

Year 2014

VCE

Specialist Mathematics

Trial Examination 2

Solutions



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SECTION 1

ANSWERS

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1

Question 1

Answer D

$$f(x) = a^2 - x^2 = (a+x)(a-x)$$

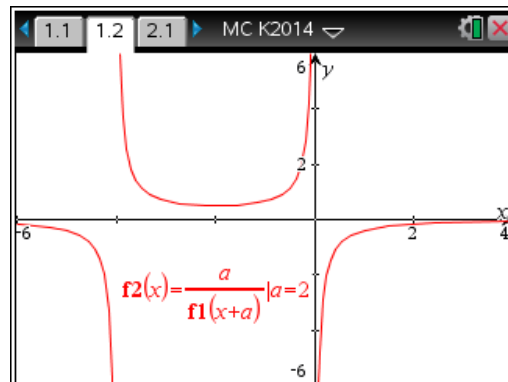
$$y = \frac{a}{f(x+a)} = \frac{a}{(x+2a)(-x)} = \frac{-a}{x(x+2a)}$$

vertical asymptotes occur when $x(x+2a) = 0$

vertical asymptotes at $x = 0$ and $x = -2a$

the turning point is at $x = -a \Rightarrow y = \frac{a}{a^2} = \frac{1}{a}$

$\left(-a, \frac{1}{a}\right)$ and is a minimum turning point.



Question 2

Answer A

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1. \text{ The asymptote is } y = -\frac{x}{2}$$

and passes through the centre, when

$y = -1 \Rightarrow x = 2$, so that the centre of the hyperbola

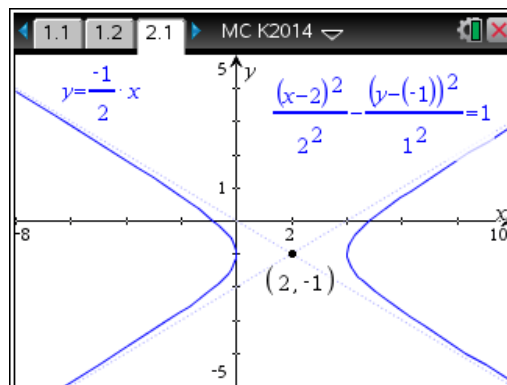
is at $(2, -1)$, so that $h = 2$ and $k = -1$.

The distance from the centre to the point where the hyperbola touches the y-axis is 2,

so that $a = 2$. The asymptotes are

$$y - k = \pm \frac{b}{a}(x - h) \text{ and have a gradient}$$

of $\frac{b}{a} = \frac{1}{2}$, since $a = 2$ it follows that $b = 1$.



Question 3

Answer D

$$y = \tan^{-1}(x) \text{ has a range } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

For $y = a \tan^{-1}(bx) + c$, the value of b does not affect the range (only the domain)

For a range equal to $\frac{6}{\pi} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = (-3, 3)$ so that $a = \frac{6}{\pi}$ and $c = 0$

Question 4**Answer C**

The roots occur in complex conjugate pairs, since the coefficients are all real.

Let $u = \alpha + i$ and $\bar{u} = \alpha - i$ be the roots of $z^2 + bz + c = 0$.

The sum of the roots $u + \bar{u} = 2\alpha = -b$ and the product of the roots are

$$u \cdot \bar{u} = (\alpha + i)(\alpha - i) = \alpha^2 + 1 = c$$

For the quadratic, let $v = \alpha + 1 - i$ and $\bar{v} = \alpha + 1 + i$.

The sum of these roots are $v + \bar{v} = 2\alpha + 2 = 2 - b$ and the product of these roots are

$$v \cdot \bar{v} = (\alpha + 1 - i)(\alpha + 1 + i) = (\alpha + 1)^2 + 1 = \alpha^2 + 2\alpha + 2 = c - b + 1. \text{ The quadratic is}$$

$$z^2 - (2 - b)z + c - b + 1 = z^2 + (b - 2)z + c - b + 1 = 0$$

Question 5**Answer E**

by the conjugate root theorem

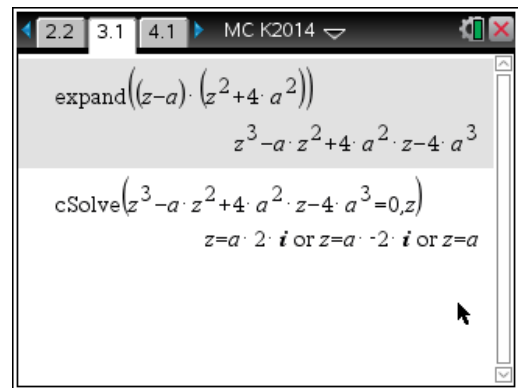
$z = 2ai$ is also a root.

$$(z + 2ai)(z - 2ai) = z^2 - 4a^2i^2 = z^2 + 4a^2$$

expanding

$$P(z) = (z - a)(z^2 + 4a^2)$$

$$P(z) = z^3 - az^2 + 4a^2z - 4a^3$$

**Question 6****Answer B**

$$a + bi = r \text{cis}(\theta) \text{ where } r = \sqrt{a^2 + b^2}$$

$$(a + bi)^2 = r^2 \text{cis}(2\theta)$$

$$a^2 + 2abi + b^2i^2 = a^2 - b^2 + 2abi = r^2 \text{cis}(2\theta) \text{ take the conjugate}$$

$$a^2 - b^2 - 2abi = r^2 \text{cis}(-2\theta) \text{ multiply both sides by } i = 1 \text{cis}\left(\frac{\pi}{2}\right)$$

$$i(a^2 - b^2 - 2abi) = ir^2 \text{cis}(-2\theta) = 1 \text{cis}\left(\frac{\pi}{2}\right) \times r^2 \text{cis}(-2\theta)$$

$$(a^2 - b^2)i - 2abi^2 = 2ab + (a^2 - b^2)i = r^2 \text{cis}\left(\frac{\pi}{2} - 2\theta\right)$$

$$\text{Arg}(2ab + (a^2 - b^2)i) = \frac{\pi}{2} - 2\theta$$

Question 7**Answer E**

Alex: $\int_0^{\frac{\pi}{4}} \sin^3(2x) \cos^3(2x) dx,$

let $u = \sin(2x) \quad \frac{du}{dx} = 2 \cos(2x) \Rightarrow \cos(2x) dx = \frac{1}{2} du$

terminals, $x = \frac{\pi}{4} \quad u = \sin\left(\frac{\pi}{2}\right) = 1 \quad x = 0 \quad u = \sin(0) = 0$

$$\int_0^{\frac{\pi}{4}} \sin^3(2x) \cos^2(2x) \cos(2x) dx = \int_0^{\frac{\pi}{4}} \sin^3(2x) (1 - \sin^2(2x)) \cos(2x) dx = \frac{1}{2} \int_0^1 u^3 (1 - u^2) du$$

Brenda: $\int_0^{\frac{\pi}{4}} \sin^3(2x) \cos^3(2x) dx,$

let $u = \cos(2x) \quad \frac{du}{dx} = -2 \sin(2x) \Rightarrow \sin(2x) dx = -\frac{1}{2} du$

terminals, $x = \frac{\pi}{4} \quad u = \cos\left(\frac{\pi}{2}\right) = 0 \quad x = 0 \quad u = \cos(0) = 1$

$$\int_0^{\frac{\pi}{4}} \sin(2x) \sin^2(2x) \cos^3(2x) dx = \int_0^{\frac{\pi}{4}} (1 - \cos^2(2x)) \cos^3(2x) \sin(2x) dx$$

$$= -\frac{1}{2} \int_1^0 (1 - u^2) u^3 du = \frac{1}{2} \int_0^1 (1 - u^2) u^3 du$$

