

Year 2014

VCE

Specialist Mathematics

Trial Examination 1

Solutions



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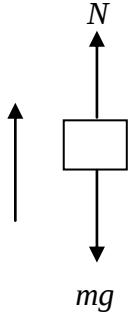
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Question 1

$$ma = N - mg$$

$$m = 60 \text{ kg}, N = 788 \text{ newtons}$$

M1

$$60a = 788 - 60 \times 9.8$$

$$60a = 788 - 588 = 200$$

$$a = \frac{10}{3} \text{ m/s}^2$$

A1

$$s = ut + \frac{1}{2}at^2$$

$$s = ?, u = 0, a = \frac{10}{3}, t = 3$$

$$s = 0 + \frac{1}{2} \times \frac{10}{3} \times 9$$

$$s = 15 \text{ metres}$$

A1

Question 2

$$y = \arccos\left(\frac{3x}{4}\right) = \cos^{-1}\left(\frac{3x}{4}\right)$$

$$\frac{dy}{dx} = \frac{-3}{\sqrt{16-9x^2}} = -3(16-9x^2)^{-\frac{1}{2}}$$

A1

$$\frac{d^2y}{dx^2} = -3 \times \frac{-1}{2} \times -18x(16-9x^2)^{-\frac{3}{2}} = \frac{-27x}{\sqrt{(16-9x^2)^3}}$$

A1

$$\frac{d^2y}{dx^2} = ax \left(\frac{dy}{dx}\right)^3 \text{ substituting}$$

$$\frac{-27x}{\sqrt{(16-9x^2)^3}} = ax \left(\frac{-3}{\sqrt{16-9x^2}}\right)^3 = \frac{-27ax}{\sqrt{(16-9x^2)^3}}$$

$$\Rightarrow a = 1$$

A1

Question 3

$xe^{2y} - y = c$ using implicit differentiation and the product rule

$$e^{2y} + 2xe^{2y} \frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$e^{2y} = (1 - 2xe^{2y}) \frac{dy}{dx} \quad \text{M1}$$

$$\frac{dy}{dx} = \frac{e^{2y}}{1 - 2xe^{2y}}$$

$$m_N = 3 \Rightarrow m_T = -\frac{1}{3} = \frac{dy}{dx} \text{ when it crosses x-axis } y = 0 \quad \text{A1}$$

$$-\frac{1}{3} = \frac{1}{1 - 2x} \Rightarrow 1 - 2x = -3 \quad 2x = 4 \Rightarrow x = 2 \quad P(2, 0)$$

$$c = xe^{2y} - y = 2e^0 - 0$$

$$c = 2 \quad \text{A1}$$

Question 4

a. $z = (1 - i)^3 (-\sqrt{3} + i)^4$

$$z = \left[\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right) \right]^3 \left[2 \operatorname{cis} \left(\frac{5\pi}{6} \right) \right]^4 \quad \text{M1}$$

$$\arg(z) = -3 \times \frac{\pi}{4} + 4 \times \frac{5\pi}{6} = -\frac{3\pi}{4} + \frac{10\pi}{3} = \frac{31\pi}{12}$$

$$\operatorname{Arg}(z) = \frac{31\pi}{12} - 2\pi$$

$$\operatorname{Arg}(z) = \frac{7\pi}{12} \quad k = \frac{7}{12} \quad \text{A1}$$

b. $z^2 = -8i = 8 \operatorname{cis} \left(-\frac{\pi}{2} + 2k\pi \right)$

$$z = \sqrt{8} \operatorname{cis} \left(-\frac{\pi}{4} + k\pi \right)$$

$$k = 0 \quad z = \sqrt{8} \operatorname{cis} \left(-\frac{\pi}{4} \right) = 2\sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right) \quad \text{A1}$$

$$= 2\sqrt{2} \left(\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) = 2 - 2i$$

$$k = 1 \quad z = \sqrt{8} \operatorname{cis} \left(\frac{3\pi}{4} \right) = 2\sqrt{2} \left(\cos \left(\frac{3\pi}{4} \right) + i \sin \left(\frac{3\pi}{4} \right) \right) \quad \text{A1}$$

$$= 2\sqrt{2} \left(-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = -2 + 2i$$

