2014 Specialist Maths Trial Exam 2 Solutions

Section 1

1	2	3	4	5	6	7	8	9	10	11
E	B	D	E	D	D	D	D	C	C	D

12	13	14	15	16	17	18	19	20	21	22
C	A	E	D	C	B	A	C	E	A	D

Q1 $\frac{(x-k)^2}{4-k}-\frac{(y-k)^2}{6-k}=\frac{1}{12}$ is a hyperbola

when $4-k>0$ and $6-k>0$ OR $4-k<0$ and $6-k<0$

$\therefore\ k<4$ OR $k>6$ D

Q2 Ran $\left(-\cos^{-1}x\right)\subseteq$ dom $(\sin x)$, $\therefore\ -\frac{\pi}{2}\le -\cos^{-1}x\le\frac{\pi}{2}$

$\frac{\pi}{2}\ge\cos^{-1}x\ge-\frac{\pi}{2}$, $\frac{\pi}{2}\ge\cos^{-1}x\ge 0$, $\therefore\ 0\le x\le 1$ B

Q3 $z=a\left[i+cis\left(-\frac{2\pi}{3}\right)\right]=a\left[-\frac{1}{2}+\left(1-\frac{\sqrt{3}}{2}\right)i\right]$

$Arg(z)=\tan^{-1}\left(-2+\sqrt{3}\right)\approx 2.8798$ D

Q4 $c=0$, no asymptote; $c<0$, 1 asymptote;
$c=1$, 2 asymptotes; $c>0$ and $c\ne 1$, 3 asymptotes E

Q5 $\frac{1-\sin 2x}{1+\sin 2x}=\frac{(1-\sin 2x)(1-\sin 2x)}{(1+\sin 2x)(1-\sin 2x)}$

$=\frac{(1-\sin 2x)^2}{1-\sin^2 2x}=\frac{(1-\sin 2x)^2}{\cos^2 2x}$

$=\left(\frac{1}{\cos 2x}-\frac{\sin 2x}{\cos 2x}\right)^2=(\sec 2x-\tan 2x)^2$ D

Q6 The graph is the dilation of $y=\cos^{-1}x$ from the x and y axis by factors of $\frac{1}{2}$ and 2 respectively, followed by translation of 1 unit to the left and translation of $\frac{\pi}{4}$ downwards.

$y=\cos^{-1}x\to 2y=\cos^{-1}\left(\frac{x}{2}\right)\to 2\left(y+\frac{\pi}{4}\right)=\cos^{-1}\left(\frac{x+1}{2}\right)$ D

Q7 $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)=\frac{2\pi}{3}+\frac{7\pi}{6}=\frac{11\pi}{6}=\arg z_6$ D

Q8 $(x+yi)^2=n-ni$, $(x^2-y^2)+2xyi=n-ni$

$\therefore\ x^2-y^2=n$ and $2xy=-n$

$\therefore\ \left(\frac{-n}{2y}\right)^2-y^2=n$, $n^2-4\left(y^2\right)^2=4ny^2$, $4\left(y^2\right)^2+4ny^2-n^2=0$

$y^2=\frac{n\left(\sqrt{2}-1\right)}{2}=\frac{n}{2\left(\sqrt{2}+1\right)}$ D

Q9 C

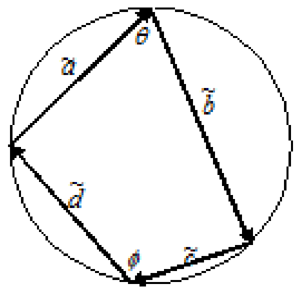
Q10 $y=\tan^{-1}x$, $\frac{dy}{dx}=\frac{1}{1+x^2}$

At $x=a$, $y=\tan^{-1}a$ and $m=\frac{dy}{dx}=\frac{1}{1+a^2}$

$\therefore\ \left(a,\tan^{-1}a\right)$ is a point on the tangent line $y=\frac{1}{1+a^2}x+\frac{\pi}{4}$

$\therefore\ \tan^{-1}a=\frac{a}{1+a^2}+\frac{\pi}{4}$, $\therefore\ a\approx 2.264$ C

Q11



$\theta+\phi=\pi$, $\theta=\pi-\phi$ (cyclic quadrilateral)

$\frac{\tilde{a}.\tilde{b}}{|\tilde{a}||\tilde{b}|}=\cos(\pi-\theta)=-\cos\theta$, $\frac{\tilde{c}.\tilde{d}}{|\tilde{c}||\tilde{d}|}=\cos(\pi-\phi)=\cos\theta$