Claire: $\int_0^{\frac{\pi}{4}} \sin^3(2x) \cos^3(2x) dx = \frac{1}{8} \int_0^{\frac{\pi}{4}} (2 \sin(2x) \cos(2x))^3 dx = \frac{1}{8} \int_0^{\frac{\pi}{4}} \sin^3(4x) dx$

$$= \frac{1}{8} \int_0^{\frac{\pi}{4}} \sin(4x) \sin^2(4x) dx = \frac{1}{8} \int_0^{\frac{\pi}{4}} \sin(4x) (1 - \cos^2(4x)) dx$$

let $u = \cos(4x) \quad \frac{du}{dx} = -4 \sin(4x) \Rightarrow \sin(4x) dx = -\frac{1}{4} du$

terminals $x = \frac{\pi}{4} \quad u = \cos(\pi) = -1 \quad x = 0 \quad u = \cos(0) = 1$

$$\int_0^{\frac{\pi}{4}} \sin^3(2x) \cos^3(2x) dx = -\frac{1}{32} \int_1^{-1} (1 - u^2) du = \frac{1}{32} \int_{-1}^1 (1 - u^2) du$$

$\int_0^{\frac{\pi}{4}} ((\sin(2 \cdot x))^3 \cdot (\cos(2 \cdot x))^3) dx$	$\frac{1}{24}$
$\frac{1}{2} \int_0^1 (u^3 \cdot (1-u^2)) du$	$\frac{1}{24}$
$\frac{1}{32} \int_{-1}^1 (1-u^2) du$	$\frac{1}{24}$

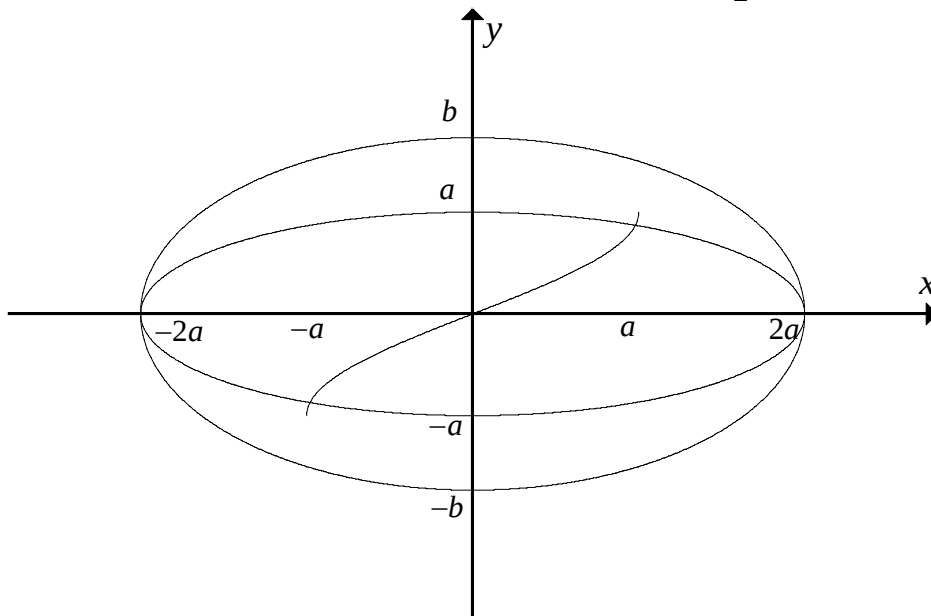
All are correct.

Question 8

Answer D

The domain of $\frac{2a}{\pi} \sin^{-1}\left(\frac{x}{a}\right)$ is $[-a, a]$ and the range is $[-a, a]$. Its endpoints are $(-a, -a)$ and (a, a) . The ellipse $\frac{x^2}{4a^2} + \frac{y^2}{b^2} = 1$ has its centre at the origin and has a domain $[-2a, 2a]$ and a range $[-b, b]$. When $x = a$ on the ellipse $\frac{a^2}{4a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow y = \pm \frac{\sqrt{3}b}{2}$.

For the ellipse to intersect exactly twice, we require $a > \frac{\sqrt{3}b}{2}$.



Question 9**Answer B**

$$|z + 2a| = 2|z - ai| \quad \text{let } z = x + yi$$

$$|(x + 2a) + yi| = 2|x + (y - a)i|$$

$$\sqrt{(x + 2a)^2 + y^2} = 2\sqrt{x^2 + (y - a)^2}$$

$$(x + 2a)^2 + y^2 = 4[x^2 + (y - a)^2]$$

$$x^2 + 4ax + 4a^2 + y^2 = 4[x^2 + y^2 - 2ay + a^2]$$

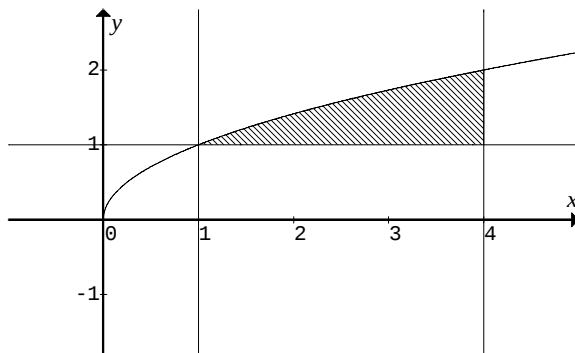
$$3x^2 - 4ax + 3y^2 - 8ay = 0$$

$$x^2 - \frac{4ax}{3} + y^2 - \frac{8ay}{3} = 0 \Rightarrow \text{a circle, however let's find the centre and radius.}$$

$$x^2 - \frac{4ax}{3} + \frac{4a^2}{9} + y^2 - \frac{8ay}{3} + \frac{16a^2}{9} = \frac{20a^2}{9}$$

$$\left(x - \frac{2a}{3}\right)^2 + \left(y - \frac{4a}{3}\right)^2 = \frac{20a^2}{9}$$

This represents a circle centre at $\left(\frac{2a}{3}, \frac{4a}{3}\right)$ and radius $\frac{\sqrt{20}a}{3}$.

Question 10**Answer B**

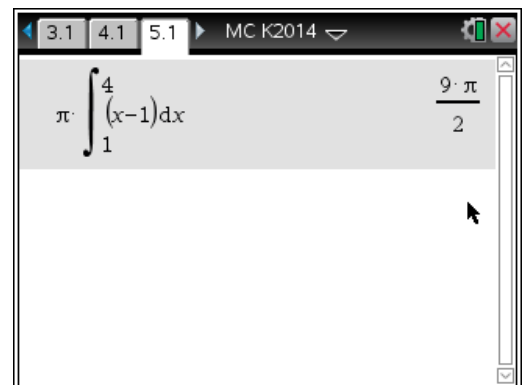
$$V_x = \pi \int_a^b (y_2^2 - y_1^2) dx$$

$$a = 1, b = 4, y_2 = \sqrt{x}, y_1 = 1$$

$$V = \pi \int_1^4 (x - 1) dx = \pi \left[\frac{1}{2} x^2 - x \right]_1^4$$

$$V = \pi \left[(8 - 1) - \left(\frac{1}{2} - 1 \right) \right]$$

$$V = \frac{9\pi}{2}$$



Question 11**Answer C**

The length of u is 2 and the length of v is 3. v is a rotation from u by 270° , that is i^3 .

$$\text{So } \frac{3ui^3}{2} = v \Rightarrow -3ui = 2v \text{ so that } 2v + 3ui = 0$$