Question 5

$$v(t) = \frac{dx}{dt} = \frac{30t}{\sqrt{25+2t^2}}$$

a. distance travelled in 10 seconds $D = \int_0^{10} \frac{30t}{\sqrt{25+2t^2}} dt$ A1

let $u = 25 + 2t^2$ $\frac{du}{dt} = 4t$

terminals when $t = 0$ $u = 25$ and when $t = 10$ $u = 225$

$$D = \frac{30}{4} \int_{25}^{225} u^{-\frac{1}{2}} du$$
 M1

$$D = \frac{15}{2} \left[2u^{\frac{1}{2}} \right]_{25}^{225} = 15 \left[\sqrt{225} - \sqrt{25} \right] = 15(15 - 5)$$

$$D = 150 \text{ metres}$$
 A1

b. $\lim_{t \rightarrow \infty} v(t)$

$$= \lim_{t \rightarrow \infty} \left(\frac{30t}{\sqrt{25+2t^2}} \right) = \lim_{t \rightarrow \infty} \left(\frac{30}{\sqrt{\frac{25}{t^2} + 2}} \right) = \frac{30}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= 15\sqrt{2} \text{ ms}^{-1}$$
 A1

Question 6

$$V = \pi \int_a^b y^2 dx \quad y = 4 \cos\left(\frac{x}{3}\right) \quad y = 0 \Rightarrow \frac{x}{3} = \frac{\pi}{2} \quad x = \frac{3\pi}{2}$$

$$V = \pi \int_0^{\frac{3\pi}{2}} 16 \cos^2\left(\frac{x}{3}\right) dx$$
 A1

$$V = 8\pi \int_0^{\frac{3\pi}{2}} \left(1 + \cos\left(\frac{2x}{3}\right) \right) dx$$

$$V = 8\pi \left[x + \frac{3}{2} \sin\left(\frac{2x}{3}\right) \right]_0^{\frac{3\pi}{2}}$$
 M1

$$V = 8\pi \left[\left(\frac{3\pi}{2} + \frac{3}{2} \sin(\pi) \right) - \left(0 + \frac{3}{2} \sin(0) \right) \right]$$

$$V = 12\pi^2 \text{ units}^3$$
 A1

Question 7

$$y = \frac{x^4 - 81}{3x^3} = \frac{x}{3} - \frac{27}{x^3} = \frac{x}{3} - 27x^{-3}$$

crosses x-axis when $y = 0 \Rightarrow x^4 - 81 = 0 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