$\frac{\tilde{a}.\tilde{b}}{|\tilde{a}||\tilde{b}|}+\frac{\tilde{c}.\tilde{d}}{|\tilde{c}||\tilde{d}|}=-\cos\theta+\cos\theta=0$

$\therefore\ \frac{\tilde{a}.\tilde{b}|\tilde{c}||\tilde{d}|+|\tilde{a}||\tilde{b}|\tilde{c}.\tilde{d}}{|\tilde{a}||\tilde{b}||\tilde{c}||\tilde{d}|}=0$

$\therefore\ \tilde{a}.\tilde{b}|\tilde{c}||\tilde{d}|+|\tilde{a}||\tilde{b}|\tilde{c}.\tilde{d}=0$ D

Q12 $\tilde{a}.\tilde{b}=0$, $\therefore\ \tilde{a}\perp\tilde{b}$, $\tilde{c}$ is parallel to $\tilde{a}$, $\therefore\ \tilde{c}\perp\tilde{b}$

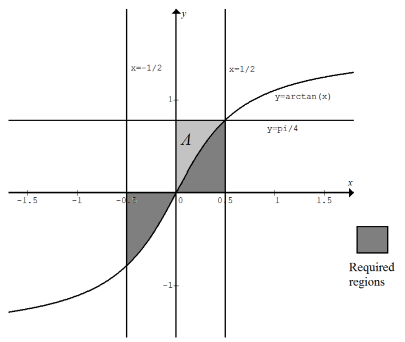
$\tilde{c}=x\tilde{i}+y\tilde{j}+z\tilde{k}$ and $\sqrt{x^2+y^2+z^2}=1$

$\tilde{c}=m\tilde{a}=m\tilde{i}+m\sqrt{2}\,\tilde{j}-m\tilde{k}$, $\therefore\ \sqrt{m^2+2m^2+(-m)^2}=1$

$\therefore\ 4m^2=1$, $m=\pm\frac{1}{2}$, $\tilde{c}=\frac{1}{2}\tilde{i}+\frac{1}{\sqrt{2}}\tilde{j}-\frac{1}{2}\tilde{k}$ C

Q13 Three non-parallel 3-D vectors cannot be dependent. A

Q14



Area of the required regions $=2\left(\frac{\pi}{4}\times\frac{1}{2}-A\right)=\frac{\pi}{4}-2A$ E

Q15 $f(x)=\frac{d}{dx}\int f(x)dx$

One of the x-intercepts of $f(x)$ corresponds to the local minimum of the anti-derivative of $f(x)$. The second x-intercept of $f(x)$ is a turning point corresponding to the stationary point of inflection of the anti-derivative of $f(x)$. D

Q16 $a\sin^{-1}x+2b\cos^{-1}x=a\sin^{-1}x+2b\left(\frac{\pi}{2}-\sin^{-1}x\right)$

$=a\sin^{-1}x-2b\sin^{-1}x+b\pi=b\pi-(2b-a)\sin^{-1}x$ C

Q17 $y=\int_{1.7}^{4.6}\cos\sqrt{x^2+1}\,dx+5.24\approx 3.20$ B

Q18 $\int\frac{1}{1+e^t}dt=\int\frac{e^{-t}}{e^{-t}+1}dt$, let $u=e^{-t}+1$, $\frac{du}{dt}=-e^{-t}$

.: $\int\frac{1}{1+e^t}dt=\int-\frac{du}{u}=-\log_e u+c=-\log_e\left(e^{-t}+1\right)+c$

.: $s=\int_1^2\frac{1}{1+e^t}dt=\left[-\log_e\left(e^{-t}+1\right)\right]_1^2=\left[\log_e\frac{1}{e^{-t}+1}\right]_1^2$

$=\left[\log_e\frac{e^t}{1+e^t}\right]_1^2$ A

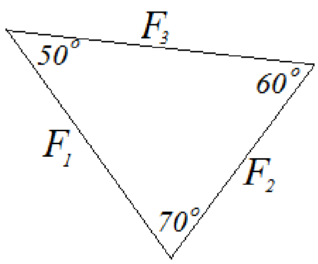
Q19 $\tilde{r}=\left(\frac{t^3}{3}-t^2\right)\tilde{i}$, $\tilde{v}=\left(t^2-2t\right)\tilde{i}$, $\tilde{a}=(2t-2)\tilde{i}$

The particle reverses direction when $\tilde{v}=\tilde{0}$ at $t=2$.

