

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2



(TSSM's 2013 trial exam updated for the current study design)

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: C

Explanation

$$\begin{aligned}y &= \frac{2x^3 - 8x^2 + 8x - 3}{x^2 - 4x + 4} \\&= 2x - \frac{3}{(x-2)^2}\end{aligned}$$

Question 2

Answer: B

Explanation

Equation required is $(x + 2)^2 + (y - 3)^2 = 25$

Expanding brackets gives:

$$x^2 + 4x + y^2 - 6y - 12 = 0$$

Question 3

Answer: E

Explanation

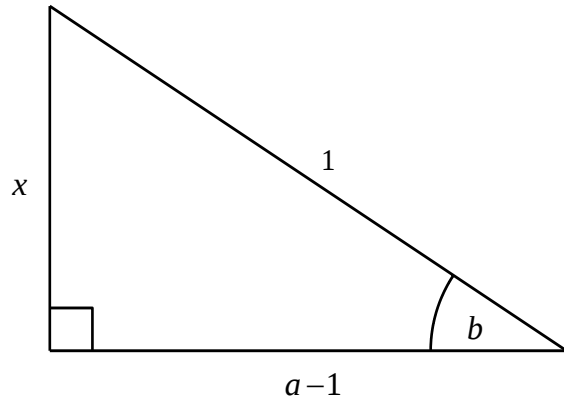
$$\cos^{-1}(a-1) = b \Rightarrow \cos b = a-1$$

From Pythagoras' theorem,

$$\begin{aligned} x &= \sqrt{1-(a-1)^2} \\ &= \sqrt{2a-a^2} \\ &= \sqrt{a(2-a)} \end{aligned}$$

Therefore,

$$\begin{aligned} \cot\left(b - \frac{\pi}{2}\right) &= \cot\left(-\left(\frac{\pi}{2} - b\right)\right) \\ &= -\cot\left(\frac{\pi}{2} - b\right) \\ &= \frac{\sqrt{a(2-a)}}{1-a} \end{aligned}$$



Question 4

Answer: C

Explanation

$$-1 \leq 2x-1 \leq 1 \Rightarrow 0 \leq x \leq 1 \Rightarrow \text{the domain is } [0, 1]$$

$$-\frac{\pi}{2} \leq \sin^{-1}(2x-1) \leq \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} \leq \frac{1}{2}\sin^{-1}(2x-1) \leq \frac{\pi}{4} \Rightarrow 2 - \frac{\pi}{4} \leq 2 - \frac{1}{2}\sin^{-1}(2x-1) \leq 2 + \frac{\pi}{4}$$

$$\text{Therefore the range of } y = \frac{1}{2}f(x) \text{ is } \left[1 - \frac{\pi}{8}, 1 + \frac{\pi}{8}\right] \text{ or } \left[\frac{8-\pi}{8}, \frac{8+\pi}{8}\right]$$

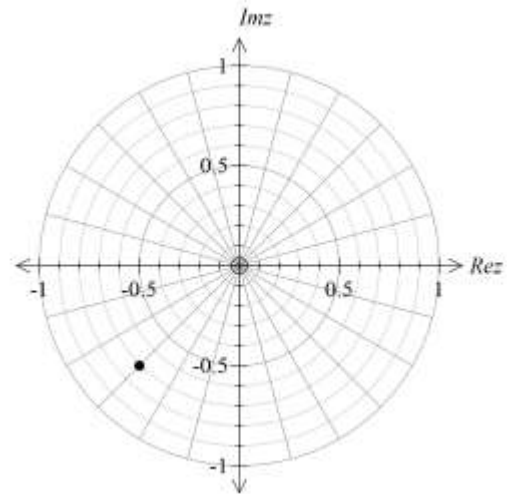
Question 5

Answer: B

Explanation

$$z = \frac{1+i}{(1-i)^2} \Rightarrow z = -\frac{1}{2} + \frac{1}{2}i \Rightarrow \bar{z} = -\frac{1}{2} - \frac{1}{2}i$$

$$\text{In polar form, } \bar{z} = \frac{\sqrt{2}}{2} \text{cis}\left(-\frac{3\pi}{4}\right)$$



Question 6

Answer: D

Explanation

$$\begin{aligned} z^3 + bz^2 + (2+i)z - 2 &= (z+i)(z^2 + mz + 2i) \\ &= z^3 + mz^2 + 2iz + z^2i + miz - 2 \\ &= z^3 + z^2(m+i) + z(mi+2i) - 2 \end{aligned}$$

Equating coefficients,

$$m+i = b \text{ and } mi+2i = 2+i$$

$$\text{It follows, } mi = 2-i \Rightarrow m = \frac{2-i}{i} \times \frac{-i}{-i} = -1-2i$$

$$\text{Therefore, } b = -1-2i+i = -1-i$$

Question 7

Answer: B

Explanation

$$1 + ax = 0 \text{ gives } x = -\frac{1}{a}$$

$$\frac{1}{2} = -\frac{1}{a} \rightarrow a = -2$$

Question 8

Answer: A

Explanation

METHOD 1: Non CAS

$$z^3 - 64i = 0$$

$$z^3 = 64i$$

$$r^3 \text{cis} 3\theta = 64 \text{cis} \frac{\pi}{2}$$

$$r^3 = 64 \Rightarrow r = 4 \quad \text{and} \quad 3\theta = \frac{\pi}{2} + 2k\pi, k = 0, 1, 2$$