$(3,0)$ $(-3,0)$

A1

does not cross the y-axis

$x = 0$ is a vertical asymptote and $y = \frac{x}{3}$ is oblique asymptote

A1

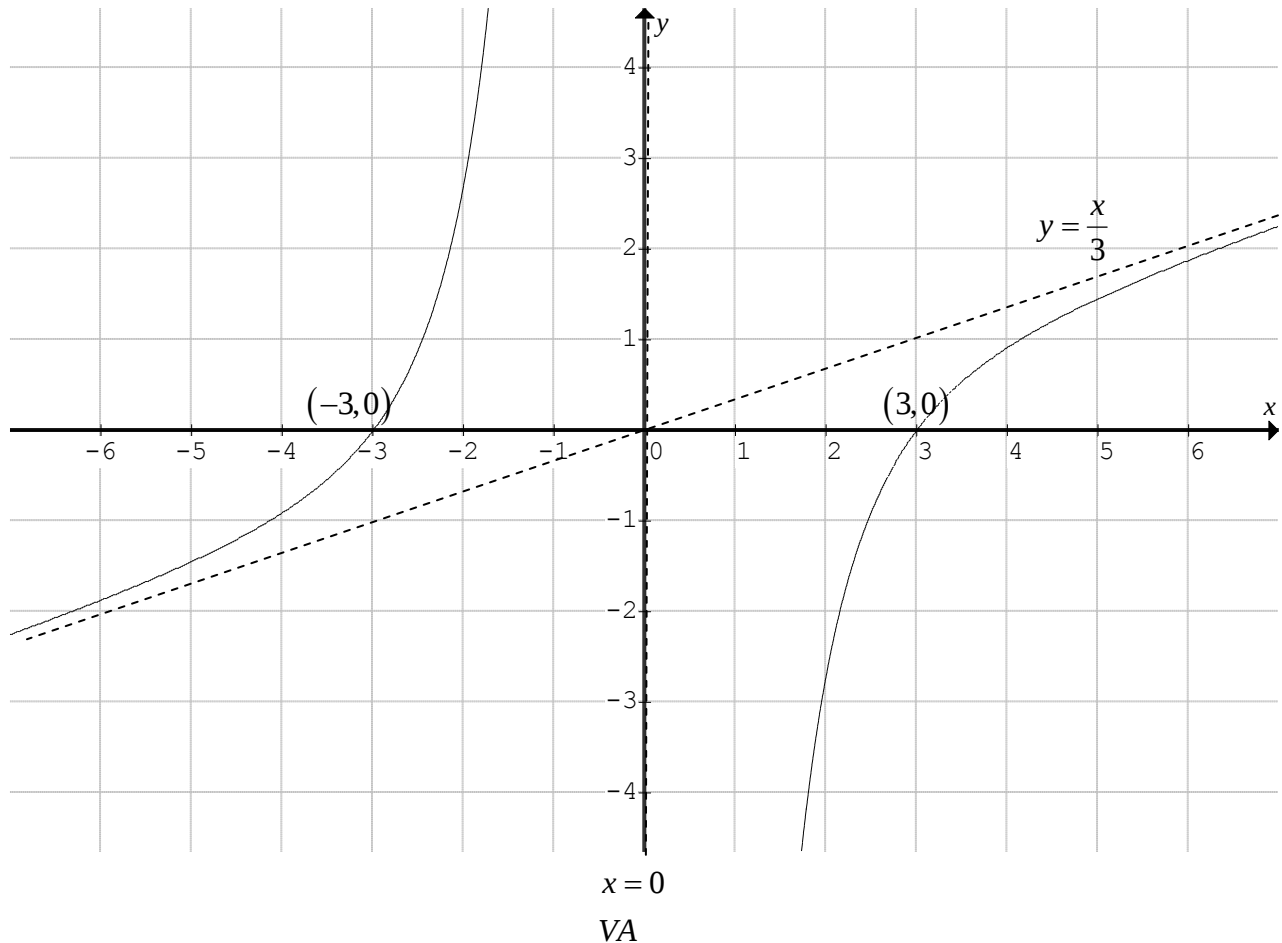
for turning points $\frac{dy}{dx} = \frac{1}{3} + 81x^{-4} = \frac{1}{3} + \frac{81}{x^4} = 0 \Rightarrow x^4 = -27$

this has no real solutions, there are no turning points

A1

correct graph, shape asymptotes

G1



Question 8

a. $\underline{a} = 2\underline{i} - \underline{j} - 2\underline{k}$, $\underline{b} = 5\underline{i} + 4\underline{j} + 3\underline{k}$ and $\underline{c} = 4\underline{i} + 11\underline{j} + z\underline{k}$.

$\underline{a}, \underline{b}$ and $\underline{c}$ are linearly dependent $\Rightarrow \underline{c} = m\underline{a} + n\underline{b}$, $m, n \in \mathbb{R} \setminus \{0\}$

$\underline{i}$: $\Rightarrow (1) \quad 4 = 2m + 5n$

$\underline{j}$: $\Rightarrow (2) \quad 11 = -m + 4n$

M1

$\underline{k}$: $\Rightarrow (3) \quad z = -2m + 3n$

(1) $4 = 2m + 5n$

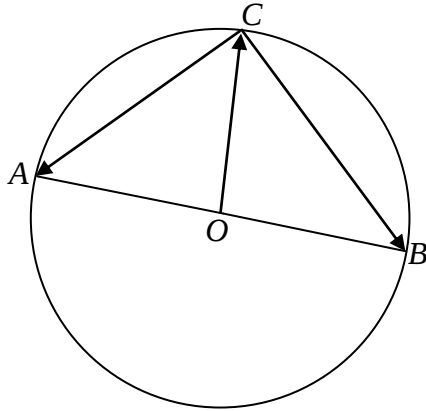
$2 \times (2) \quad 22 = -2m + 8n$ adding $26 = 13n \Rightarrow n = 2$

and $2m = 4 - 5n = 4 - 10 = -6 \Rightarrow m = -3$

into (3) $z = 12$

A1

b.



