

Question 1 (3 marks)

$$\int \frac{3+x}{4-x^2} dx$$

Let $\frac{3+x}{4-x^2} \equiv \frac{A}{(2-x)} + \frac{B}{(2+x)}$ (1 mark)

$$\equiv \frac{A(2+x) + B(2-x)}{(2-x)(2+x)}$$

True iff $3+x \equiv A(2+x) + B(2-x)$

Put $x = -2$, $1 = 4B$, $B = \frac{1}{4}$

Put $x = 2$, $5 = 4A$, $A = \frac{5}{4}$

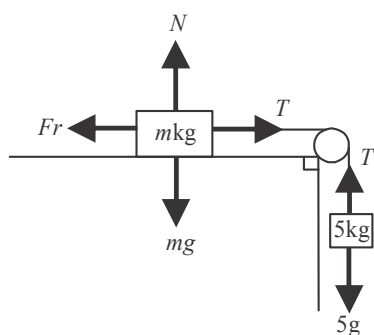
$$\int \frac{3+x}{4-x^2} dx = \int \frac{5}{4(2-x)} dx + \int \frac{1}{4(2+x)} dx$$
 (1 mark)

$$= -\frac{5}{4} \log_e |2-x| + \frac{1}{4} \log_e |2+x| + c$$
 (1 mark)

Note: you can go on and simplify this using log laws but you risk losing a mark if you make a mistake. (see 2011 Exam 1 Question 1 Examiners report on VCAA website)

Question 2 (4 marks)

a.



(1 mark) – for 3 correct forces
(1 mark) – for 3 more correct forces

b. At the point of moving $Fr = \mu N$.

Around the 5kg mass: $T = 5g$

(1 mark)

Around the m kg mass:

$$Fr = T \text{ and } N = mg$$

$$\text{So } \mu N = 5g$$

$$\frac{1}{5} \times mg = 5g$$

$$mg = 25g$$

$$m = 25$$

(1 mark)

Question 3 (3 marks)

Since $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly dependent $\alpha(\underline{i} - 3\underline{j} + 3\underline{k}) + \gamma(2\underline{i} - \underline{j} + 2\underline{k}) = x\underline{i} + y\underline{j}$

(1 mark)

$$\text{So } \alpha + 2\gamma = x \quad - (1)$$

$$-3\alpha - \gamma = y \quad - (2)$$

$$3\alpha + 2\gamma = 0$$

$$\alpha = \frac{-2\gamma}{3} \quad - (3)$$

(1 mark)

$$(3) \text{ in } (1) \quad -\frac{2\gamma}{3} + 2\gamma = x$$

$$\frac{4\gamma}{3} = x$$

$$\gamma = \frac{3x}{4}$$

$$(3) \text{ in } (2) \quad -3 \times -\frac{2\gamma}{3} - \gamma = y$$

$$2\gamma - \gamma = y$$

$$\gamma = y$$

$$\text{So } y = \frac{3x}{4}$$

$$\text{So } p = \frac{3}{4}$$

(1 mark)**Question 4** (3 marks)

$$2 \cot(x) = -\operatorname{cosec}(x)$$

$$2 \frac{\cos(x)}{\sin(x)} = \frac{-1}{\sin(x)}, \quad \sin(x) \neq 0$$

$$2 \cos(x) \sin(x) = -\sin(x)$$

$$2 \cos(x) \sin(x) + \sin(x) = 0$$

$$\sin(x)(2 \cos(x) + 1) = 0 \quad \textbf{(1 mark)}$$

$$\sin(x) \neq 0 \text{ (from above)} \quad 2 \cos(x) + 1 = 0$$

$$\cos(x) = -\frac{1}{2}$$

$$x = \frac{2\pi}{3} + 2k\pi \quad \text{or} \quad x = \frac{4\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$$

(1 mark)**(1 mark)**

Question 5 (3 marks)

$$\begin{aligned}
 a &= \frac{-x}{(x^2+1)^2} \\
 \frac{d}{dx} \left(\frac{1}{2} v^2 \right) &= \frac{-x}{(x^2+1)^2} \\
 \frac{1}{2} v^2 &= \int \frac{-x}{(x^2+1)^2} dx && \text{(1 mark)} \\
 &= -\frac{1}{2} \int \frac{du}{dx} u^{-2} dx && \text{where } u = x^2 + 1 \\
 &= -\frac{1}{2} \int u^{-2} du && \frac{du}{dx} = 2x \\
 &= -\frac{1}{2} \times \frac{u^{-1}}{-1} + c \\
 \frac{1}{2} v^2 &= \frac{1}{2(x^2+1)} + c
 \end{aligned}$$