Question 12**Answer E**

$$\cos(\theta) = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| |\underline{v}|} \Rightarrow \cos(120^\circ) = -\frac{1}{2} = \frac{\underline{u} \cdot \underline{v}}{3 \times 4} \Rightarrow \underline{u} \cdot \underline{v} = -6 \quad \text{A. is true.}$$

$$\text{The scalar resolute of } \underline{u} \text{ in the direction of } \underline{v} \text{ is equal to } \underline{u} \cdot \hat{\underline{v}} = \frac{\underline{u} \cdot \underline{v}}{|\underline{v}|} = \frac{-6}{4} = -\frac{3}{2} \quad \text{B. is true}$$

$$\text{The scalar resolute of } \underline{v} \text{ in the direction of } \underline{u} \text{ is equal to } \underline{v} \cdot \hat{\underline{u}} = \frac{\underline{v} \cdot \underline{u}}{|\underline{u}|} = \frac{-6}{3} = -2 \quad \text{C. is true.}$$

$$|\underline{u} + \underline{v}|^2 = (\underline{u} + \underline{v}) \cdot (\underline{u} + \underline{v}) = \underline{u} \cdot \underline{u} + 2\underline{u} \cdot \underline{v} + \underline{v} \cdot \underline{v} = |\underline{u}|^2 + 2\underline{u} \cdot \underline{v} + |\underline{v}|^2 = 3^2 + 2 \times -6 + 4^2 = 13$$

$$|\underline{u} + \underline{v}| = \sqrt{13} \quad \text{D. is true}$$

$$|\underline{v} - \underline{u}| = 1 \quad \text{E. is false.}$$

Question 13**Answer D**

use implicit differentiation on $x^2 + 2 \tan^{-1}\left(\frac{y}{2}\right) + y^2 = 5 + \frac{\pi}{2}$, gives

$$2x + \left(\frac{4}{4+y^2} + 2y\right) \frac{dy}{dx} = 0 \quad \text{when } x = -1 \text{ and } y = 2$$

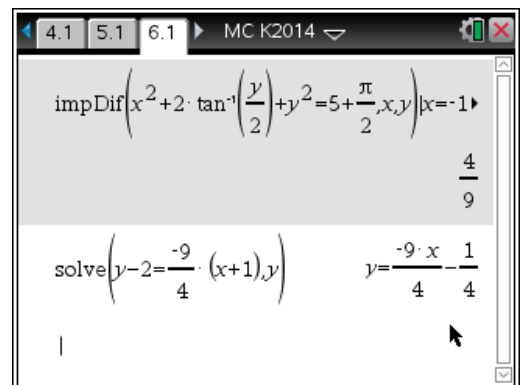
$$-2 + \left(\frac{4}{4+4} + 4\right) \frac{dy}{dx} = 0$$

$$\frac{9}{2} \frac{dy}{dx} = 2 \quad m_T = \frac{dy}{dx} = \frac{4}{9}$$

$$m_N = -\frac{9}{4} \text{ at } (-1, 2)$$

$$N: y - 2 = -\frac{9}{4}(x + 1)$$

$$y = -\frac{9x}{4} - \frac{1}{4}$$



Question 14**Answer D**

$$\underline{a} = x\underline{i} + \underline{j} - \underline{k}, \underline{b} = \underline{i} - 2\underline{j} + 2\underline{k}, \underline{a} \cdot \underline{b} = x - 2 - 2 = x - 4$$

When $x = 4$ then $\underline{a} \cdot \underline{b} = 0$, so that the vector $\underline{a}$ is perpendicular to the vector $\underline{b}$. **A.** is true.

$-2\underline{a} = \underline{b} \Rightarrow -2x = 1 \Rightarrow x = -\frac{1}{2}$, then the vector $\underline{a}$ is parallel to the vector $\underline{b}$. **B.** is true.

$$|\underline{a}| = \sqrt{x^2 + 1 + 1} = \sqrt{x^2 + 2} \quad |\underline{b}| = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

$$\text{If } |\underline{a}| = |\underline{b}| \Rightarrow \sqrt{x^2 + 2} = 3 \Rightarrow x^2 + 2 = 9 \Rightarrow x^2 = 7$$

If $x = \pm\sqrt{7}$ then the vectors $\underline{a}$ and $\underline{b}$ are equal in length, **C.** is true.

If $x = \sqrt{2}$ then $|\underline{a}| = 2$, so that $|\underline{a}| + |\underline{b}| = 5$, **E.** is true.

$\underline{a} + \underline{b} = (x+1)\underline{i} - \underline{j} + \underline{k}$, if $x = 0$, $|\underline{a} + \underline{b}| = \sqrt{1+1+1} = \sqrt{3}$, **D.** is false

Question 15**Answer C**

$$\frac{d^2y}{dx^2} = 5x^4 - 5$$

$$\frac{dy}{dx} = \int (5x^4 - 5) dx = x^5 - 5x + c$$

$$\text{when } x = 1 \quad \frac{dy}{dx} = 0$$

$$0 = 1 - 5 + c \Rightarrow c = 4$$

$$\frac{dy}{dx} = x^5 - 5x + 4$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow x^4 = 1 \quad x^2 = 1 \quad x = \pm 1$$

$$\text{when } x = 1 \quad y'' = 0 \text{ and } y' = 0 \Rightarrow x = 1$$

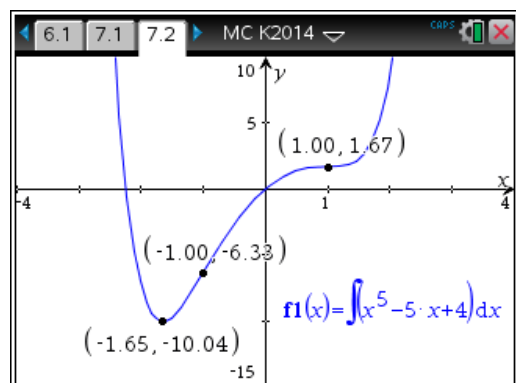
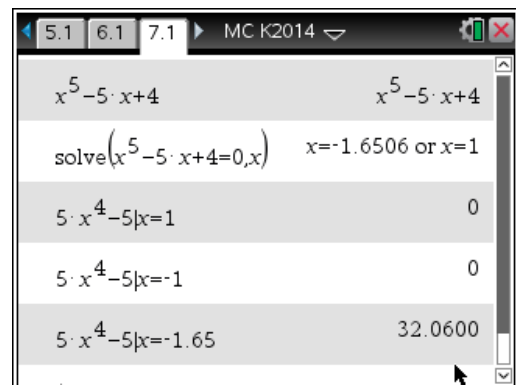
is a stationary point of inflexion.

$$\text{when } x = -1 \quad y'' = 0 \text{ and } y' > 0 \Rightarrow x = -1$$

is a point of inflexion.

$$y' = 0 \Rightarrow x = -1.65 \quad y'' > 0 \Rightarrow x = -1.65$$

is a minimum turning point.



