

2014 VCAA Specialist Math Exam 2 Solutions

© 2014 itute.com

SECTION 1

1	2	3	4	5	6	7	8	9	10	11
B	E	D	C	A	D	E	C	B	A	D

12	13	14	15	16	17	18	19	20	21	22
C	E	C	B	A	D	E	C	B	A	E

Q1 Asymptotes are $y = \pm \frac{2}{3}(x-3)$

x -intercept: $(3, 0)$; y -intercepts: $(0, -2)$ and $(0, 2)$

B

Q2 $x^2 - 6x + 2y^2 + 8y + 16 = 0$,

$x^2 - 6x + 9 + 2(y^2 + 4y + 4) + 16 = 9 + 8$

$(x-3)^2 + 2(y+2)^2 = 1$, $\frac{(x-3)^2}{1^2} + \frac{(y+2)^2}{(\frac{1}{\sqrt{2}})^2} = 1$

E

Q3 $f(x) = \frac{x^2 - 4x + 3}{x^2 - x - 6} = 1 - \frac{3(x-3)}{(x+2)(x-3)}$

or $1 - \frac{3}{x+2}$ where $x \neq 3$

D

Q4 For $\arcsin(2x-1)$, $-1 \leq 2x-1 \leq 1$, $\therefore 0 \leq x \leq 1$

C

Q5 $z^2 = (2\sqrt{2})^2 \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) = -8i$

A

Q6 $i^{2n+3} = i^{2n} i^3 = (i^2)^n i^3 = -i p^2$

D

Q7 $z^3 - 5z^2 + 11z - 7 = (z-\alpha)(z-\beta)(z-\gamma) = 0$

Coefficient of z^2 : $-(\alpha + \beta + \gamma) = -5$, $\therefore \alpha + \beta + \gamma = 5$

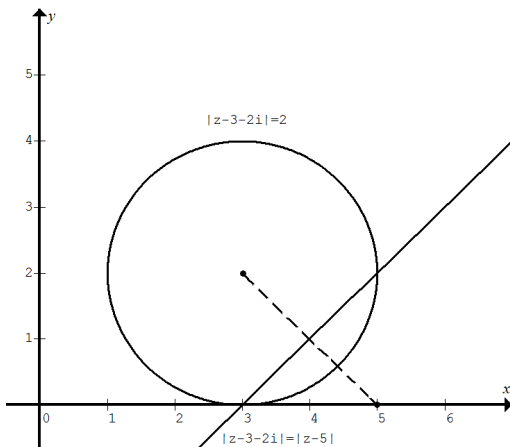
E

Q8 $\frac{-3\sqrt{2} - i\sqrt{6}}{2 + 2i} = \frac{A \operatorname{cis} \frac{7\pi}{6}}{B \operatorname{cis} \frac{\pi}{4}} = C \operatorname{cis} \frac{11\pi}{12}$

C

Q9

B



Q10 After t minutes, Q kg of salt is in $1500 - 2t$ litres of

solution, $\therefore$ concentration is $\frac{Q}{1500 - 2t}$ kg per litre

Rate of inflow = $2 \times 8 = 16$ kg per minute (2 kg of salt per litre?)

Rate of outflow = $\frac{Q}{1500 - 2t} \times 10 = \frac{5Q}{750 - t}$ kg per minute

$\therefore \frac{dQ}{dt} = 16 - \frac{5Q}{750 - t}$

A

Q11 $y_{n+1} \approx y_n + h \times \frac{dy}{dx}$

$x_0 = 1$ $y_0 = 2$ $\frac{dy}{dx} = x^3 - xy = -1$

$x_1 = 1.1$ $y_1 \approx 2 + 0.1(-1) = 1.9$

D

Q12

C

Q13 $u = \sqrt{x+1}$, $\frac{du}{dx} = \frac{1}{2\sqrt{x+1}}$, $u^2 + 1 = x + 2$

When $x = 0$, $u = 1$; when $x = 2$, $u = \sqrt{3}$

$\int_0^2 \frac{dx}{(x+2)\sqrt{x+1}} = 2 \int_1^{\sqrt{3}} \frac{1}{u^2 + 1} du$

E

Q14 At $y = x$, $\frac{dy}{dx}$ is undefined; when $y < x$, $\frac{dy}{dx} < 0$;

when $y > x$, $\frac{dy}{dx} > 0$

C

Q15 $|\tilde{a}| = \sqrt{20}$, $|\tilde{b}| = \sqrt{20}$, $\cos \theta = \frac{\tilde{a} \cdot \tilde{b}}{|\tilde{a}| |\tilde{b}|} = -\frac{4}{5}$