Given $v=1$ when $x=0$,

$$\begin{aligned}
 \frac{1}{2} &= \frac{1}{2} + c \\
 c &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{So } \frac{1}{2} v^2 &= \frac{1}{2(x^2+1)} && \text{(1 mark)} \\
 v^2 &= \frac{1}{x^2+1} \\
 v &= \pm \frac{1}{\sqrt{x^2+1}}
 \end{aligned}$$

Since $v=1$ when $x=0$, reject the negative branch.

$$v = \frac{1}{\sqrt{x^2+1}} \quad \text{(1 mark)}$$

Question 6 (4 marks)

$$2xy + \frac{\log_e(y)}{x} = k$$

When $y=1$,

$$2x + 0 = k$$

$$x = \frac{k}{2}$$

(1 mark)

Use implicit differentiation to differentiate the function.

$$2y + 2x \frac{dy}{dx} + \frac{x \times \frac{1}{y} \times \frac{dy}{dx} - \log_e(y)}{x^2} = 0$$

$$2y + 2x \frac{dy}{dx} + \frac{1}{xy} \frac{dy}{dx} - \frac{1}{x^2} \log_e(y) = 0$$

(1 mark) – product rule**(1 mark)** – quotient rule

When $y=1$, $\frac{dy}{dx} = \frac{1}{2}$ (given) and $x = \frac{k}{2}$.

$$\text{So, } 2 + 2 \times \frac{k}{2} \times \frac{1}{2} + \frac{2}{k} \times \frac{1}{2} - \frac{4}{k^2} \times 0 = 0$$

$$2 + \frac{k}{2} + \frac{1}{k} = 0$$

$$4k + k^2 + 2 = 0$$

$$k^2 + 4k + 2 = 0$$

$$k = \frac{-4 \pm \sqrt{16 - 4 \times 1 \times 2}}{2} \quad (\text{quadratic formula})$$

$$= \frac{-4 \pm \sqrt{8}}{2}$$

$$= \frac{-4 \pm 2\sqrt{2}}{2}$$

$$k = -2 \pm \sqrt{2}$$

(1 mark)

Question 7 (4 marks)

a. $\underline{r}(t) = (\cos(t) + 2)\underline{i} + 4\sin(t)\underline{j}, \quad t \in \left[0, \frac{3\pi}{2}\right]$

$$x = \cos(t) + 2 \quad y = 4\sin(t)$$

(1 mark)

$$x - 2 = \cos(t) \quad \frac{y}{4} = \sin(t)$$

$$(x - 2)^2 = \cos^2(t) \quad \frac{y^2}{16} = \sin^2(t)$$

$$(x - 2)^2 + \frac{y^2}{16} = \cos^2(t) + \sin^2(t)$$

$$(x - 2)^2 + \frac{y^2}{16} = 1$$

(1 mark)

- b.** From part **a.**, we have an ellipse with centre at $(2, 0)$, semi-major axis length of 4 and semi-minor axis length of 1.

Since $t \in \left[0, \frac{3\pi}{2}\right]$,

when $t = 0$, $\underline{r}(0) = 3\underline{i} + 0\underline{j}$,

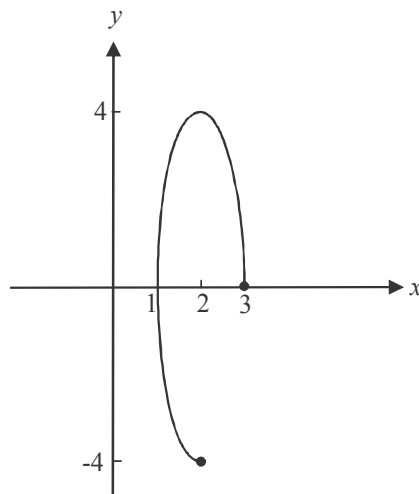
when $t = \frac{\pi}{2}$, $\underline{r}\left(\frac{\pi}{2}\right) = 2\underline{i} + 4\underline{j}$,

when $t = \pi$, $\underline{r}(\pi) = \underline{i} + 0\underline{j}$ and

when $t = \frac{3\pi}{2}$, $\underline{r}\left(\frac{3\pi}{2}\right) = 2\underline{i} - 4\underline{j}$.