.: $\tilde{a}=2\tilde{i}$ C

Q20 The reading on the bathroom scale is lowered. The lift can be moving upwards with decreasing speed, or moving downwards with increasing speed. E

Q21 A



Q22 Acceleration of the particle is zero, .: the vector sum of the only two forces, the weight force and the reaction force, on the particle is zero, .: the reaction force is equal and opposite to the weight force. D

Section 2

Q1a $9(y-b)^2=4(x-a)^2-36$, $y-b=\pm\frac{2}{3}\sqrt{(x-a)^2-9}$

$\frac{dy}{dx}=\pm\frac{2(x-a)}{3\sqrt{(x-a)^2-9}}$

Q1b $y-b=\pm\frac{2}{3}\sqrt{(x-a)^2-9}$ and $\frac{dy}{dx}=\pm\frac{2(x-a)}{3\sqrt{(x-a)^2-9}}$

When $a=b=0$ and at $x=k$,

$y=\pm\frac{2}{3}\sqrt{k^2-9}$ and $\frac{dy}{dx}=\pm\frac{2k}{3\sqrt{k^2-9}}$

Tangent: $y-\left(\pm\frac{2}{3}\sqrt{k^2-9}\right)=\pm\frac{2k}{3\sqrt{k^2-9}}(x-k)$

.: $y=\pm\frac{2k}{3\sqrt{k^2-9}}(x-k)\pm\frac{2}{3}\sqrt{k^2-9}$

$y=\pm\frac{2k}{3\sqrt{k^2-9}}x\pm\frac{6}{\sqrt{k^2-9}}$

Q1c y-intercepts: $x=0$, $y=\pm\frac{6}{\sqrt{k^2-9}}$

As $k\to\infty$, $y\to 0$, the same y-intercept $(0,0)$

Q1d Asymptotes of $4x^2-9y^2=36$:

$\frac{x^2}{9}-\frac{y^2}{4}=1$, the asymptotes are $y=\pm\frac{2}{3}x$

The tangents: $y=\pm\frac{2k}{3\sqrt{k^2-9}}x\pm\frac{6}{\sqrt{k^2-9}}$

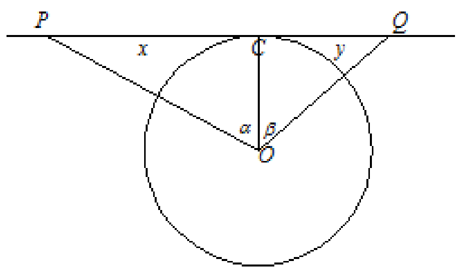
$y=\pm\frac{\frac{2k}{k}}{\frac{3\sqrt{k^2-9}}{k}}x\pm\frac{6}{\sqrt{k^2-9}}$, $y=\pm\frac{2}{3\sqrt{\frac{k^2-9}{k^2}}}x\pm\frac{6}{\sqrt{k^2-9}}$

$y=\pm\frac{2}{3\sqrt{1-\frac{9}{k^2}}}x\pm\frac{6}{\sqrt{k^2-9}}$

As $k\to\infty$, $\sqrt{1-\frac{9}{k^2}}\to 1$, $\frac{6}{\sqrt{k^2-9}}\to 0$

.: the tangents approach $y=\pm\frac{2}{3}x$, the asymptotes of $4x^2-9y^2=36$.

Q2a Let $\theta=\alpha+\beta$



$\tan\theta=\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{x+y}{1-xy}$

Q2b $\theta=\frac{\pi}{4}$, $\frac{x+y}{1-xy}=1$, $y=\frac{1-x}{1+x}$, $\frac{dy}{dx}=-\frac{2}{(1+x)^2}$

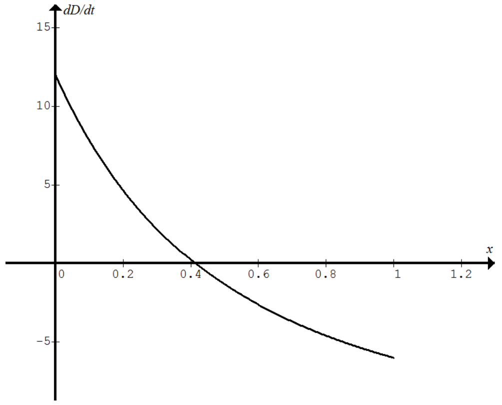
When $x=1$, $\frac{dy}{dx}=-\frac{1}{2}$,