It follows,

$$3\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$$

$$\text{The roots are } z_1 = 4 \text{cis} \frac{\pi}{6}, z_2 = 4 \text{cis} \frac{5\pi}{6}, z_3 = 4 \text{cis} \left(-\frac{\pi}{2} \right)$$

$$\text{It follows that } \frac{2z_1 z_3}{z_2} = 8 \text{cis} \left(\frac{\pi}{6} - \frac{\pi}{2} - \frac{5\pi}{6} \right) = 8 \text{cis} \left(-\frac{7\pi}{6} \right) = 8 \text{cis} \left(\frac{5\pi}{6} \right)$$

$$\text{Therefore } \text{Arg} \left(\frac{2z_1 z_3}{z_2} \right) = \frac{5\pi}{6}$$

METHOD 2: Using CAS

The CAS calculator can be used to solve the expression $z^3 - 64i = 0$ and subsequently find the complex roots. The expression $\frac{2z_1z_3}{z_2}$ can then be calculated using CAS and the principal argument determined.

Question 9

Answer: E

Explanation

$$\begin{aligned}\sec^2(2\pi x) &= 2 \tan(2\pi x) \\ \sec^2(2\pi x) - 2 \tan(2\pi x) &= 0 \\ 1 + \tan^2(2\pi x) - 2 \tan(2\pi x) &= 0 \\ \tan^2(2\pi x) - 2 \tan(2\pi x) + 1 &= 0 \\ (\tan(2\pi x) - 1)^2 &= 0 \\ \tan(2\pi x) - 1 &= 0 \\ \tan(2\pi x) &= 1\end{aligned}$$

Question 10

Answer: D

Explanation

$$\begin{aligned}c &= b - a \\ c \cdot c &= (b - a) \cdot (b - a) \\ |c|^2 &= (b - a) \cdot (b - a)\end{aligned}$$

Question 11

Answer: B

Explanation

Since the object is in equilibrium, then

$$\begin{aligned} \underline{F}_1 + \underline{F}_2 + \underline{F}_3 &= \underline{0} \\ -5\underline{i} + 4\underline{j} + 3\underline{i} + 7\underline{j} + a\underline{i} + b\underline{j} &= \underline{0} \\ (-5 + 3 + a)\underline{i} + (4 + 7 + b)\underline{j} &= \underline{0} \\ (a - 2)\underline{i} + (b + 11)\underline{j} &= \underline{0} \\ a = 2 \text{ and } b = -11 \end{aligned}$$

And so $\underline{F}_3 = 2\underline{i} - 11\underline{j}$

If the angle between $\underline{F}_1$ and $\underline{F}_3$ is represented by θ , then

$$\begin{aligned} \theta &= \cos^{-1} \left(\frac{(-5\underline{i} + 4\underline{j}) \cdot (2\underline{i} - 11\underline{j})}{|-5\underline{i} + 4\underline{j}| |2\underline{i} - 11\underline{j}|} \right) \\ &= \cos^{-1} \left(\frac{-10 - 44}{\sqrt{41} \times 125} \right) \\ &= \cos^{-1} \left(\frac{-54}{\sqrt{5125}} \right) \\ &= 139^\circ \end{aligned}$$

Question 12

Answer: C

Explanation

$$\begin{aligned} \sqrt{a^2 + b^2} &= 3 \Rightarrow a^2 + b^2 = 9 \\ (a\underline{i} + b\underline{j})(2\underline{i} - 3\underline{j} - \underline{k}) &= 0 \\ 2a - b &= 0 \\ b &= 2a \end{aligned}$$

Therefore,

$$\begin{aligned} a^2 + (2a)^2 &= 9 \\ 5a^2 &= 9 \\ a &= \pm \frac{3\sqrt{5}}{5} \Rightarrow b = \pm \frac{6\sqrt{5}}{5} \end{aligned}$$

Question 13

Answer: B

Explanation

$$\underline{r}(t) = (t^2 - 3)\underline{i} + \sqrt{3t^2 + 1}\underline{j}$$

$$\dot{\underline{r}}(t) = 2t\underline{i} + \frac{3t}{\sqrt{3t^2 + 1}}\underline{j}$$

$$|\dot{\underline{r}}(t)| = \sqrt{4t^2 + \frac{9t^2}{3t^2 + 1}}$$

$$|\dot{\underline{r}}(4)| = 8.2 \text{ ms}^{-1}$$

Question 14

Answer: A

Explanation

$$x = 2 \sin t \Rightarrow \frac{dx}{dt} = 2 \cos t$$

$$y = 3 \tan t \Rightarrow \frac{dy}{dt} = \frac{3}{\cos^2 t}$$

It follows,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \frac{dt}{dx} \\ &= \frac{3}{\cos^2 t} \times \frac{1}{2 \cos t} \\ &= \frac{3}{2 \cos^3 t} \end{aligned}$$

$$\text{When } t = \frac{5\pi}{6}, \frac{dy}{dx} = \frac{3}{2 \left(-\frac{\sqrt{3}}{2} \right)^3} = -\frac{4\sqrt{3}}{3}$$

Question 15*Answer: D**Explanation*

$$\begin{aligned}
 u &= \log_e(4x) & x=4 & \quad u = \log_e 16 = 4\log_e 2 \\
 \frac{du}{dx} &= \frac{1}{x} & x=8 & \quad u = \log_e 32 = 5\log_e 2
 \end{aligned}$$

It follows,

$$\begin{aligned}
 \int_4^8 \frac{2}{x \log_e(4x)} dx &= \int_4^8 2 \times \frac{1}{x} \times \frac{1}{\log_e(4x)} dx \\
 &= \int_{4\log_e 2}^{5\log_e 2} 2 \times \frac{du}{dx} \times \frac{1}{u} dx \\
 &= \int_{4\log_e 2}^{5\log_e 2} \frac{2}{u} du
 \end{aligned}$$

Question 16*Answer: E**Explanation*

$$\begin{aligned}
 \frac{dV}{dt} &= \frac{k}{\sqrt{V}} \\
 \frac{dt}{dV} &= \frac{\sqrt{V}}{k} \\
 t &= \frac{1}{k} \int V^{\frac{1}{2}} dV \\
 &= \frac{2k}{3} V^{\frac{3}{2}} + c & t=0, V=9 \Rightarrow c = -\frac{18}{k} \\
 &= \frac{2}{3k} V^{\frac{3}{2}} - \frac{18}{k} \\
 &= \frac{2}{k} \left(\frac{V^{\frac{3}{2}}}{3} - 9 \right) \\
 V &= \left(\frac{3kt + 54}{2} \right)^{\frac{2}{3}} \\
 \text{When } t=5, \quad V &= \left(\frac{15k + 54}{2} \right)^{\frac{2}{3}}
 \end{aligned}$$

Question 17

Answer: A

Explanation

The solution could be $y = \log_e |x-2| + c \Rightarrow \frac{dy}{dx} = \frac{1}{x-2}$

Question 18

Answer: B

Explanation

Let $g(x_n) = x_n \log_e x_n$

Tabulating, we have

n	x_n	y_n	$g(x_n)$
0	2	1	1.386294
1	2.2	1.277259	1.734606
2	2.4	1.62418	

Therefore $y(2.4) = 1.624$

Question 19

Answer: B

Explanation

During the third second of motion

$$20 = (u + 2a) + \frac{a}{2}$$

$$40 = 2u + 5a$$

During the fifth second of motion

$$10 = (u + 4a) + \frac{a}{2}$$

$$20 = 2u + 9a$$

Solving for a and u , $a = -5\text{ms}^{-2}$ and $u = 32.5\text{ms}^{-1}$

Question 20

Answer: C

Explanation

$$a = 24 - 6t$$

Therefore,

$$\begin{aligned} v &= \int (24 - 6t) dt \\ &= 24t - 3t^2 + c \\ &= 24t - 3t^2 + 24 \end{aligned} \quad t = 0, v = 24 \Rightarrow c = 24$$

It follows, the distance, s is

$$\begin{aligned} s &= \int_0^6 (24t - 3t^2 + 24) dt \\ &= 360 \text{ metres} \end{aligned}$$

Question 21

Answer: C

Explanation

$$\begin{aligned} 2a &= 2v^2 - 1 \\ a &= \frac{2v^2 - 1}{2} \\ v \frac{dv}{dx} &= \frac{2v^2 - 1}{2} \\ \frac{dx}{dv} &= \frac{2v}{2v^2 - 1} \\ x &= \int \frac{2v}{2v^2 - 1} dv \\ &= \frac{1}{2} \int \frac{1}{u} du \\ &= \frac{1}{2} \log_e |2v^2 - 1| + c \\ 1 &= \frac{1}{2} \log_e (1) + c \Rightarrow c = 1 \\ x &= \frac{1}{2} \log_e |2v^2 - 1| + 1 \\ v &= \sqrt{\frac{1}{2} (e^{2(x-1)} + 1)} \end{aligned} \quad \begin{aligned} u &= 2v^2 - 1 \\ \frac{du}{dv} &= 4v \Rightarrow \frac{1}{2} \frac{du}{dv} = 2v \end{aligned}$$

Since $v = 1$ when $x = 1$ we take the **positive** square root

Question 22

Answer: A

Explanation

The equation of motion is

$$2.5v - 10g = -10a$$

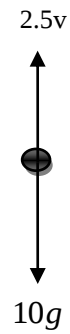
$$a = \frac{10g - 2.5v}{10}$$

$$\frac{dv}{dt} = \frac{10g - 2.5v}{10}$$

$$\frac{dt}{dv} = \frac{10}{10g - 2.5v}$$

$$t = \int_0^{20} \frac{10}{10g - 2.5v} dv$$

$$= 2.85 \text{ seconds}$$



SECTION 2

Question 1

a.

From CAS

$$(\cos \theta + i \sin \theta)^3 = \cos \theta (4 \cos^2 \theta - 3) + i \sin \theta (4 \cos^2 \theta - 1) \quad [A1]$$

b.

$$\operatorname{cis}(3\theta) = \cos(3\theta) + i \sin(3\theta)$$

Since $(\operatorname{cis}(\theta))^3 = \operatorname{cis}(3\theta)$, then

$$\cos(3\theta) + i \sin(3\theta) = \cos \theta (4 \cos^2 \theta - 3) + i \sin \theta (4 \cos^2 \theta - 1)$$

Equating coefficients,

$$\begin{aligned} \sin(3\theta) &= \sin \theta (4 \cos^2 \theta - 1) \\ &= \sin \theta (4(1 - \sin^2 \theta) - 1) \\ &= 3 \sin \theta - 4 \sin^3 \theta \end{aligned}$$

} [M1]

} [M1]

c.

From CAS and simplifying

$$(\cos \theta + i \sin \theta)^5 = \cos \theta (1 + 4 \sin^2 \theta - 16 \sin^2 \theta \cos^2 \theta) - i \sin \theta ((16 \sin^2 \theta - 4) \cos^2 \theta - 1) \quad [A1]$$

d.

$$\operatorname{cis}(5\theta) = \cos(5\theta) + i \sin(5\theta)$$

$$\cos(5\theta) + i \sin(5\theta) = \cos \theta (1 + 4 \sin^2 \theta - 16 \sin^2 \theta \cos^2 \theta) - i \sin \theta ((16 \sin^2 \theta - 4) \cos^2 \theta - 1) \quad \left. \vphantom{\cos(5\theta)} \right\} [M1]$$

Equating coefficients,

$$\begin{aligned} \sin(5\theta) &= -\sin \theta ((16 \sin^2 \theta - 4) \cos^2 \theta - 1) \\ &= -\sin \theta ((16 \sin^2 \theta - 4)(1 - \sin^2 \theta) - 1) \\ &= -\sin \theta (16 \sin^2 \theta - 16 \sin^4 \theta - 5 + 4 \sin^2 \theta) \\ &= -\sin \theta (20 \sin^2 \theta - 16 \sin^4 \theta - 5) \\ &= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta \end{aligned} \quad \left. \vphantom{\sin(5\theta)} \right\} [M2]$$

e.

$$\begin{aligned}\sin(5\theta) + \sin(3\theta) + \sin\theta &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta + 3\sin\theta - 4\sin^3\theta + \sin\theta \\ &= 16\sin^5\theta - 24\sin^3\theta + 9\sin\theta \\ &= \sin\theta(16\sin^4\theta - 24\sin^2\theta + 9) \\ &= \sin\theta(4\sin^2\theta - 3)^2\end{aligned}$$

$$\sin(5\theta) + \sin(3\theta) + \sin\theta = 0 \Rightarrow \sin\theta(4\sin^2\theta - 3)^2 = 0$$

It follows,

$$\sin\theta = 0 \text{ or } \sin\theta = \pm \frac{\sqrt{3}}{2}$$

Solving for θ

$$\theta = 0 \text{ or } \theta = \pm \frac{\pi}{3}$$

[M2] [A1]

f.

$$\begin{aligned}16\sin^5\theta - 20\sin^3\theta + 5\sin\theta &= 0 \\ \sin\theta(16\sin^4\theta - 20\sin^2\theta + 5) &= 0\end{aligned}$$

Since $\sin\theta \neq 0$, then

$$16\sin^4\theta - 20\sin^2\theta + 5 = 0$$

$$\begin{aligned}\sin^2\theta &= \frac{20 \pm \sqrt{400 - 320}}{32} \\ &= \frac{5 \pm \sqrt{5}}{8} \\ \sin\theta &= \pm \sqrt{\frac{5 \pm \sqrt{5}}{8}}\end{aligned}$$

[M2]

It follows,

$$\sin(5\theta) = 0$$

$$5\theta = n\pi, n \in \mathbb{Z}$$

$$\theta = \frac{n\pi}{5}$$

[M1]

$$\text{Therefore, } \sin \frac{2\pi}{5} = \sqrt{\frac{5 + \sqrt{5}}{8}}$$

Question 2**a.**

$$\left. \begin{array}{l} x=0 \Rightarrow y=2 \text{ giving } (0, 2) \\ y=0 \Rightarrow x^2-2=0 \Rightarrow x=\pm\sqrt{2} \text{ giving } (-\sqrt{2}, 0) \text{ and } (\sqrt{2}, 0) \end{array} \right\} \text{ [A1]}$$

b.

$$\left. \begin{array}{l} f(x) = \frac{\frac{1}{2}(2x^2-1) - \frac{3}{2}}{2x^2-1} \\ = \frac{1}{2} - \frac{3}{2(2x^2-1)} \end{array} \right\} \text{ [M1]}$$

Therefore, the equations of the asymptotes are $x = -\frac{\sqrt{2}}{2}$, $x = \frac{\sqrt{2}}{2}$ and $y = \frac{1}{2}$ [A1]

c.

$$\left. \begin{array}{l} f(x) = \frac{1}{2} - \frac{3}{2(2x^2-1)} \\ = \frac{1}{2} - \frac{3}{2}(2x^2-1)^{-1} \\ f'(x) = \frac{3}{2}(2x^2-1)^{-2} \times 4x \\ = \frac{6x}{(2x^2-1)^2} \\ f'(x) = 0 \Rightarrow x = 0 \text{ and } y = 2 \\ x < 0, f'(x) < 0 \text{ and } x > 0, f'(x) > 0 \end{array} \right\} \text{ [M2]}$$

Therefore, $(0, 2)$ is a local minimum point [A1]

d.

$$f'(x) = \frac{6x}{(2x^2 - 1)^2}$$

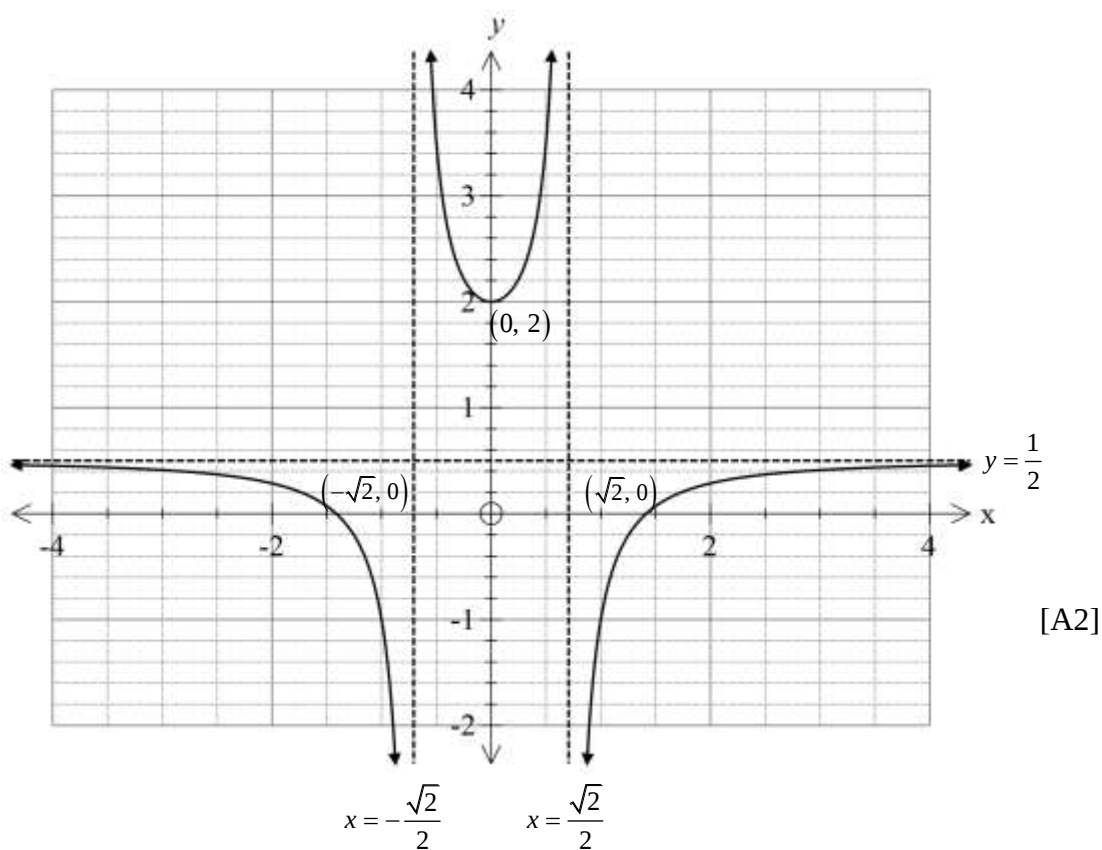
From CAS or otherwise,

$$f''(x) = -\frac{6(6x^2 + 1)}{(2x^2 - 1)^3}$$

Since $f''(x) \neq 0$ for all $x \in \mathbb{R}$

then the curve $y = f(x)$ does not have any points of inflection

[M1] [A1]

e.

f.

At the points of intersection,

$$\frac{x^2 - 2}{2x^2 - 1} = 2 + \log_e(x + 3) \quad [\text{M1}]$$

From CAS,

The relevant points of intersection are $(-0.4389, 2.9404)$ and $(0.4763, 3.2460)$ Therefore, the volume of the solid of revolution, V is equal to

$$V = \pi \int_{-0.4389}^{0.4763} \left((2 + \log_e(x + 3))^2 - \left(\frac{x^2 - 2}{2x^2 - 1} \right)^2 \right) dx$$

$$= 12.35 \text{ unit}^3$$

[M2] [A1]

Question 3**a.**

$$\begin{aligned} \overrightarrow{AB} &= 700\mathbf{i} + 400\mathbf{j} \\ \overrightarrow{BC} &= 350\mathbf{i} - 1550\mathbf{j} \\ \overrightarrow{AC} &= 1050\mathbf{i} - 1150\mathbf{j} \end{aligned} \quad \left. \vphantom{\begin{aligned} \overrightarrow{AB} &= 700\mathbf{i} + 400\mathbf{j} \\ \overrightarrow{BC} &= 350\mathbf{i} - 1550\mathbf{j} \\ \overrightarrow{AC} &= 1050\mathbf{i} - 1150\mathbf{j} \end{aligned}} \right\} [\text{A2}]$$

b.

$$\begin{aligned} |\overrightarrow{AB}| &= \sqrt{700^2 + 400^2} = 806.23 \\ |\overrightarrow{BC}| &= \sqrt{350^2 + (-1550)^2} = 1589.02 \\ |\overrightarrow{AC}| &= \sqrt{1050^2 + (-1150)^2} = 1557.24 \end{aligned} \quad \left. \vphantom{\begin{aligned} |\overrightarrow{AB}| &= \sqrt{700^2 + 400^2} = 806.23 \\ |\overrightarrow{BC}| &= \sqrt{350^2 + (-1550)^2} = 1589.02 \\ |\overrightarrow{AC}| &= \sqrt{1050^2 + (-1150)^2} = 1557.24 \end{aligned}} \right\} [\text{M1}]$$

Therefore, the perimeter of the paddock $= 806.23 + 1589.02 + 1557.24 = 3952.49$.

Correct to the nearest metre, the perimeter is 3952 m. [A1]

c.

Let $\angle BAC = \theta$

It follows,

$$\begin{aligned}\cos \theta &= \frac{(700\hat{i} + 400\hat{j}) \cdot (1050\hat{i} - 1150\hat{j})}{\sqrt{650000} \times \sqrt{2425000}} \\ &= \frac{275000}{\sqrt{650000} \times \sqrt{2425000}} \\ \theta &= 77.35^\circ\end{aligned}$$

[M1] [A1]

The area of the paddock, A is equal to

$$\begin{aligned}A &= \frac{1}{2} \times \sqrt{650000} \times \sqrt{2425000} \sin 77.35^\circ \\ &= 612506 \text{ m}^2 \\ &= 61.3 \text{ ha}\end{aligned}$$

[A1]

d.

One of a number of possible methods

A unit vector in the direction of $\overrightarrow{AC}$ is $\frac{1050\hat{i} - 1150\hat{j}}{\sqrt{2425000}}$

Let $\underline{v}$ = the vector projection of $\overrightarrow{AB}$ in a direction perpendicular to $\overrightarrow{AC}$.

It follows,

$$\begin{aligned}\underline{v} &= 700\hat{i} + 400\hat{j} - (700\hat{i} + 400\hat{j}) \cdot \left(\frac{1050\hat{i} - 1150\hat{j}}{\sqrt{2425000}} \right) \times \frac{1050\hat{i} - 1150\hat{j}}{\sqrt{2425000}} \\ &= 700\hat{i} + 400\hat{j} - \frac{275000}{2425000} (1050\hat{i} - 1150\hat{j}) \\ &= 580.928\hat{i} + 530.412\hat{j}\end{aligned}$$

[M2] [A1]

Therefore, the minimum distance from B to $[AC]$ is

$$|\underline{v}| = \sqrt{580.928^2 + 530.412^2} = 786.65 \text{ m}$$

e.Therefore, the area of the paddock, A is equal to

$$\begin{aligned}
 A &= \frac{1}{2} \times |\overrightarrow{AC}| \times |y| \\
 &= \frac{1}{2} \times 1557.24 \times 786.65 \\
 &= 612501 \text{ m}^2 \\
 &= 61.3 \text{ ha}
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} A &= \frac{1}{2} \times |\overrightarrow{AC}| \times |y| \\ &= \frac{1}{2} \times 1557.24 \times 786.65 \\ &= 612501 \text{ m}^2 \\ &= 61.3 \text{ ha} \end{aligned}} \right\} \text{ [A1]}$$

f.

$$\begin{aligned}
 \overrightarrow{AB} &= 700\hat{i} + 400\hat{j} \\
 \overrightarrow{OD} &= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AC} \\
 &= -200\hat{i} + 800\hat{j} + \frac{2}{3}(1050\hat{i} - 1150\hat{j}) \\
 &= 500\hat{i} + \frac{100}{3}\hat{j} \\
 \overrightarrow{OE} &= \overrightarrow{OB} + \frac{2}{3}\overrightarrow{BC} \\
 &= 500\hat{i} + 1200\hat{j} + \frac{2}{3}(350\hat{i} - 1550\hat{j}) \\
 &= \frac{2200}{3}\hat{i} + \frac{500}{3}\hat{j}
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \overrightarrow{OD} &= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AC} \\ &= -200\hat{i} + 800\hat{j} + \frac{2}{3}(1050\hat{i} - 1150\hat{j}) \\ &= 500\hat{i} + \frac{100}{3}\hat{j} \\ \overrightarrow{OE} &= \overrightarrow{OB} + \frac{2}{3}\overrightarrow{BC} \\ &= 500\hat{i} + 1200\hat{j} + \frac{2}{3}(350\hat{i} - 1550\hat{j}) \\ &= \frac{2200}{3}\hat{i} + \frac{500}{3}\hat{j} \end{aligned}} \right\} \text{ [M2]}$$

Therefore,

$$\begin{aligned}
 \overrightarrow{DE} &= \frac{2200}{3}\hat{i} + \frac{500}{3}\hat{j} - \left(500\hat{i} + \frac{100}{3}\hat{j} \right) \\
 &= \frac{700}{3}\hat{i} + \frac{400}{3}\hat{j} \\
 &= \frac{1}{3}(700\hat{i} + 400\hat{j}) \\
 &= \frac{1}{3}\overrightarrow{AB}
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \overrightarrow{DE} &= \frac{2200}{3}\hat{i} + \frac{500}{3}\hat{j} - \left(500\hat{i} + \frac{100}{3}\hat{j} \right) \\ &= \frac{700}{3}\hat{i} + \frac{400}{3}\hat{j} \\ &= \frac{1}{3}(700\hat{i} + 400\hat{j}) \\ &= \frac{1}{3}\overrightarrow{AB} \end{aligned}} \right\} \text{ [A1]}$$

Since $\overrightarrow{DE} = k \times \overrightarrow{AB}$, where $k = \frac{1}{3}$ then the fence $[DE]$ is parallel to the boundary $[AB]$

Question 4

a.

$$2 \frac{dv}{dt} + v^2 + 1 = 0$$

$$\frac{dv}{dt} = -\frac{(v^2 + 1)}{2}$$

$$\frac{dt}{dv} = -\frac{2}{v^2 + 1}$$

$$t = -\int \frac{2}{v^2 + 1} dv$$

$$= -2 \int \frac{1}{v^2 + 1} dv$$

$$= -2 \tan^{-1} v + c$$

$$v = 1 \text{ when } t = 0$$

It follows,

$$0 = -2 \tan^{-1} v + c \Rightarrow c = 2 \tan^{-1} 1 = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$$

Therefore,

$$t = \frac{\pi}{2} - 2 \tan^{-1} v$$

$$2 \tan^{-1} v = \frac{\pi}{2} - t$$

$$\tan^{-1} v = \frac{\pi}{4} - \frac{t}{2}$$

$$v = \tan\left(\frac{\pi}{4} - \frac{t}{2}\right)$$

b.

$$\frac{\pi}{4} - \frac{t}{2} = \frac{\pi}{2}$$

$$t = -\frac{\pi}{2} + n \times 2\pi, n \in \mathbb{Z}$$

Since $\pi \leq a \leq 2\pi$

$$n = 1 \Rightarrow t = \frac{3\pi}{2} \Rightarrow a = \frac{3\pi}{2}$$

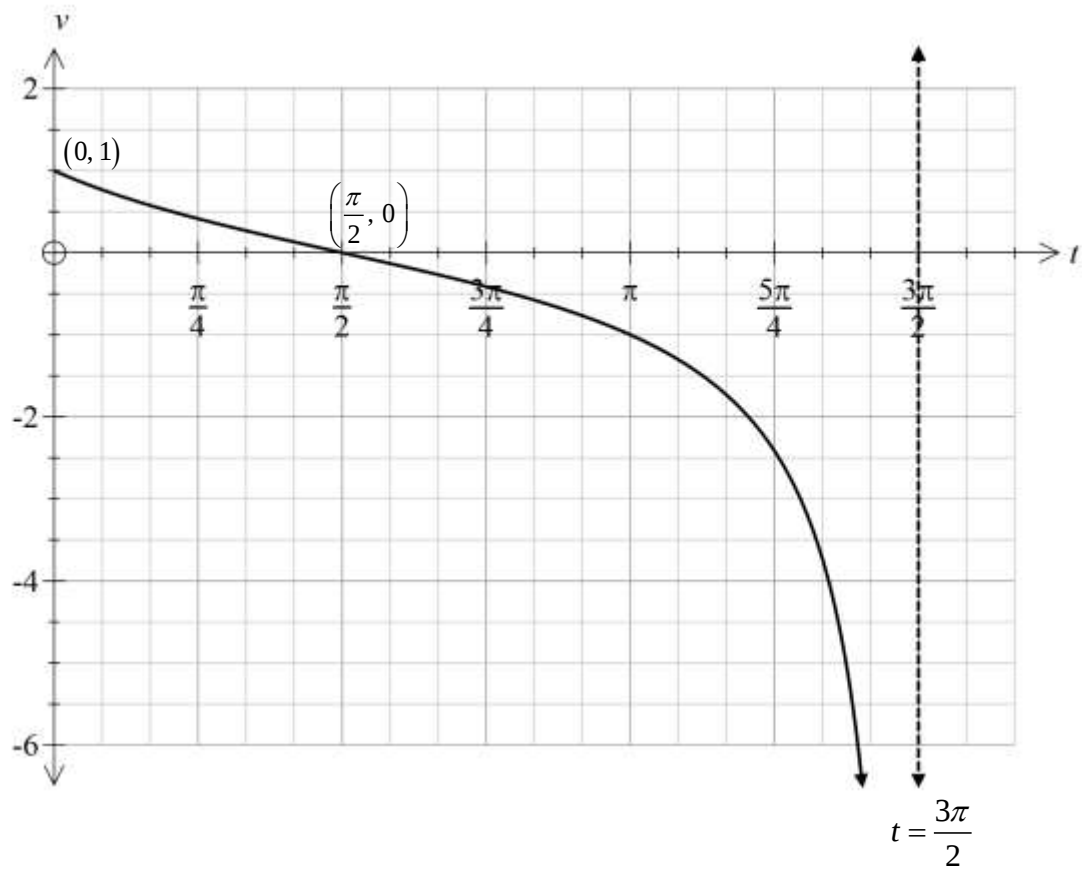
$$t = 0 \Rightarrow v = 1 \text{ and } v = 0 \Rightarrow t = \frac{\pi}{2}$$

The vertical asymptote is $t = \frac{3\pi}{2}$

[M2]

[M1]

[M1]



[A2]

c.

From CAS,

$$\text{Distance} = \int_0^{\frac{5\pi}{4}} \left| \tan\left(\frac{\pi}{4} - \frac{t}{2}\right) \right| dt = 2.61 \text{ metres} \quad [\text{A1}]$$

d.

Let T = the time taken to travel a distance of 0.75 metres

$$\int_0^T \left| \tan\left(\frac{\pi}{4} - \frac{t}{2}\right) \right| dt = 0.75$$

Solving for T

$$T = 2.05 \text{ sec}$$

} [M2] [A1]

e.

$$v = \tan\left(\frac{\pi}{4} - \frac{t}{2}\right)$$

$$\frac{dx}{dt} = \tan\left(\frac{\pi}{4} - \frac{t}{2}\right)$$

$$x = \int \tan\left(\frac{\pi}{4} - \frac{t}{2}\right) dt$$

$$= \int \frac{\sin\left(\frac{\pi}{4} - \frac{t}{2}\right)}{\cos\left(\frac{\pi}{4} - \frac{t}{2}\right)} dt$$

$$= \int \frac{2}{u} du$$

$$= 2 \log_e \left| \cos\left(\frac{\pi}{4} - \frac{t}{2}\right) \right| + c$$

$$u = \cos\left(\frac{\pi}{4} - \frac{t}{2}\right)$$

$$\frac{du}{dt} = \frac{1}{2} \sin\left(\frac{\pi}{4} - \frac{t}{2}\right) \Rightarrow \sin\left(\frac{\pi}{4} - \frac{t}{2}\right) = 2 \frac{du}{dt}$$

[M2]

$$x = 0 \text{ when } t = 0$$

It follows,

$$0 = 2 \log_e \left| \cos\left(\frac{\pi}{4}\right) \right| + c \Rightarrow c = -2 \log_e \cos\left(\frac{\pi}{4}\right) = -2 \log_e \frac{1}{\sqrt{2}}$$

[M1]

Therefore,

$$\begin{aligned} x &= 2 \log_e \left| \cos\left(\frac{\pi}{4} - \frac{t}{2}\right) \right| - 2 \log_e \frac{1}{\sqrt{2}} \\ &= 2 \log_e \left| \sqrt{2} \cos\left(\frac{\pi}{4} - \frac{t}{2}\right) \right| \end{aligned} \quad \text{[A1]}$$

f.

$$\begin{aligned} x &= 2 \log_e \left| \sqrt{2} \cos\left(\frac{\pi}{4} - \frac{5\pi}{8}\right) \right| \\ &= -1.23 \end{aligned}$$

Therefore, the particle is 1.23 metres to the left of the starting point. [A1]

g.

One of a number of possible methods

$$2 \frac{dv}{dt} + v^2 + 1 = 0$$

$$\frac{dv}{dt} = -\frac{(v^2 + 1)}{2}$$

$$v \frac{dv}{dx} = -\frac{(v^2 + 1)}{2}$$

$$\frac{dv}{dx} = -\frac{v^2 + 1}{2v}$$

$$\frac{dx}{dv} = -\frac{2v}{v^2 + 1}$$

$$x = -\int \frac{2v}{v^2 + 1} dv$$

$$= -\int \frac{1}{u} du$$

$$= -\log_e(v^2 + 1) + c$$

$$v = 1 \text{ when } x = 0$$

It follows,

$$0 = -\log_e(2) + c \Rightarrow c = \log_e 2$$

Therefore,

$$x = \log_e 2 - \log_e(v^2 + 1)$$

$$= \log_e \left(\frac{2}{v^2 + 1} \right)$$

$$u = v^2 + 1$$

$$\frac{du}{dv} = 2v$$

[M2]

[M1]