Question 16**Answer C**

$$\frac{dy}{dx} = f(x) = \cos^2(2x)$$

$$h = \frac{\pi}{8}, x_0 = 0 \text{ and } y_0 = 2, x_1 = \frac{\pi}{8}, x_2 = \frac{\pi}{4}$$

using Euler's Method

$$y_1 = y_0 + h f(x_0)$$

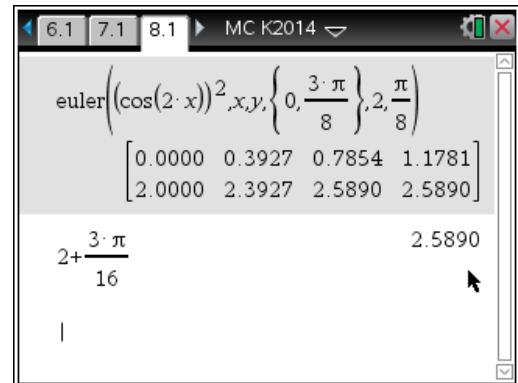
$$y_1 = 2 + \frac{\pi}{8} \cos^2(0) = 2 + \frac{\pi}{8}$$

$$y_2 = y_1 + h f(x_1)$$

$$y_2 = 2 + \frac{\pi}{8} + \frac{\pi}{8} \cos^2\left(\frac{\pi}{4}\right) = 2 + \frac{\pi}{8} + \frac{\pi}{16} = 2 + \frac{3\pi}{16}$$

$$y_3 = y_2 + h f(x_2)$$

$$y_3 = 2 + \frac{3\pi}{16} + \frac{\pi}{8} \cos^2\left(\frac{\pi}{2}\right) = 2 + \frac{3\pi}{16} + 0 = 2 + \frac{3\pi}{16}$$

**Question 17****Answer B**

$$v = \frac{dx}{dt} = e^{\sqrt{t}}$$

$$x = \int_0^t e^{\sqrt{u}} du + c$$

$$\text{when } x = 3, t = 1$$

$$3 = \int_0^1 e^{\sqrt{u}} du + c \Rightarrow c = 3 - \int_0^1 e^{\sqrt{u}} du$$

$$x = x(t) = \int_0^t e^{\sqrt{u}} du + 3 - \int_0^1 e^{\sqrt{u}} du$$

$$x(2) = \int_0^2 e^{\sqrt{u}} du - \int_0^1 e^{\sqrt{u}} du + 3$$

$$x(2) = \int_0^2 e^{\sqrt{u}} du + \int_1^0 e^{\sqrt{u}} du + 3$$

$$x(2) = \int_1^2 e^{\sqrt{u}} du + 3$$

Question 18**Answer A**

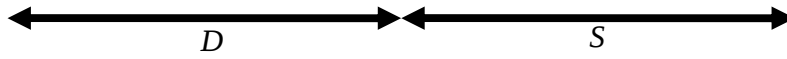
$$u = 3V$$

$$t = 0$$

$$v = V$$

$$v = 0$$

$$t = T$$



$$\text{use } v^2 = u^2 + 2as$$

$$\text{with } v = V, u = 3V \text{ } s = D$$

$$V^2 = 9V^2 + 2aD$$

$$a = -\frac{4V^2}{D}$$

$$\text{use } s = \left(\frac{u+v}{2}\right)t$$

$$\text{with } v = 0, u = 3V \text{ } s = D + S \text{ and } t = T$$

$$D + S = \frac{3V}{2} \times T \Rightarrow T = \frac{2(D+S)}{3V}$$

$$\text{use } v^2 = u^2 + 2as$$

$$\text{with } v = 0, u = V \text{ } s = S \text{ and } a = -\frac{4V^2}{D}$$

$$0 = V^2 - 2 \times \frac{4V^2}{D} \times S \Rightarrow D = 8S$$

Question 19**Answer B**

resolving downwards for the 2 kg mass hanging over the edge of the table

$$(1) \quad 2g - T = 2a$$

resolving for the mass M on the table

$$(2) \quad N - Mg = 0$$

$$(3) \quad T - \mu N = Ma$$

$$(2) \Rightarrow N = Mg \text{ into } (3) \quad T - \mu Mg = Ma$$

adding to eliminate T

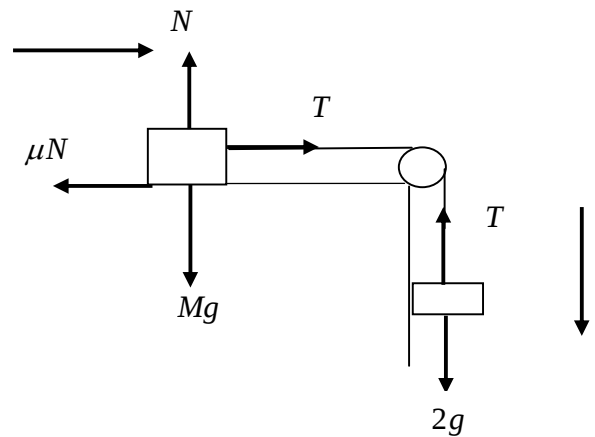
$$2g - \mu Mg = (M + 2)a \text{ but } \mu = \frac{1}{3}$$

$$2g - \frac{1}{3}Mg = \frac{g(6-M)}{3} = (M + 2)a$$

$$a = \frac{g(6-M)}{3(M+2)}$$

$$a > 0 \Rightarrow 0 < M < 6$$

$$a = 0 \Rightarrow M = 6$$



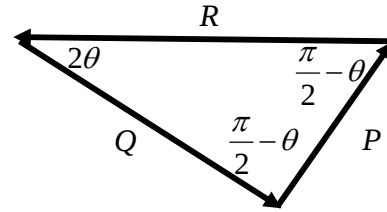
Question 20

Answer A

$$\frac{Q}{\sin\left(\frac{\pi}{2}-\theta\right)} = \frac{R}{\sin\left(\frac{\pi}{2}-\theta\right)} = \frac{P}{\sin(2\theta)}$$

$$\frac{Q}{\cos(\theta)} = \frac{R}{\cos(\theta)} = \frac{P}{2\sin(\theta)\cos(\theta)}$$

$$Q = R \text{ and } P = 2R\sin(\theta)$$



Alternatively resolving vertically $P\cos(\theta) - Q\sin(2\theta) = 0$

$$P\cos(\theta) = 2Q\sin(\theta)\cos(\theta) \Rightarrow P = 2Q\sin(\theta) \quad (1)$$

resolving horizontally $P\sin(\theta) + Q\cos(2\theta) - R = 0$

$$P\sin(\theta) + Q(1 - 2\sin^2(\theta)) = R \text{ from (1) } P = 2Q\sin(\theta)$$

$$2Q\sin^2(\theta) + Q - 2Q\sin^2(\theta) = R \Rightarrow Q = R \text{ and } P = 2R\sin(\theta)$$

Question 21

Answer E

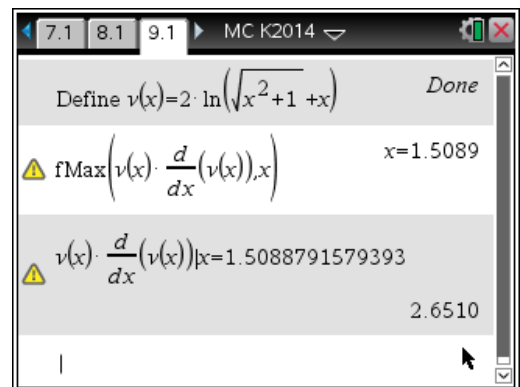
$$v = 2\log_e(\sqrt{x^2+1} + x)$$

$$\frac{dv}{dx} = \frac{2}{\sqrt{x^2+1}}$$

$$a(x) = v \frac{dv}{dx} = \frac{4\log_e(\sqrt{x^2+1} + x)}{\sqrt{x^2+1}}$$

$a(x)$ has a maximum when $x = 1.51$

$$F_{\max} = ma = 10 \times a(1.51) = 26.5$$



Question 22

Answer A

when $x = 2$ and $y = 1 \Rightarrow m = 1$ when $x = 2$ and $y = -1 \Rightarrow m = -1$

when $x = 0$ and $y = 1 \Rightarrow m = -1$ when $x = 0$ and $y = -1 \Rightarrow m = 1$

is only satisfied by $m = \frac{dy}{dx} = \frac{x-1}{y}$

alternatively, the solution curves are hyperbolas, of the form,

$$\frac{(x-1)^2}{a^2} - y^2 = 1 \text{ using implicit differentiation, } \frac{2(x-1)}{a^2} - 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x-1}{a^2 y} \text{ let } a^2 = 1$$