So the particle starts at the point $(3, 0)$ and follows an elliptical path passing through the points $(2, 4)$, and $(1, 0)$ before finishing at the point $(2, -4)$.

Note that the endpoints are included.

**(1 mark)** – correct shape**(1 mark)** – correct endpoints

Question 8 (4 marks)

$$z^4 - 8z^2 + 49 = 0$$

$$(z^4 - 14z^2 + 49) + 6z^2 = 0 \quad (\text{complete the square})$$

(1 mark)

$$(z^2 - 7)^2 - (i\sqrt{6}z)^2 = 0$$

$$(z^2 - 7 - i\sqrt{6}z)(z^2 - 7 + i\sqrt{6}z) = 0$$

(1 mark)

$$z^2 - i\sqrt{6}z - 7 = 0 \quad \text{or} \quad z^2 + i\sqrt{6}z - 7 = 0$$

$$z = \frac{i\sqrt{6} \pm \sqrt{-6 - 4 \times 1 \times -7}}{2} \quad \text{or} \quad z = \frac{-i\sqrt{6} \pm \sqrt{-6 - 4 \times 1 \times -7}}{2}$$

$$= \frac{i\sqrt{6} \pm \sqrt{22}}{2} \quad \quad \quad = \frac{-i\sqrt{6} \pm \sqrt{22}}{2}$$

$$\text{So } z = \pm \frac{\sqrt{22}}{2} + \frac{\sqrt{6}i}{2} \quad \quad \text{or} \quad \quad z = \pm \frac{\sqrt{22}}{2} - \frac{\sqrt{6}i}{2}$$

(1 mark)**(1 mark)****Question 9** (5 marks)

a. $g(x) = \frac{4}{\pi} \arcsin\left(\frac{x}{3} - 2\right) + 1$

Finding the domain:

$$\text{For } g \text{ to be defined, } -1 \leq \frac{x}{3} - 2 \leq 1$$

$$1 \leq \frac{x}{3} \leq 3$$

$$3 \leq x \leq 9$$

$$\text{So } d_g = [3, 9]$$

(1 mark)Finding the range:Method 1The range of the function $y = \arcsin(x)$

$$\text{is } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\text{so, } r_g = \left[-\frac{\pi}{2} \times \frac{4}{\pi} + 1, \frac{\pi}{2} \times \frac{4}{\pi} + 1\right]$$

$$= [-1, 3]$$

(1 mark)

Method 2

$$g(x) = \frac{4}{\pi} \arcsin\left(\frac{x}{3} - 2\right) + 1$$

$$\text{So, } \frac{\pi}{4}(g(x) - 1) = \arcsin\left(\frac{x}{3} - 2\right)$$

The range of the function $y = \arcsin(x)$ is $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\begin{aligned} \text{So } -\frac{\pi}{2} &\leq \frac{\pi}{4}(g(x) - 1) \leq \frac{\pi}{2} \\ -2 &\leq g(x) - 1 \leq 2 \\ -1 &\leq g(x) \leq 3 \end{aligned}$$

$$\text{So } r_g = [-1, 3]$$

(1 mark)

b. Let $y = \frac{4}{\pi} \arcsin\left(\frac{x}{3} - 2\right) + 1$

and let $y = \frac{4}{\pi} \arcsin(u) + 1$ where $u = \frac{x}{3} - 2$

$$\frac{dy}{du} = \frac{4}{\pi\sqrt{1-u^2}} \qquad \frac{du}{dx} = \frac{1}{3}$$

Now, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ (Chain rule)

$$= \frac{4}{\pi\sqrt{1-u^2}} \times \frac{1}{3}$$

(1 mark)

$$= \frac{4}{\pi\sqrt{1-\left(\frac{x}{3}-2\right)^2}} \times \frac{1}{3}$$

$$= \frac{4}{3\pi\sqrt{1-\left(\frac{x^2}{9}-\frac{4x}{3}+4\right)}}$$

(1 mark)

$$= \frac{4}{3\pi\sqrt{-\frac{x^2}{9}+\frac{4x}{3}-3}}$$

$$= \frac{4}{3\pi\sqrt{\frac{-x^2+12x-27}{9}}}$$

$$= \frac{4}{\pi\sqrt{-(x^2-12x+27)}}$$

$$= \frac{4}{\pi\sqrt{-(x-3)(x-9)}}$$

Re-read the question!