Let $\overrightarrow{OC} = \underline{c}$ and $\overrightarrow{OB} = \underline{b}$

$\overrightarrow{OA} = -\underline{b}$, since O is the midpoint of AB

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = -\underline{b} - \underline{c} = -(\underline{b} + \underline{c})$

M1

$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = \underline{b} - \underline{c}$

$\overrightarrow{CA} \cdot \overrightarrow{CB} = -(\underline{b} + \underline{c}) \cdot (\underline{b} - \underline{c}) = -(\underline{b} \cdot \underline{b} + \underline{c} \cdot \underline{b} - \underline{b} \cdot \underline{c} - \underline{c} \cdot \underline{c})$

A1

but $\underline{c} \cdot \underline{b} = \underline{b} \cdot \underline{c}$ and $\underline{b} \cdot \underline{b} = |\underline{b}|^2$ and $\underline{c} \cdot \underline{c} = |\underline{c}|^2$

$\overrightarrow{CA} \cdot \overrightarrow{CB} = |\underline{c}|^2 - |\underline{b}|^2$

however $|\underline{b}| = |\underline{c}|$ since both are radii of the circle

so that $\overrightarrow{CA} \cdot \overrightarrow{CB} = 0 \Rightarrow \overrightarrow{CA}$ is perpendicular to $\overrightarrow{CB}$