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

- a. $\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$
 $\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$ M1
 $\cos\left(\frac{5\pi}{12}\right) = \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}$
 $\cos\left(\frac{5\pi}{12}\right) = \frac{1}{4}(\sqrt{6} - \sqrt{2})$ A1
- b.i. $\overrightarrow{OA} = \underline{i} + \underline{j}$ and $\overrightarrow{OB} = -\underline{i} + \sqrt{3}\underline{j}$ A1
- ii. $|\overrightarrow{OA}| = \sqrt{2}$ $|\overrightarrow{OB}| = 2$ $\overrightarrow{OA} \cdot \overrightarrow{OB} = -1 + \sqrt{3}$
 $\cos(\theta) = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OA}| |\overrightarrow{OB}|} = \frac{-1 + \sqrt{3}}{2\sqrt{2}}$
 $\cos(\theta) = \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{1}{4}(\sqrt{6} - \sqrt{2})$ M1
 $\theta = \cos^{-1}\left(\frac{1}{4}(\sqrt{6} - \sqrt{2})\right) = \frac{5\pi}{12}$
 $\theta = 75^\circ$ A1
- iii. $\text{Area} = \frac{1}{2} |\overrightarrow{OA}| |\overrightarrow{OB}| \sin(\theta)$
 $\text{Area} = \frac{1}{2} \times \sqrt{2} \times 2 \times \sin\left(\frac{5\pi}{12}\right) = \sqrt{2} \times \frac{1}{4}(\sqrt{6} + \sqrt{2})$ M1
 $\text{Area} = \frac{1 + \sqrt{3}}{2}$ A1
- iv. $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$
 $= \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA}) = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$ M1
 $= \frac{1}{2}((\underline{i} + \underline{j}) + (-\underline{i} + \sqrt{3}\underline{j}))$
 $= \frac{1}{2}(1 + \sqrt{3})\underline{j}$ A1

c.i. $a = 1 + i = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right) \quad |a| = \sqrt{2} \quad \alpha = \frac{\pi}{4}$ A1

$b = -1 + i\sqrt{3} = 2 \operatorname{cis}\left(\frac{2\pi}{3}\right) \quad |b| = 2 \quad \beta = \frac{2\pi}{3}$ A1

ii. If $z = x + iy$ $\operatorname{Re}(z) = x$ $\operatorname{Im}(z) = y$,
the line $\operatorname{Im}(z) = m \operatorname{Re}(z) + k$ is $y = mx + k$
 m is the gradient of the line joining $A(1,1)$ and $B(-1, \sqrt{3})$ M1

$$m(AB) = \frac{1 - \sqrt{3}}{2} \quad \text{the line is } y - 1 = \left(\frac{1 - \sqrt{3}}{2}\right)(x - 1)$$

$$y = \left(\frac{1 - \sqrt{3}}{2}\right)x - \left(\frac{1 - \sqrt{3}}{2}\right) + 1 = \left(\frac{1 - \sqrt{3}}{2}\right)x + \frac{1}{2}(\sqrt{3} + 1)$$

so that $m = \frac{1}{2}(1 - \sqrt{3})$ and $k = \frac{1}{2}(\sqrt{3} + 1)$ A1

iii. $R = \frac{1}{4}(a\bar{b} - \bar{a}b)i$
 $R = \frac{1}{4}((1+i)(-1-i\sqrt{3}) - (1-i)(-1+i\sqrt{3}))i$
 $R = \frac{1}{4}(-1 - i\sqrt{3} - i - i^2\sqrt{3} - (-1 + i\sqrt{3} + i - i^2\sqrt{3}))i$ M1

$$R = \frac{1}{4}(2i(-\sqrt{3} - 1))i$$

$R = \frac{1 + \sqrt{3}}{2}$ and is equal to the area of the triangle AOB from **b.iii.** A1

iv. The circle $|z - c| = r$, has its centre at C ,
the midpoint of AB , so $c = \frac{1}{2}(1 + \sqrt{3})i$ A1

$$\overline{AB} = -2i + (\sqrt{3} - 1)j \quad \text{and} \quad |\overline{AB}| = \sqrt{(-2)^2 + (\sqrt{3} - 1)^2} = \sqrt{8 - 2\sqrt{3}}$$

and the radius of the circle is $r = \frac{1}{2}|\overline{AB}| = \frac{1}{2}\sqrt{2(4 - \sqrt{3})}$ A1

Question 2

a.
$$N = \frac{1000}{1+9e^{-kt}} = 1000(1+9e^{-kt})^{-1}$$

$$\frac{dN}{dt} = 1000 \times 9ke^{-kt} (1+9e^{-kt})^{-2} = \frac{1000 \times 9ke^{-kt}}{(1+9e^{-kt})^2}$$

but $1+9e^{-kt} = \frac{1000}{N} \Rightarrow 9e^{-kt} = \frac{1000}{N} - 1 = \frac{1000-N}{N}$ M1

$$\frac{dN}{dt} = \frac{1000 \times 9ke^{-kt}}{(1+9e^{-kt})^2}$$

$$\frac{dN}{dt} = k \left(\frac{1000}{1+9e^{-kt}} \right)^2 \times \frac{9e^{-kt}}{1000}$$
A1

$$\frac{dN}{dt} = \frac{kN^2}{1000} \left(\frac{1000-N}{N} \right)$$

$$\frac{dN}{dt} = kN \left(1 - \frac{N}{1000} \right)$$

b. initial number $N_0 = N(0) = \frac{1000}{1+9e^0} = 100$

ultimate number, since $k > 0 \quad \lim_{t \rightarrow \infty} N(t) = 1000$ A1

c. $N(20) = 900$

$$900 = \frac{1000}{1+9e^{-20k}} \Rightarrow 900 + 8100e^{-20k} = 1000$$

$$e^{-20k} = \frac{100}{8100} = \frac{1}{81} \Rightarrow e^{20k} = 81$$

$$k = \frac{1}{20} \log_e(81) = \frac{1}{5} \log_e(3)$$
A1

d.
$$\frac{dN}{dt} = k \left(N - \frac{N^2}{1000} \right)$$

$$\frac{d^2N}{dt^2} = \frac{d}{dt} \left(\frac{dN}{dt} \right) = \frac{d}{dN} \left(k \left(N - \frac{N^2}{1000} \right) \right) \frac{dN}{dt}$$
M1

$$\frac{d^2N}{dt^2} = k^2 N \left(1 - \frac{N}{500} \right) \left(1 - \frac{N}{1000} \right)$$
A1

$$\frac{d^2N}{dt^2} = 0 \Rightarrow N = 500 \Rightarrow t = 10$$

inflexion point at (10, 500) A1

e. $\frac{dN}{dt} = rN \Rightarrow N = N_0 e^{rt} \quad N(0) = 50 \Rightarrow N_0 = 50$

$$N(20) = 450 \Rightarrow 450 = 50e^{20r}$$

$$r = \frac{1}{20} \log_e(9) = \frac{1}{10} \log_e(3)$$

$$r \approx 0.1099$$

$$N = N(t) = 50e^{0.1099t}$$

M1

A1

f. solving using CAS $50e^{0.1099t} = \frac{1000}{1+9e^{-0.2197t}}$ using exact values gives