$\cos 2\theta = 2 \cos^2 \theta - 1 = \frac{7}{25}$

B

Q16 $\tilde{a} = 4\tilde{i} + m\tilde{j} - 3\tilde{k}$, $\tilde{b} = -2\tilde{i} + n\tilde{j} - \tilde{k}$, where $m, n \in \mathbb{R}^+$

$|\tilde{a}|^2 = 100$, $\therefore m = 5\sqrt{3}$; $\tilde{a} \cdot \tilde{b} = 0$, $\therefore mn = 5$, $\therefore n = \frac{\sqrt{3}}{3}$

A

Q17 $\tilde{v}(t) = \int (-4 \sin 2t \tilde{i} + 20 \cos 2t \tilde{j} - 20e^{-2t} \tilde{k}) dt$

$= 2 \cos 2t \tilde{i} + 10 \sin 2t \tilde{j} + 10e^{-2t} \tilde{k} + \tilde{c}$

Given $\tilde{v}(0) = 0$, $\therefore \tilde{c} = -2\tilde{i} - 10\tilde{k}$ and

$\tilde{v}(t) = (2 \cos 2t - 2)\tilde{i} + 10 \sin 2t \tilde{j} + (10e^{-2t} - 10)\tilde{k}$

D

Q18 North-south: $1 + 2 \cos 60^\circ + 4 \cos 120^\circ = 0$

East-west: $2 \sin 60^\circ + 4 \sin 120^\circ - 5 = 3\sqrt{3} - 5 > 0$

$\therefore$ the net force acts in a easterly direction.

The initial state of motion is not specified!

Assume that the body is initially at rest (or moving to the east), it will move to the east.

E

Q19 $\tilde{v}(t) = 3 \sin 2t \tilde{i} + 4 \cos 2t \tilde{j}$, $\tilde{a}(t) = \frac{d\tilde{v}}{dt} = 6 \cos 2t \tilde{i} - 8 \sin 2t \tilde{j}$

Net force $= m\tilde{a} = 30 \cos 2t \tilde{i} - 40 \sin 2t \tilde{j}$

$|\text{Net force}| = \sqrt{900 \cos^2 2t + 1600 \sin^2 2t} = \sqrt{900 + 700 \sin^2 2t}$

$\therefore$ the max. magnitude of the net force $= \sqrt{900 + 700} = 40$

C

Q20 Net force $= 5 \times 9.8 - 3 \times 9.8 = 19.6 \text{ N}$

Acceleration $= \frac{\text{net force}}{\text{total mass}} = \frac{19.6}{8} = 2.45 \text{ m s}^{-2}$

After 2 seconds, $v = u + at = 0 + 2.45 \times 2 = 4.9 \text{ m s}^{-1}$

B

Q21 $a = -4x$, $\frac{1}{2} \frac{dv^2}{dx} = -4x$, $\frac{dv^2}{dx} = -8x$, $v^2 = -4x^2 + c$

Given $v = 0$ at $x = 5$, $\therefore c = 100$ and $v^2 = 100 - 4x^2$

At $x = 3$, $v^2 = 64$, $\therefore |v| = 8$

A

Q22 In $0 \leq t \leq 4$, distance $= \frac{1}{2}(2+4)(9) = 27$

In $4 \leq t \leq 8$, distance $= \int_4^8 \left(-\frac{9}{16}(t-4)^2 + 9 \right) dt = 24$

In $8 \leq t \leq 9$, distance $= -\int_8^9 \left(-\frac{9}{16}(t-4)^2 + 9 \right) dt = 2.4375$

