = 3n.$$

The first two asymptotes for which $x > 1$ are $x = 3$ and $x = 6$.Therefore $a = 3$ and $b = 6$.

Both values.

[A1]**Total 1 mark****b.**

• Vertical asymptotes:

 $x = 3$ and $x = 6$ (consequential on answer to **part a.**: $x = a$ and $x = b$).**[A1]**• Minimum turning point at $\left(\frac{9}{2}, 3\right)$ and maximum turning point at $\left(\frac{3}{2}, -1\right)$.**[A1]**

Calculation:**Method 1:**

Maximum turning point when $\sin\left(\frac{\pi x}{3}\right) = 1$: $x = \frac{3}{2}$ and $y = -1$.

Minimum turning point when $\sin\left(\frac{\pi x}{3}\right) = -1$: $x = \frac{9}{2}$ and $y = 3$.

Method 2:

The turning points lie halfway between the vertical asymptotes.

- Endpoint: $x = 1$ and $y = 1 - \frac{4}{\sqrt{3}}$.

Correct 'ball park' location.

[A1]

Comment: $1 - \frac{4}{\sqrt{3}} \approx 1 - \frac{4}{2} = -1$ so the y -coordinate of the endpoint should be shown in the 'ball park' of $y = -1$.

- Shape.

[A1]

Total 4 marks