$$N = 977 \text{ and } t = 27.1$$

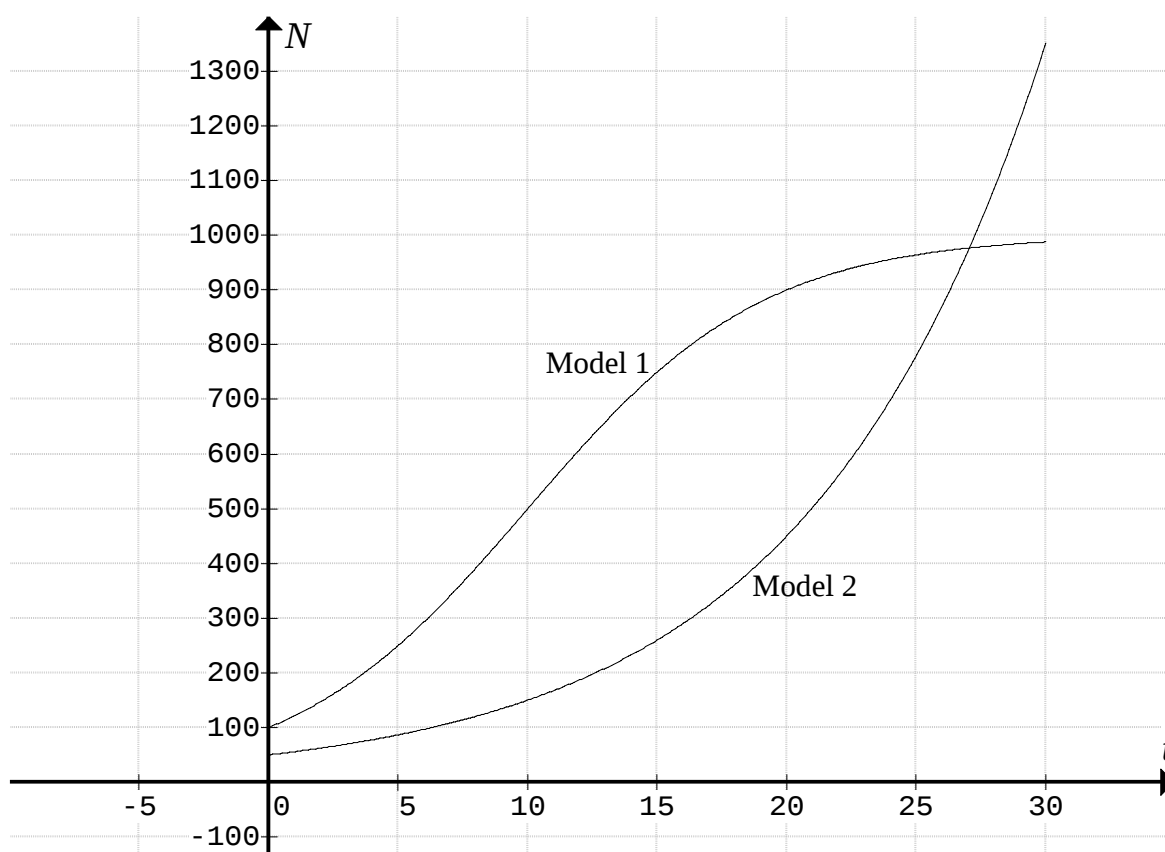
A1

g. both graphs correct shape, passing through exact points

G2

Model 1 $(0,100), (10,500), (20,900), (27,977), (30,988)$ asymptotes to $N = 1000$

Model 2 $(0,50), (20,450), (27,977), (30,1350)$



Question 3

a. $x = 9\cos(t) \quad y = \frac{9\sin^2(t)}{2 + \sin(t)}$

$LHS = y^2(81 - x^2)$ substituting

$$= \frac{81\sin^4(t)}{(2 + \sin(t))^2} (81 - 81\cos^2(t))$$

$$= \frac{81\sin^4(t)}{(2 + \sin(t))^2} (81(1 - \cos^2(t)))$$

M1

$$= \frac{81^2 \sin^6(t)}{(2 + \sin(t))^2}$$

$RHS = (x^2 + 18y - 81)^2$ substituting

$$= \left(81\cos^2(t) + \frac{18 \times 9\sin^2(t)}{2 + \sin(t)} - 81 \right)^2$$

$$= \left(81(\cos^2(t) - 1) + \frac{162\sin^2(t)}{2 + \sin(t)} \right)^2$$

M1

$$= \left(-81\sin^2(t) + \frac{162\sin^2(t)}{2 + \sin(t)} \right)^2$$

$$= \left(\frac{-81\sin^2(t)(2 + \sin(t)) + 162\sin^2(t)}{2 + \sin(t)} \right)^2$$

$$= \left(\frac{-162\sin^2(t) - 81\sin^3(t) + 162\sin^2(t)}{2 + \sin(t)} \right)^2$$

A1

$$= \frac{81^2 \sin^6(t)}{(2 + \sin(t))^2} = LHS$$

b. the gradient is zero $\frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}} = 0$

$x(t) = 9\cos(t)$ $\frac{dx}{dt} = \dot{x} = -9\sin(t)$ for $t \in [0, 2\pi]$.

$y(t) = \frac{9\sin^2(t)}{2 + \sin(t)}$ $\frac{dy}{dt} = \dot{y} = \frac{9\sin(t)(\sin(t) + 4)\cos(t)}{(2 + \sin(t))^2}$ M1

solving, $\frac{dy}{dx} = \frac{-(\sin(t) + 4)\cos(t)}{(2 + \sin(t))^2} = 0$ since $\sin(t) \neq -4$

$\Rightarrow \cos(t) = 0$ with $t \in [0, 2\pi] \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$ A1

$x\left(\frac{\pi}{2}\right) = 0, x\left(\frac{3\pi}{2}\right) = 0$ and $y\left(\frac{\pi}{2}\right) = 3, y\left(\frac{3\pi}{2}\right) = 9$

turning points are $t = \frac{\pi}{2} \Rightarrow (0, 3)$ $t = \frac{3\pi}{2} \Rightarrow (0, 9)$ A1

c. $y^2(81 - x^2) = (x^2 + 18y - 81)^2$

$y\sqrt{81 - x^2} = \pm(x^2 + 18y - 81)$

consider the case when taking the positive sign

$y\sqrt{81 - x^2} = x^2 + 18y - 81$

$81 - x^2 = 18y - y\sqrt{81 - x^2} = y(18 - \sqrt{81 - x^2})$

$y = f(x) = \frac{81 - x^2}{18 - \sqrt{81 - x^2}}$ and $f(0) = 9$

consider the case when taking the negative sign M1

$y\sqrt{81 - x^2} = -x^2 - 18y + 81$

$81 - x^2 = y\sqrt{81 - x^2} + 18y = y(\sqrt{81 - x^2} + 18)$

$y = g(x) = \frac{81 - x^2}{18 + \sqrt{81 - x^2}}$ and $g(0) = 3$

both have maximal domain of $[-9, 9]$

$g: [-9, 9] \rightarrow R, g(x) = \frac{81 - x^2}{18 + \sqrt{81 - x^2}}$ A1

- d. graphs correct shape, plotting $f(x)$ and $g(x)$ or parametric $x_1(t), y_1(t)$
over correct domain $[-9, 9]$ passing through exact points