So $a = 4$, $b = -1$ and $c = 3$.

1 mark

Question 10 (7 marks)

a. $y = \frac{1}{x^2 + 3}$

When $x = 0$, $y = \frac{1}{3}$

range = $\left(0, \frac{1}{3}\right]$

(1 mark)

b. area = $\int_0^1 \frac{1}{x^2 + 3} dx$

(1 mark)

$$= \frac{1}{\sqrt{3}} \int_0^1 \frac{\sqrt{3}}{(\sqrt{3})^2 + x^2} dx$$

$$= \frac{1}{\sqrt{3}} \left[\arctan\left(\frac{x}{\sqrt{3}}\right) \right]_0^1$$

$$= \frac{1}{\sqrt{3}} \left(\arctan\left(\frac{1}{\sqrt{3}}\right) - \arctan(0) \right)$$

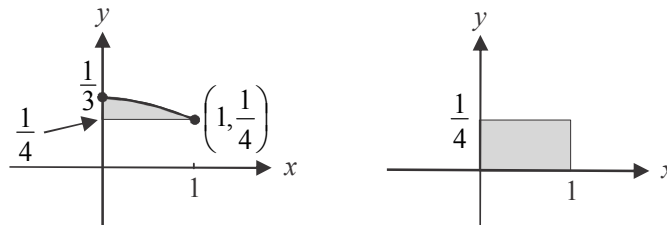
$$= \frac{1}{\sqrt{3}} \left(\frac{\pi}{6} - 0 \right)$$

$$= \frac{\pi}{6\sqrt{3}} \text{ square units}$$

(1 mark)

c. When $x = 1$, $y = \frac{1}{4}$.

The volume generated can be broken up into two parts.



The first is obtained by rotating the shaded region in the left hand diagram around the y-axis.

This volume = $\pi \int_{\frac{1}{4}}^{\frac{1}{3}} x^2 dy$

(1 mark)

Since $y = \frac{1}{x^2 + 3}$, then $x^2 = \frac{1}{y} - 3$

so, volume = $\pi \int_{\frac{1}{4}}^{\frac{1}{3}} \left(\frac{1}{y} - 3 \right) dy$

(1 mark)

The second part is obtained by rotating the shaded region in the right hand diagram around the y -axis.

This forms a cylinder with radius 1 and height $\frac{1}{4}$,

$$\text{so volume} = \pi r^2 h \quad (\text{formula sheet})$$

$$= \pi \times 1 \times \frac{1}{4}$$

$$= \frac{\pi}{4}$$

(1 mark)

$$\text{Alternatively, volume} = \pi \int_0^{\frac{1}{4}} x^2 dy$$

$$= \pi \int_0^{\frac{1}{4}} 1 dy$$

(since we are rotating the line $x = 1$ around the y -axis)

$$= \pi [y]_0^{\frac{1}{4}}$$

$$= \frac{\pi}{4}$$

Combining these two volumes we have,

$$\text{total volume} = \pi \int_{\frac{1}{4}}^{\frac{1}{3}} \left(\frac{1}{y} - 3 \right) dy + \frac{\pi}{4}$$

$$= \pi [\log_e |y| - 3y]_{\frac{1}{4}}^{\frac{1}{3}} + \frac{\pi}{4}$$

$$= \pi \left\{ \left(\log_e \left(\frac{1}{3} \right) - 1 \right) - \left(\log_e \left(\frac{1}{4} \right) - \frac{3}{4} \right) \right\} + \frac{\pi}{4}$$

$$= \pi \left\{ \log_e \left(\frac{4}{3} \right) - \frac{1}{4} \right\} + \frac{\pi}{4}$$

$$= \pi \log_e \left(\frac{4}{3} \right) \text{ cubic units}$$

(1 mark)