A1

Question 9

a. $\vec{r}(t) = (3 - 2\cos(2t))\vec{i} + (4 - 3\sin(2t))\vec{j}$ for $t \geq 0$

$$x = 3 - 2\cos(2t) \quad y = 4 - 3\sin(2t)$$

$$\cos(2t) = \frac{3-x}{2}, \quad \sin(2t) = \frac{4-y}{3}$$

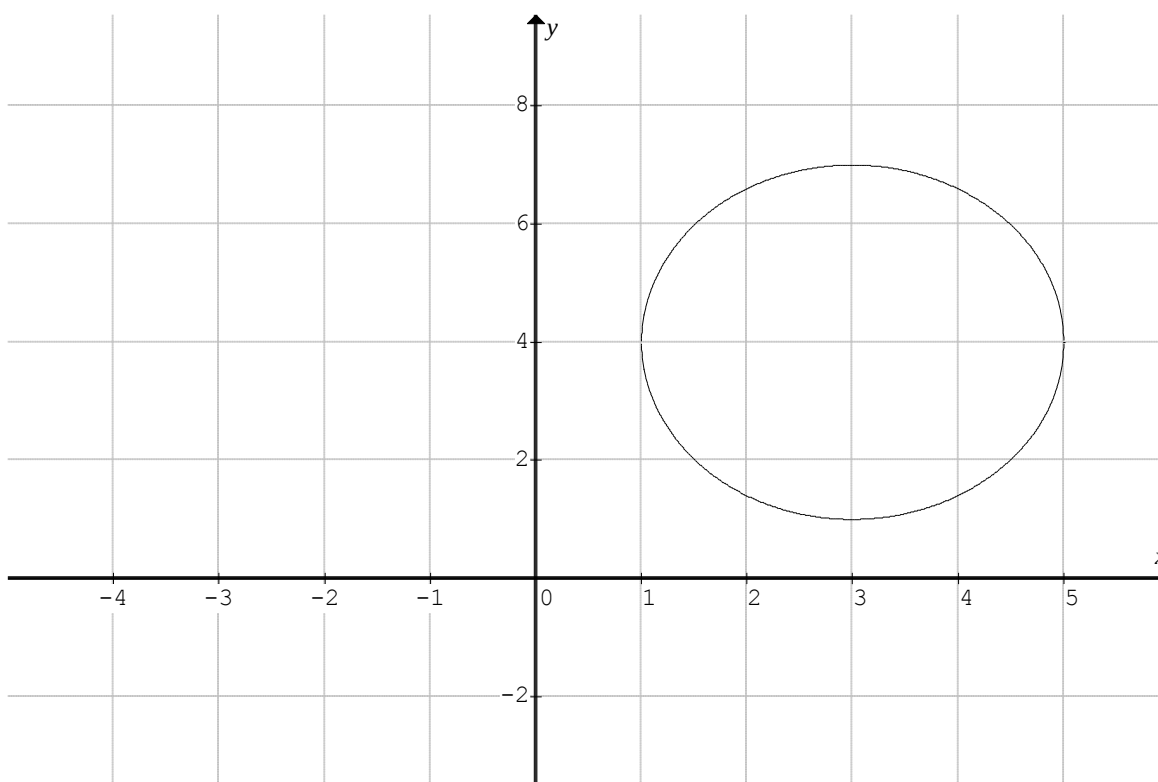
$$\sin^2(2t) + \cos^2(2t) = 1$$

$$\frac{(x-3)^2}{4} + \frac{(y-4)^2}{9} = 1 \quad \text{ellipse centre } (3,4)$$

$$\text{domain } [1,5] \quad \text{range } [1,7]$$

graph correct

G1



b. $\vec{r}(t) = (3 - 2\cos(2t))\vec{i} + (4 - 3\sin(2t))\vec{j}$

$\dot{\vec{r}}(t) = 4\sin(2t)\vec{i} - 6\cos(2t)\vec{j}$ A1

$|\dot{\vec{r}}(t)| = \sqrt{16\sin^2(2t) + 36\cos^2(2t)}$

$= \sqrt{16(1 - \cos^2(2t)) + 36\cos^2(2t)}$ M1

$= \sqrt{20\cos^2(2t) + 16}$

$|\dot{\vec{r}}(t)|_{\max} = \sqrt{36} = 6$

when $\cos(2t) = 1 \Rightarrow 2t = 2k\pi \quad t = k\pi$ A1

$|\dot{\vec{r}}(t)|_{\min} = \sqrt{16} = 4$

when $\cos(2t) = 0 \Rightarrow 2t = (2k+1)\frac{\pi}{2} \quad t = (2k+1)\frac{\pi}{4}, k \in \mathbb{Z}$ A1

Question 10

$\int \frac{5x+3}{4x^2+81} dx$ separate out into two integrals

$= 5 \int \frac{x}{4x^2+81} dx + 3 \int \frac{1}{4x^2+81} dx$

let $u = 4x^2 + 81$ let $v = 2x$

$\frac{du}{dx} = 8x$ $\frac{dv}{dx} = 2$ M1

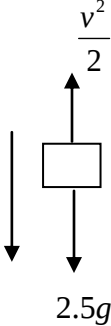
$= \frac{5}{8} \int \frac{1}{u} du + \frac{3}{2} \int \frac{1}{81+v^2} dv$

$= \frac{5}{8} \log_e(|u|) + \frac{3}{2} \times \frac{1}{9} \tan^{-1}\left(\frac{v}{9}\right) + c$

$= \frac{5}{8} \log_e(4x^2+9) + \frac{3}{2} \times \frac{1}{9} \tan^{-1}\left(\frac{v}{9}\right) + c$ since $4x^2+9 > 0$

$= \frac{5}{8} \log_e(4x^2+81) + \frac{1}{6} \tan^{-1}\left(\frac{2x}{9}\right) + c$ A1

Question 11

a. 

$$2.5a = 2.5 \times 9.8 - \frac{v^2}{2}$$

$$\frac{5a}{2} = \frac{5 \times 9.8}{2} - \frac{v^2}{2}$$

$$5a = 49 - v^2$$

$$a = \frac{dv}{dt} = \frac{49 - v^2}{5}$$

M1

b. $\frac{dt}{dv} = \frac{5}{49 - v^2}$ inverting both sides

$$t = \int \frac{5}{49 - v^2} dv \quad \text{by partial fractions}$$

$$\frac{5}{49 - v^2} = \frac{A}{7 + v} + \frac{B}{7 - v} = \frac{A(7 - v) + B(7 + v)}{(7 + v)(7 - v)} = \frac{7(A + B) + v(B - A)}{49 - v^2}$$

M1

$$7(A + B) = 5$$

$$B - A = 0 \Rightarrow A = B = \frac{5}{14}$$

$$t = \frac{5}{14} \int \left(\frac{1}{7 + v} + \frac{1}{7 - v} \right) dv$$

$$t = \frac{5}{14} \left[\log_e(|7 + v|) - \log_e(|7 - v|) \right] + c$$

A1

when $t = 0 \quad v = 0 \Rightarrow c = 0$

$$t = \frac{5}{14} \log_e \left(\frac{|7 + v|}{|7 - v|} \right)$$

A1

END OF SUGGESTED SOLUTIONS