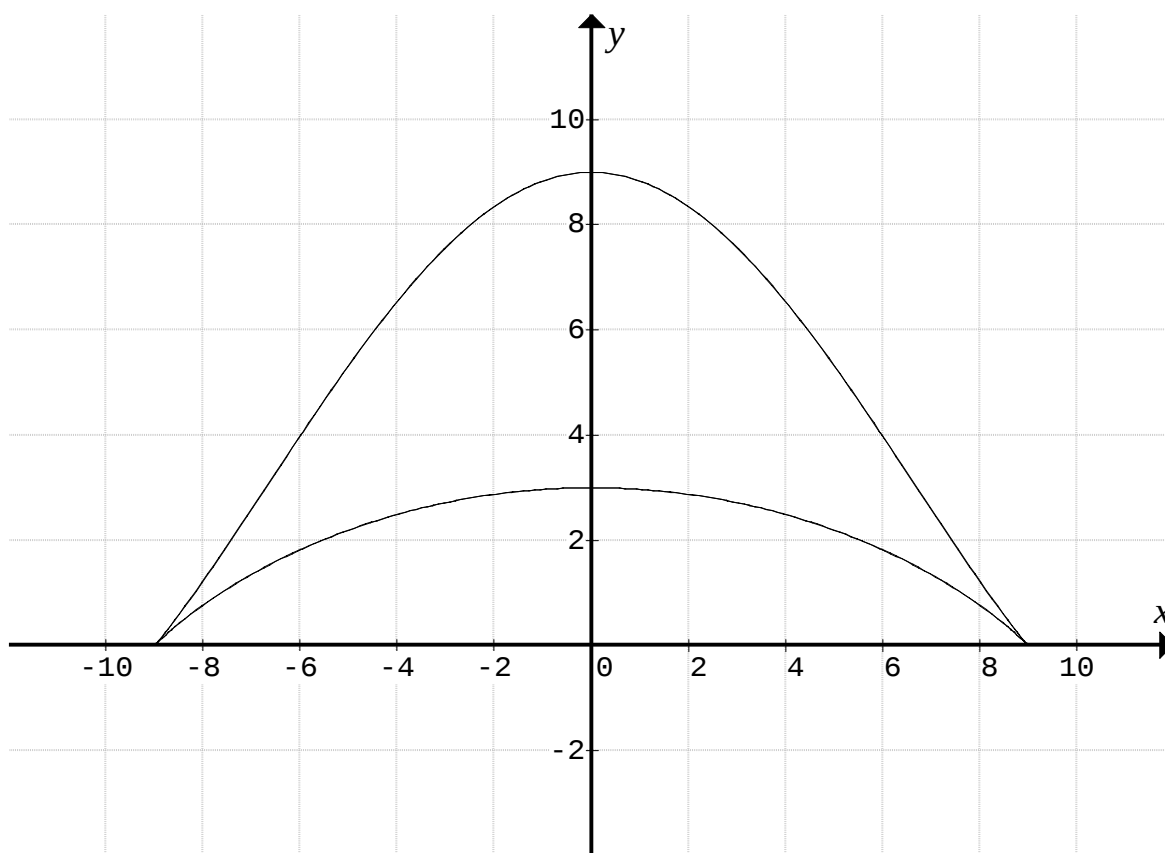
$$t = 0, 2\pi \Rightarrow A(9, 0), t = \frac{\pi}{2} \Rightarrow B(0, 3), t = \pi \Rightarrow C(-9, 0), t = \frac{3\pi}{2} \Rightarrow D(0, 9) \quad A1$$

$B(0, 3)$ and $D(0, 9)$ are a maximum turning points.

The function $f(x)$ passes through ADC

The function $g(x)$ passes through ABC

A1



e. $A = \int_{-9}^9 (f(x) - g(x)) dx = 2 \int_0^9 (f(x) - g(x)) dx$ by symmetry

$$A = 2 \int_0^9 \left(\frac{81 - x^2}{18 - \sqrt{81 - x^2}} - \frac{81 - x^2}{18 + \sqrt{81 - x^2}} \right) dx$$

$$A = 2 \int_0^9 \left(\frac{(81 - x^2) \left[(18 + \sqrt{81 - x^2}) - (18 - \sqrt{81 - x^2}) \right]}{(18 - \sqrt{81 - x^2})(18 + \sqrt{81 - x^2})} \right) dx \quad \text{M1}$$

$$A = 2 \int_0^9 \left(\frac{(81 - x^2) 2\sqrt{81 - x^2}}{(18^2 - (81 - x^2))} \right) dx$$

$$A = \int_0^9 \frac{4(81 - x^2)^{\frac{3}{2}}}{x^2 + 243} dx \Rightarrow n = \frac{3}{2} \quad b = 243 \quad \text{A1}$$

f.i. $y = \frac{81 - x^2}{18 - \sqrt{81 - x^2}}$ let $u = x^2$

$$y = \frac{81 - u}{18 - \sqrt{81 - u}} \text{ solving CAS gives}$$

$$u = x^2 = \frac{1}{2} \left(\sqrt{y^3(y + 72)} - y^2 - 36y + 162 \right)$$

$$V = \pi \int_0^9 x_2^2 dy - \pi \int_0^3 x_1^2 dy$$

$$V = \pi \int_3^9 x^2 dy$$

$$V = \frac{\pi}{2} \int_3^9 \left(\sqrt{y^3(y + 72)} - y^2 - 36y + 162 \right) dy \quad \text{A1}$$

f.ii. $V = 391.2 \text{ units}^3 \quad \text{A1}$

Define $x1(t) = 9 \cdot \cos(t)$	Done
Define $y1(t) = \frac{9 \cdot (\sin(t))^2}{2 + \sin(t)}$	Done
$\frac{d}{dt}(x1(t))$	$-9 \cdot \sin(t)$
$\frac{d}{dt}(y1(t))$	$\frac{9 \cdot \sin(t) \cdot (\sin(t) + 4) \cdot \cos(t)}{(\sin(t) + 2)^2}$
Δ solve $\left(\frac{\frac{d}{dt}(y1(t))}{\frac{d}{dt}(x1(t))} = 0, t \right) 0 \leq t \leq 2 \cdot \pi$	$t = \frac{\pi}{2}$ or $t = \frac{3 \cdot \pi}{2}$
$x1\left(\frac{\pi}{2}\right)$	0
$x1\left(\frac{3 \cdot \pi}{2}\right)$	0
$y1\left(\frac{\pi}{2}\right)$	3
$y1\left(\frac{3 \cdot \pi}{2}\right)$	9
$y^2 \cdot (81 - x^2) = (x^2 + 18 \cdot y - 81)^2$	$-(x^2 - 81) \cdot y^2 = (x^2 + 9 \cdot (2 \cdot y - 9))^2$
impDif $\left(y^2 \cdot (81 - x^2) = (x^2 + 18 \cdot y - 81)^2, x, y \right)$	$\frac{-x \cdot (2 \cdot x^2 + y^2 + 36 \cdot y - 162)}{x^2 \cdot (y + 18) + 243 \cdot (y - 6)}$
Define $f1(x) = \frac{81 - x^2}{18 - \sqrt{81 - x^2}}$	Done
Define $f2(x) = \frac{81 - x^2}{18 + \sqrt{81 - x^2}}$	Done
$f1(0)$	9
$f2(0)$	3
$y = \frac{81 - x^2}{18 + \sqrt{81 - x^2}} x^2 = u$	$y = \frac{-(u - 81)}{\sqrt{81 - u} + 18}$
Δ solve $\left(y = \frac{-(u - 81)}{\sqrt{81 - u} + 18}, u \right)$	$u = \frac{\sqrt{y^3 \cdot (y + 72)} - y^2 - 36 \cdot y + 162}{2}$ or $u = \frac{-(\sqrt{y^3 \cdot (y + 72)} + y^2 + 36 \cdot y - 162)}{2}$
$\pi \int_3^9 \frac{\sqrt{y^3 \cdot (y + 72)} - y^2 - 36 \cdot y + 162}{2} dy$	391.2027