$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}=-\frac{1}{2}\times(-12)=6$ km h^{-1}

Q2ci Let $D = x + y$, $\frac{dD}{dt} = \frac{dx}{dt} + \frac{dy}{dx} \times \frac{dx}{dt} = \frac{dx}{dt}\left(1 + \frac{dy}{dx}\right)$

$= -12\left(1 - \frac{2}{(1+x)^2}\right) = \frac{24}{(1+x)^2} - 12$

Q2cii



When $x > 0.41$ approximately, $\frac{dD}{dt} < 0$, i.e. D decreases with t. When $x < 0.41$ approximately, $\frac{dD}{dt} > 0$, i.e. D increases with t.

Q2d $\tan\theta = \frac{x+y}{1-xy}$ and $x + y = \frac{2\sqrt{3}}{3} = \frac{2}{\sqrt{3}}$

.: $y = \frac{2}{\sqrt{3}} - x$ and $\tan\theta = \frac{\frac{2}{\sqrt{3}}}{1 - x\left(\frac{2}{\sqrt{3}} - x\right)} = \frac{2}{\sqrt{3} - 2x + \sqrt{3}\,x^2}$

$\frac{d}{dx}(\tan\theta) = \frac{d}{dx}\left(\frac{2}{\sqrt{3} - 2x + \sqrt{3}\,x^2}\right)$

$\frac{d\tan\theta}{d\theta} \times \frac{d\theta}{dx} = \frac{-2\left(-2 + 2\sqrt{3}\,x\right)}{\left(\sqrt{3} - 2x + \sqrt{3}\,x^2\right)^2}$

Let $\frac{d\theta}{dx} = 0$, .: $\frac{-2\left(-2 + 2\sqrt{3}\,x\right)}{\left(\sqrt{3} - 2x + \sqrt{3}\,x^2\right)^2} = 0$, $-2 + 2\sqrt{3}\,x = 0$

.: $x = \frac{1}{\sqrt{3}}$ and $y = \frac{1}{\sqrt{3}}$, .: $\tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}$

Q2e When $\theta = \frac{\pi}{3}$, $\frac{d\theta}{dx} = 0$, .: $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} = 0$

Q3a $\tilde{r}(t) = \int \tilde{v}\,dt = \int\left(\frac{1}{1+t}\tilde{i} + \frac{1}{1+t^2}\tilde{j}\right)dt$

$= (\log_e(1+t))\tilde{i} + (\tan^{-1} t)\tilde{j}$, given $\tilde{r}(0) = \tilde{0}$

Q3b $x = \log_e(1+t)$, $t = e^x - 1$

$y = \tan^{-1} t$, $t = \tan y$.: $\tan y = e^x - 1$

Q3ci $\tan y = e^x - 1$, $\frac{d}{dx}(\tan y) = \frac{d}{dx}\left(e^x - 1\right)$

$\frac{d}{dy}(\tan y) \times \frac{dy}{dx} = e^x$, $\sec^2 y \times \frac{dy}{dx} = e^x$, $\frac{dy}{dx} = e^x \cos^2 y$

Note: $y \neq \frac{\pi}{2}$, .: $\frac{dy}{dx} \neq 0$

$\frac{d^2y}{dx^2} = e^x(2\cos y)(-\sin y)\frac{dy}{dx} + e^x\cos^2 y$

.: $\frac{d^2y}{dx^2} = -e^x(2\sin y\cos y)\frac{dy}{dx} + \frac{dy}{dx}$

$= \frac{dy}{dx}\left(1 - e^x \sin 2y\right)$

For points of inflection, $\frac{d^2y}{dx^2} = 0$.: $\frac{dy}{dx}\left(1 - e^x\sin 2y\right) = 0$

.: $1 - e^x \sin 2y = 0$, $e^x \sin 2y = 1$, $e^x \sin 2\left(\tan^{-1}\left(e^x - 1\right)\right) = 1$

By CAS, $x \approx 0.35$ (0.34657) and $y = \tan^{-1}\left(e^x - 1\right) \approx 0.39$