Total distance ≈ 53.4

E

SECTION 2

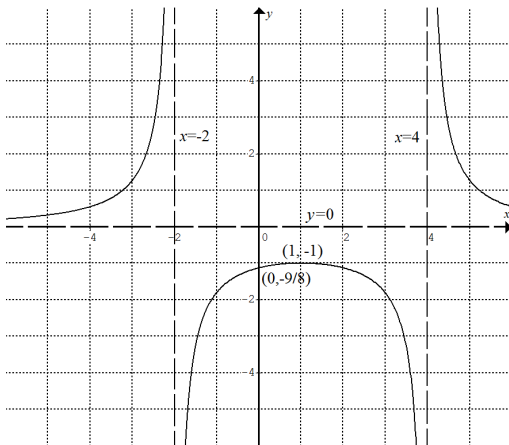
Q1a $y = \frac{9}{(x+2)(x-4)} = \frac{9}{x^2 - 2x - 8}$, $\frac{1}{y} = \frac{x^2 - 2x - 8}{9}$,
 $-\frac{1}{y^2} \frac{dy}{dx} = \frac{2x-2}{9}$.

Let $\frac{dy}{dx} = 0$. $\therefore x = 1$ and $y = -1$

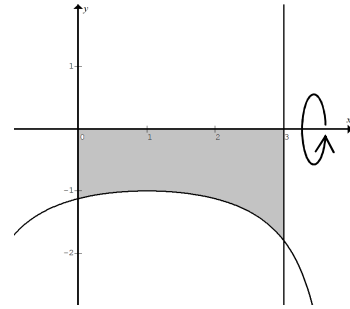
The stationary point is $(1, -1)$.

Q1b $x = -2$, $x = 4$, $y = 0$

Q1c



Q1di $V = \int_0^3 \pi y^2 dx = \int_0^3 \frac{81\pi}{(x+2)^2(x-4)^2} dx$



Q1dii By CAS, $V = 12.85$ cubic units

Q2ai $z_1 = \sqrt{3} - 3i$, $|z_1| = \sqrt{(\sqrt{3})^2 + (-3)^2} = \sqrt{12} = 2\sqrt{3}$

$\text{Arg}(z_1) = \tan^{-1}\left(\frac{-3}{\sqrt{3}}\right) = -\frac{\pi}{3}$, $\therefore z_1 = 2\sqrt{3} \text{cis}\left(-\frac{\pi}{3}\right)$

Q2aai $z_1 = 2\sqrt{3} \text{cis}\left(-\frac{\pi}{3}\right)$, $\arg(z_1^4) = 4 \arg(z_1) = -\frac{4\pi}{3}$

$\therefore \text{Arg}(z_1^4) = \frac{2\pi}{3}$

Q2aiii $z_1 = \sqrt{3} - 3i$ is a root of $z^3 + 24\sqrt{3} = 0$, $\therefore z = \sqrt{3} + 3i$ is also a root.

$\therefore z^3 + 24\sqrt{3} = (z - (\sqrt{3} - 3i))(z - (\sqrt{3} + 3i))(z - p) = 0$ where $p \in \mathbb{R}$

$z^3 + 24\sqrt{3} = (z^2 - 2\sqrt{3}z + 12)(z - p) = 0$

$\therefore -12p = 24\sqrt{3}$, $\therefore p = -2\sqrt{3}$

The other 2 roots are $\sqrt{3} + 3i$ and $-2\sqrt{3}$.

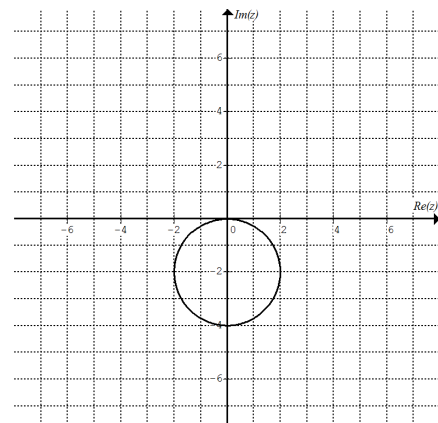
Q2bi $(z_1 + 2i)(\bar{z}_1 - 2i) = (\sqrt{3} - 3i + 2i)(\sqrt{3} + 3i - 2i)$
 $= (\sqrt{3} - i)(\sqrt{3} + i) = 4$

Q2bii Let $z = x + yi$.

$(z + 2i)(z - 2i) = (x + (y+2)i)(x - (y+2)i) = 4$

$\therefore x^2 + (y+2)^2 = 4$

Q2biii



Q2c Let the line be $y = mx + c$. It passes through $(k, -2)$ and $(0, -(2+k))$, where $k < 0$.

$$\therefore c = -(2+k) \text{ and } m = \frac{-2+(2+k)}{k-0} = 1$$

$\therefore y = x - (2+k)$ is a tangent line to $x^2 + (y+2)^2 = 4$.

Solve the two equations simultaneously:

$$x^2 + (x - (2+k) + 2)^2 = 4, \therefore 2x^2 - 2kx + k^2 - 4 = 0 \text{ and its}$$

discriminant $\Delta = 0$, i.e. $(-2k)^2 - 4(2)(k^2 - 4) = 0$

$$\therefore -4k^2 + 32 = 0, \therefore k = -2\sqrt{2} \text{ since } k < 0.$$

$$\text{Q3a } \tilde{a} = 3\tilde{i} + 2\tilde{j} + \tilde{k}, \hat{b} = \frac{\tilde{b}}{|\tilde{b}|} = \frac{2\tilde{i} - 2\tilde{j} - \tilde{k}}{3}, \tilde{a} \cdot \hat{b} = \frac{1}{3},$$

$\therefore$ the parallel vector resolute is $(\tilde{a} \cdot \hat{b})\hat{b} = \frac{2}{9}\tilde{i} - \frac{2}{9}\tilde{j} - \frac{1}{9}\tilde{k}$,

and the perpendicular vector resolute is

$$\tilde{a} - (\tilde{a} \cdot \hat{b})\hat{b} = 3\tilde{i} + 2\tilde{j} + \tilde{k} - \left(\frac{2}{9}\tilde{i} - \frac{2}{9}\tilde{j} - \frac{1}{9}\tilde{k}\right) = \frac{25}{9}\tilde{i} + \frac{20}{9}\tilde{j} + \frac{10}{9}\tilde{k}$$

$$\therefore \tilde{a} = \left(\frac{2}{9}\tilde{i} - \frac{2}{9}\tilde{j} - \frac{1}{9}\tilde{k}\right) + \left(\frac{25}{9}\tilde{i} + \frac{20}{9}\tilde{j} + \frac{10}{9}\tilde{k}\right) \\ = \frac{1}{9}(2\tilde{i} - 2\tilde{j} - \tilde{k}) + \frac{5}{9}(5\tilde{i} + 4\tilde{j} + 2\tilde{k})$$

$$\text{Q3bi } \overrightarrow{AP} = \alpha \overrightarrow{AD} = \alpha(\overrightarrow{AB} + \overrightarrow{BD}) = \alpha\left(\overrightarrow{OC} - \frac{1}{2}\overrightarrow{OA}\right) = \alpha\tilde{c} - \frac{1}{2}\alpha\tilde{a}$$

$$\text{Q3bii } \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \beta\overrightarrow{OB} - \overrightarrow{OA} = \beta(\overrightarrow{OA} + \overrightarrow{OC}) - \overrightarrow{OA} \\ = \beta(\tilde{a} + \tilde{c}) - \tilde{a} = \beta\tilde{c} - (1-\beta)\tilde{a}$$

$$\text{Q3biii From parts i and ii, } \alpha\tilde{c} - \frac{1}{2}\alpha\tilde{a} = \beta\tilde{c} - (1-\beta)\tilde{a}$$

$$\therefore \alpha = \beta \text{ and } \frac{1}{2}\alpha = 1 - \beta, \therefore \alpha = \beta = \frac{2}{3}$$

$$\text{Q4a } \frac{r}{h} = \frac{0.5}{1}, r = \frac{h}{2}, \therefore V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

$$\text{Q4b } \frac{dV}{dh} = \frac{\pi}{4}h^2; \frac{dV}{dt} = 0.02\pi - 0.01\pi\sqrt{h} = 0.01\pi(2 - \sqrt{h})$$

$$\frac{dV}{dh} \times \frac{dh}{dt} = \frac{dV}{dt}, \frac{\pi}{4}h^2 \times \frac{dh}{dt} = 0.01\pi(2 - \sqrt{h}), \frac{dh}{dt} = \frac{0.04(2 - \sqrt{h})}{h^2}$$

$$\text{When } h = 0.25, \frac{dh}{dt} = 0.96 \text{ m/min}$$

$$\text{Q4c } \frac{dh}{dt} = \frac{0.04(2 - \sqrt{h})}{h^2}, \frac{dt}{dh} = \frac{25h^2}{2 - \sqrt{h}}$$

$$t = \int_0^1 \frac{25h^2}{2 - \sqrt{h}} dh \approx 7.4 \text{ minutes (By CAS)}$$

$$\text{Q4d } V = \frac{\pi}{48}(x^3 + 6x^2 + 12x), \frac{dV}{dx} = \frac{\pi}{16}(x^2 + 4x + 4) = \frac{\pi}{16}(x+2)^2$$

$$\frac{dV}{dx} \times \frac{dx}{dt} = \frac{dV}{dt}, \frac{\pi}{16}(x+2)^2 \frac{dx}{dt} = 0.05\pi$$

$$\therefore \frac{dx}{dt} = \frac{4}{5(x+2)^2}, \therefore \frac{dt}{dx} = \frac{5}{4}(x+2)^2 \text{ and } x = 0 \text{ at } t = 0$$

$$\therefore t = \int_0^x \frac{5}{4}(x+2)^2 dx = \left[\frac{5(x+2)^3}{12}\right]_0^x = \frac{5(x+2)^3}{12} - \frac{10}{3}$$

$$\therefore \frac{12}{5}\left(t + \frac{10}{3}\right) = (x+2)^3, \therefore (x+2)^3 = 8(0.3t + 1)$$

$$\therefore x = 2(0.3t + 1)^{\frac{1}{3}} - 2$$

$$\text{Q5ai } T_1 - 2g = 2a$$

$$\text{Q5aii } T_2 + 5g \sin \theta - T_1 = 5a; 3g \sin \theta - T_2 = 3a$$

Q5aiii Add up the three equations:

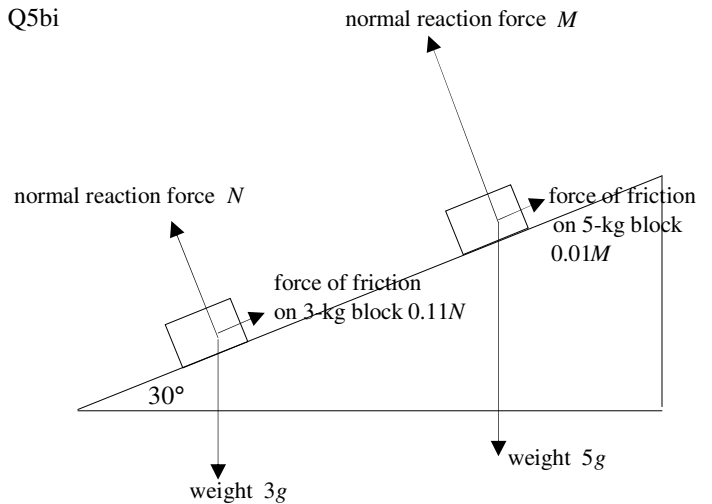
Sum of the left sides = sum of the right sides

$$8g \sin \theta - 2g = 10a, \therefore a = \frac{g(4 \sin \theta - 1)}{5}$$

Q5aiv Net force is zero for the system to be in equilibrium,

$$\therefore a = 0, \frac{g(4 \sin \theta - 1)}{5} = 0, \sin \theta = \frac{1}{4}, \theta \approx 14.5^\circ$$

Q5bi



$$\text{Q5bii } 3 \text{ kg: } 3g \sin 30^\circ - 0.11 \times 3g \cos 30^\circ = 3a, a \approx 3.97 \text{ m/s}^2$$

$$5 \text{ kg: } 5g \sin 30^\circ - 0.01 \times 5g \cos 30^\circ = 5a, a \approx 4.82 \text{ m/s}^2$$

Q5biii Both blocks start from rest. The 5 kg block moves 3 extra metres when it collides with the 3 kg block at time t seconds.

$$\therefore \frac{1}{2} \times 3.97t^2 + 3 = \frac{1}{2} \times 4.82t^2, \therefore t \approx 2.66 \text{ s}$$

Please inform mathline@itute.com re conceptual and/or mathematical errors