Question 4

a. $\dot{x} = \frac{dx}{dt} = 12 - 0.1x$

$$\frac{dt}{dx} = \frac{1}{12 - 0.1x} = \frac{10}{120 - x}$$

$$\frac{1}{10} \int 1 dt = \int \frac{1}{120 - x} dx \quad \text{M1}$$

$$\frac{t}{10} = -\log_e(120 - x) + C$$

but when $t = 0$ $x = 0$

$$0 = -\log_e(120) + C \Rightarrow C = \log_e(120)$$

$$\frac{t}{10} = -\log_e(120 - x) + \log_e(120) \quad \text{M1}$$

$$\frac{t}{10} = -\log_e\left(\frac{120 - x}{120}\right)$$

$$\frac{120 - x}{120} = e^{-\frac{t}{10}}$$

$$120 - x = 120e^{-\frac{t}{10}} \quad \text{A1}$$

$$x = x(t) = 120\left(1 - e^{-\frac{t}{10}}\right)$$

b. $\dot{y} = \frac{dy}{dt} = 12 - gt = 12 - 9.8t$

$$y = \int (12 - 9.8t) dt$$

$$y = 12t - 4.9t^2 + C \quad \text{A1}$$

but when $t = 0$ $y = 1 \Rightarrow C = 1$

$$y = 12t - 4.9t^2 + 1$$

c. hits the ground when $y = 0 \Rightarrow 12t - 4.9t^2 + 1 = 0$

solving $t = 2.52966$

correct to 3 decimal places $T = 2.530$ s A1

d. the range $R = x(T) = 120\left(1 - e^{-\frac{2.52966}{10}}\right) = 26.821\text{m}$ A1

e. at the maximum height $\dot{y} = 12 - 9.8t = 0 \Rightarrow t = \frac{12}{9.8} = 1.22449$

so $t = 1.224$ s

A1

$$H = y(1.22449) = 1 + 12 \times 1.22449 - 4.9(1.22449)^2 = 8.347 \text{ m}$$

and $x(1.22449) = 120 \left(1 - e^{-\frac{1.22449}{10}} \right) = 13.830 \text{ m}$

A1

f. when it hits the ground $T = 2.5297$

$$\dot{x}(t) = 12 - 0.1x = 12 - 0.1 \left(120 \left(1 - e^{-\frac{t}{10}} \right) \right) = 12e^{-\frac{t}{10}}$$

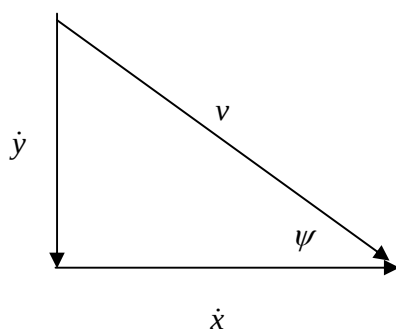
$$\dot{x}(2.5297) = 12e^{-\frac{2.5297}{10}} = 9.318$$

$$\dot{y}(t) = 12 - 9.8t$$

$$\dot{y}(2.52966) = 12 - 9.8 \times 2.52966 = -12.791$$

The speed $|\dot{r}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{(9.318)^2 + (-12.791)^2} = 15.825 \text{ m/s}$

A1



The angle at which it hits the ground $\tan(\psi) = \left| \frac{\dot{y}}{\dot{x}} \right|$

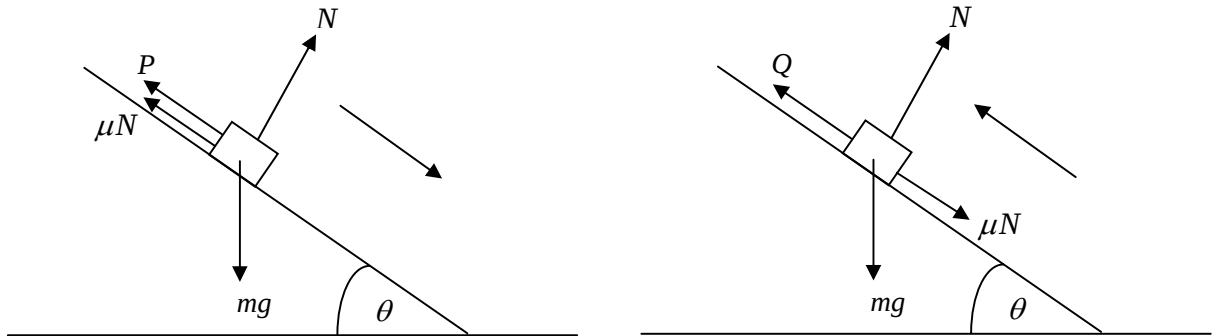
$$\psi = \tan^{-1} \left(\frac{12.791}{9.318} \right) = 53.927^\circ$$

$$\psi = 53^\circ 56'$$

A1

Define $xI(t)=120 \cdot \left(1-e^{-\frac{t}{10}}\right)$	Done
Define $yI(t)=1+12 \cdot t-4 \cdot 9 \cdot t^2$	Done
$\text{solve}(yI(t)=0,t) t \geq 0$	$t=2.5297$
$tg:=2.5296552616588$	2.5297
$xI(tg)$	26.8206
$\frac{d}{dt}(yI(t))$	$12.0000-9.8000 \cdot t$
$\text{solve}\left(\frac{d}{dt}(yI(t))=0,t\right)$	$t=1.2245$
$tt:=1.2244897959184$	1.2245
$yI(tt)$	8.3469
$xI(tt)$	13.8299
$\frac{d}{dt}(xI(t)) t=tg$	9.3179
$\frac{d}{dt}(yI(t)) t=tg$	-12.7906
$\sqrt{(-12.79062)^2+(9.31793)^2}$	15.8248
$\tan^{-1}\left(\frac{-12.79062}{9.31793}\right)$	53.9268
$(53.926769293273) \blacktriangleright \text{DMS}$	53°55'36.3695"

g.



diagrams, with correct forces

in both situations resolving perpendicular to the plane

$$N - mg \cos(\theta) = 0 \quad \Rightarrow \quad N = mg \cos(\theta) \quad \text{A1}$$

when the football is on the point of moving down the grassy slope

$$(1) \quad P + \mu N - mg \sin(\theta) = 0$$

when the football is on the point of moving up the grassy slope

$$(2) \quad Q - \mu N - mg \sin(\theta) = 0 \quad \text{M1}$$

adding (1)+(2) gives

$$P + Q = 2mg \sin(\theta)$$

$$\sin(\theta) = \frac{P + Q}{2mg} \quad \text{A1}$$

subtracting (2)-(1) gives

$$Q - P = 2\mu N = 2\mu mg \cos(\theta)$$

$$\cos(\theta) = \frac{Q - P}{2\mu mg}$$

eliminating θ using $\sin^2(\theta) + \cos^2(\theta) = 1$ M1

$$\frac{(P + Q)^2}{4m^2 g^2} + \frac{(Q - P)^2}{4\mu^2 m^2 g^2} = 1$$

$$\frac{(Q - P)^2}{4\mu^2 m^2 g^2} = 1 - \frac{(P + Q)^2}{4m^2 g^2} = \frac{4m^2 g^2 - (P + Q)^2}{4m^2 g^2} \quad \text{since } Q > P$$

$$\mu = \frac{Q - P}{\sqrt{4m^2 g^2 - (P + Q)^2}} \quad \text{A1}$$

END OF SECTION 2 SUGGESTED ANSWERS