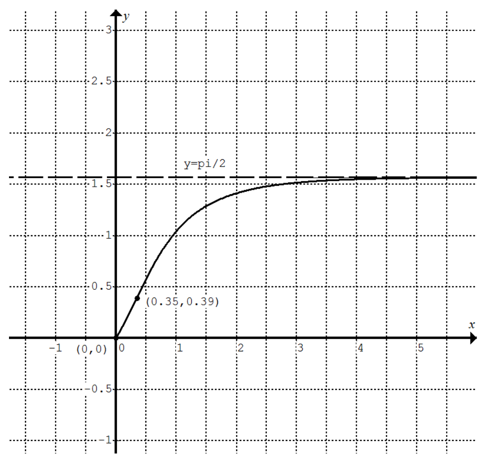
Q3cii $t = e^x - 1 \approx 0.41$ (0.4142)

Q3ciii $\tilde{v}(0.4142) = \frac{1}{1+0.4142}\tilde{i} + \frac{1}{1+0.4142^2}\tilde{j}$

$= 0.7071\tilde{i} + 0.8536\tilde{j}$

Speed $= \sqrt{0.7071^2 + 0.8536^2} \approx 1.11$

Q3d



Q4a $P(x) = (x-5)(x-3)(x-1)(x+1) - c$

Let $f(x) = (x-5)(x-3)(x-1)(x+1) = x^4 - 8x^3 + 14x^2 + 8x - 15$

The absolute minimum of $f(x)$ is -16.

For $P(x)$ to have non-real roots (i.e. no x-intercepts), $P(x) > 0$ for $x \in R$, .: $c < -16$

Q4b $P(x) = (x-5)(x-3)(x-1)(x+1) - 105 = 0$

$(x-5)(x+1)(x-3)(x-1) - 105 = 0$

$\left(x^2 - 4x - 5\right)\left(x^2 - 4x + 3\right) - 105 = 0$ and let $p = x^2 - 4x$

.: $(p-5)(p+3) - 105 = 0$, $p^2 - 2p - 120 = 0$

$(p-12)(p+10) = 0$, .: $x^2 - 4x - 12 = 0$ or $x^2 - 4x + 10 = 0$

.: $x = -2$, 6, or $x = 2 \pm \sqrt{6}\,i$

Q4ci As in Q4b with $c=-17$, $(p-5)(p+3)+17=0$

$\therefore\ p^2-2p+2=0$, $\therefore\ p=1-i$ or $p=1+i$

$\therefore\ x^2-4x-(1-i)=0$ or $x^2-4x-(1+i)=0$

$x=2\pm\sqrt{5-i}$ or $x=2\pm\sqrt{5+i}$

$x=2\pm(2.2471-0.2225i)$ or $x=2\pm(2.2471+0.2225i)$

$x=4.2471-0.2225i$, $-0.2471+0.2225i$, $4.2471+0.2225i$ or $-0.2471-0.2225i$

The two pairs of conjugate roots are:
$4.2471\pm0.2225i$ and $-0.2471\pm0.2225i$

Q4cii The roots are equidistant from $2+0i$ in the Argand plane, $\therefore$ the centre of the circle is $2+0i$ (or $(2,0)$) and the radius is $\left|\sqrt{5+i}\right|=\left|\sqrt{\sqrt{5^2+1^2}\,cis\theta}\right|=\left|\sqrt[4]{26}\,cis\frac{\theta}{2}\right|=\sqrt[4]{26}$

Q5a Force of friction $=\mu N=0.30\times1500\times9.8=4410$ N

Q5b Friction force F_f between the tyres and the ground is the driving force.
$F_f-200-4410=(3000+150+1500)\times0.20$, $F_f=5540$ N

Q5c The log has the same acceleration as the truck, 0.20 m s^{-2}.

Q5d Motion of the truck:
$5540-200-T_1=3000\times0.20$, $T_1=4740$ N
Motion of the log:
$T_2-4410=1500\times0.20$, $T_2=4710$ N
Maximum tension of 4740 N at the truck end of the rod, and minimum tension of 4710 N at the log end.

Q5e When the truck moves at constant speed in a straight line, the driving force equals to the friction between the log and the ground, and the total of air resistance and other resistive forces.
$F_f=4410+200=4610$ N

Q5f Uniform tension of 4410 N in the rod

Q5g Trucks slows down at 0.10 m s^{-2}:
$F_f-200-4410=(3000+150+1500)\times(-0.10)$, $F_f=4145$ N
Motion of the truck:
$4145-200-T_1=3000\times(-0.10)$, $T_1=4245$ N
Motion of the log:
$T_2-4410=1500\times(-0.10)$, $T_2=4260$ N
The tension in the rod is greater at the log end than the tension at the truck end.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors

