

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1



(TSSM's 2013 trial exam updated for the current study design)

SOLUTIONS

Question 1

$$\begin{aligned}
 \int \frac{4x-1}{x^2+9} dx &= \int \left(\frac{4x}{x^2+9} - \frac{1}{x^2+9} \right) dx & u = x^2 + 9 \\
 &= \int \frac{4x}{x^2+9} dx - \int \frac{1}{x^2+9} dx & \frac{du}{dx} = 2x \\
 &= \int \frac{2}{u} du - \frac{1}{3} \int \frac{3}{x^2+9} dx & 2 \frac{du}{dx} = 4x \\
 &= 2 \log_e(x^2+9) - \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + c & \text{[M2]} \\
 & & \text{[A1]}
 \end{aligned}$$

Question 2

$$\begin{aligned}
 2x^2 - x \sin y + y &= 10 \\
 4x - \left(\sin y + x \cos y \frac{dy}{dx} \right) + \frac{dy}{dx} &= 0 \\
 4x - \sin y - x \cos y \frac{dy}{dx} + \frac{dy}{dx} &= 0 & \text{[M2]} \\
 \frac{dy}{dx} (1 - x \cos y) &= \sin y - 4x \\
 \frac{dy}{dx} &= \frac{\sin y - 4x}{1 - x \cos y}
 \end{aligned}$$

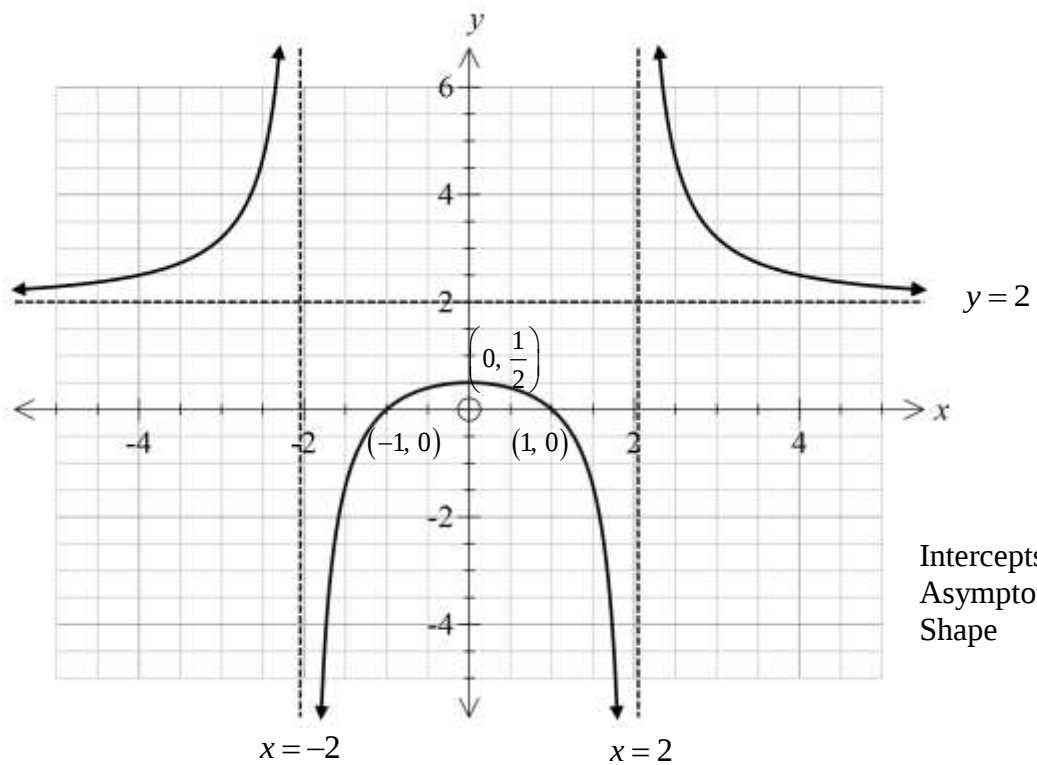
$$\text{At } \left(1, \frac{\pi}{4}\right), \frac{dy}{dx} = \frac{\frac{\sqrt{2}}{2} - 4}{1 - \frac{\sqrt{2}}{2}} = \frac{\sqrt{2} - 8}{2 - \sqrt{2}} = -3\sqrt{2} - 7 \quad \text{[A1]}$$

Question 3

a.

$$\left. \begin{aligned} f(x) &= \frac{2x^2 - 2}{x^2 - 4} \\ &= \frac{2(x^2 - 4) + 6}{x^2 - 4} \\ &= 2 + \frac{6}{x^2 - 4} \end{aligned} \right\} \text{ [A1]}$$

b.



Intercepts [A1]
Asymptotes [A1]
Shape [A1]

c.

$$\begin{aligned} A &= 2 \times \int_0^1 \left(2 + \frac{6}{x^2 - 4} \right) dx \\ &= 2 \times \int_0^1 \left(2 + \frac{3}{2(x-2)} - \frac{3}{2(x+2)} \right) dx \\ &= \left[4x + 3 \log_e \left| \frac{x-2}{x+2} \right| \right]_0^1 \\ &= 4 + 3 \log_e \left(\frac{1}{3} \right) \end{aligned} \quad \text{[A1]}$$

$$\begin{aligned} \frac{6}{x^2 - 4} &\equiv \frac{A}{x-2} + \frac{B}{x+2} \\ 6 &= A(x+2) + B(x-2) \\ A &= \frac{3}{2}, \quad B = -\frac{3}{2} \end{aligned}$$

[M2]

Question 4

$$\begin{aligned}
 z &= \left(\frac{-2-2i}{\sqrt{3}-i} \right)^3 \\
 &= \left(\frac{2\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)}{2\text{cis}\left(-\frac{\pi}{6}\right)} \right)^3 \\
 &= \left(\sqrt{2}\text{cis}\left(-\frac{7\pi}{12}\right) \right)^3 \\
 &= 2\sqrt{2}\text{cis}\left(-\frac{7\pi}{4}\right) \\
 &= 2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right) \\
 &= 2+2i
 \end{aligned}
 \quad \left. \begin{array}{l} \text{[M2]} \\ \text{[A1]} \end{array} \right\}$$

Question 5

$$\frac{dh}{dt} = \frac{dh}{dV} \frac{dV}{dt}, \text{ where } \frac{dV}{dt} = 2 \text{ and } h = 2r \Rightarrow r = \frac{h}{2} \quad \text{[M1]}$$

$$\begin{aligned}
 V &= \frac{1}{3}\pi r^2 h \\
 &= \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h \\
 &= \frac{\pi h^3}{12} \\
 \frac{dV}{dh} &= \frac{\pi h^2}{4} \Rightarrow \frac{dh}{dV} = \frac{4}{\pi h^2} \Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2} \times 2 = \frac{8}{\pi h^2} \\
 \text{After 1 minute, } V &= 2\text{m}^3 \Rightarrow \frac{\pi h^3}{12} = 2 \Rightarrow h = \sqrt[3]{\frac{24}{\pi}}
 \end{aligned}
 \quad \left. \begin{array}{l} \text{[M1]} \end{array} \right\}$$

Therefore,

$$\frac{dh}{dt} = \frac{8}{\pi \left(\frac{24}{\pi}\right)^{\frac{2}{3}}} = \sqrt[3]{\frac{8^3}{24^2 \pi}} = \sqrt[3]{\frac{2^9}{2^6 \times 9 \pi}} = \sqrt[3]{\frac{8}{9 \pi}} \quad \text{[M1]}$$

Question 6**a.**

$$\frac{d}{dx}(x \tan^{-1} x) = \tan^{-1} x + \frac{x}{1+x^2} \quad [\text{M1}]$$

b.

$$\begin{aligned} \int \left(\tan^{-1} x + \frac{x}{1+x^2} \right) dx &= x \tan^{-1} x + c \\ \int \tan^{-1} x \, dx + \int \frac{x}{1+x^2} dx &= x \tan^{-1} x + c \\ \int \tan^{-1} x \, dx &= x \tan^{-1} x - \int \frac{x}{1+x^2} dx + c \\ &= x \tan^{-1} x - \frac{1}{2} \int \frac{1}{u} du + c \\ &= x \tan^{-1} x - \frac{1}{2} \log_e (1+x^2) + c \end{aligned}$$

$$\begin{aligned} u &= 1+x^2 \\ \frac{du}{dx} &= 2x \\ \frac{1}{2} \frac{du}{dx} &= x \end{aligned}$$

}

[M2]

Therefore,

$$\begin{aligned} \int_0^1 \tan^{-1} x \, dx &= \left[x \tan^{-1} x - \frac{1}{2} \log_e (1+x^2) \right]_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2} \log_e 2 \\ &= \frac{\pi}{4} - \log_e \sqrt{2} \quad [\text{A1}] \end{aligned}$$

Question 7

Resolving forces,

$$2g \times \sin(30^\circ) = 2 \times a$$

$$a = 4.9 \text{ m/s}^2$$

 $\left. \begin{array}{l} \\ \end{array} \right\} \text{ [M1]}$

$$v^2 = 0 + 2 \times 4.9 \times 2 \rightarrow v^2 = 4 \times 4.9$$

[M1]

$$v = \frac{14}{\sqrt{10}} = \frac{14\sqrt{10}}{10} = \frac{7\sqrt{10}}{5} \text{ m/s}$$

[A1]

Question 8

a.

$$\begin{aligned}
 \underline{a} \cdot \underline{b} &= |\underline{a}| |\underline{b}| \cos 60 \\
 &= |\underline{a}| \times 3 |\underline{a}| \times \frac{1}{2} \\
 &= \frac{3}{2} |\underline{a}|^2
 \end{aligned}
 \left. \vphantom{\begin{aligned} \underline{a} \cdot \underline{b} &= |\underline{a}| |\underline{b}| \cos 60 \\ &= |\underline{a}| \times 3 |\underline{a}| \times \frac{1}{2} \\ &= \frac{3}{2} |\underline{a}|^2 \end{aligned}} \right\} \text{ [M1]}$$

$$\text{b. } \overrightarrow{\underline{BA}} = \underline{a} - \underline{b} \quad \text{[A1]}$$

$$\begin{aligned}
 \overrightarrow{\underline{OD}} &= \overrightarrow{\underline{OB}} + \overrightarrow{\underline{BC}} + \overrightarrow{\underline{CD}} \\
 &= \underline{b} + \underline{a} + m(\underline{a} - \underline{b}) \\
 &= \underline{a}(1+m) + \underline{b}(1-m) \quad \text{[A1]}
 \end{aligned}$$

c.

$$\begin{aligned}
 \overrightarrow{\underline{OD}} \cdot \overrightarrow{\underline{BA}} &= 0 \\
 (\underline{a}(1+m) + \underline{b}(1-m)) \cdot (\underline{a} - \underline{b}) &= 0 \\
 (1+m)\underline{a} \cdot \underline{a} - (1+m)\underline{a} \cdot \underline{b} + (1-m)\underline{a} \cdot \underline{b} - (1-m)\underline{b} \cdot \underline{b} &= 0 \\
 (1+m)|\underline{a}|^2 - (1+m) \times \frac{3}{2} |\underline{a}|^2 + (1-m) \times \frac{3}{2} |\underline{a}|^2 - (1-m)|\underline{b}|^2 &= 0 \\
 (1+m)|\underline{a}|^2 - (1+m) \times \frac{3}{2} |\underline{a}|^2 + (1-m) \times \frac{3}{2} |\underline{a}|^2 - (1-m) \times 9 |\underline{a}|^2 &= 0 \\
 (1+m) - \frac{3}{2}(1+m) + \frac{3}{2}(1-m) - 9(1-m) &= 0 \\
 m &= \frac{8}{7}
 \end{aligned}
 \left. \vphantom{\begin{aligned} \overrightarrow{\underline{OD}} \cdot \overrightarrow{\underline{BA}} &= 0 \\ (\underline{a}(1+m) + \underline{b}(1-m)) \cdot (\underline{a} - \underline{b}) &= 0 \\ (1+m)\underline{a} \cdot \underline{a} - (1+m)\underline{a} \cdot \underline{b} + (1-m)\underline{a} \cdot \underline{b} - (1-m)\underline{b} \cdot \underline{b} &= 0 \\ (1+m)|\underline{a}|^2 - (1+m) \times \frac{3}{2} |\underline{a}|^2 + (1-m) \times \frac{3}{2} |\underline{a}|^2 - (1-m)|\underline{b}|^2 &= 0 \\ (1+m)|\underline{a}|^2 - (1+m) \times \frac{3}{2} |\underline{a}|^2 + (1-m) \times \frac{3}{2} |\underline{a}|^2 - (1-m) \times 9 |\underline{a}|^2 &= 0 \\ (1+m) - \frac{3}{2}(1+m) + \frac{3}{2}(1-m) - 9(1-m) &= 0 \end{aligned}} \right\} \text{ [M2]}$$

Question 9

$$\left. \begin{aligned} y &= \cos^{-1}(2x) \\ x &= \frac{1}{2} \cos y \\ x^2 &= \frac{1}{4} \cos^2 y \end{aligned} \right\} \text{ [M1]}$$

$$\left. \begin{aligned} V &= \frac{\pi}{4} \int_0^{\pi/4} \cos^2 y \, dy \\ &= \frac{\pi}{8} \int_0^{\pi/4} (\cos(2y) + 1) \, dy \\ &= \frac{\pi}{8} \left[\frac{1}{2} \sin(2y) + y \right]_0^{\pi/4} \\ &= \frac{\pi}{8} \left(\frac{1}{2} + \frac{\pi}{4} \right) \\ &= \frac{\pi(\pi + 2)}{32} \end{aligned} \right\} \text{ [M1]} \quad \text{[A1]}$$

Question 10

a.

$$\left. \begin{aligned} x &= 2 \sec t + 1 \Rightarrow \sec t = \frac{x-1}{2} \\ y &= 3 \tan t - 2 \Rightarrow \tan t = \frac{y+2}{3} \end{aligned} \right\} \text{ [M1]}$$

Since $\sec^2 t - \tan^2 t = 1$, then

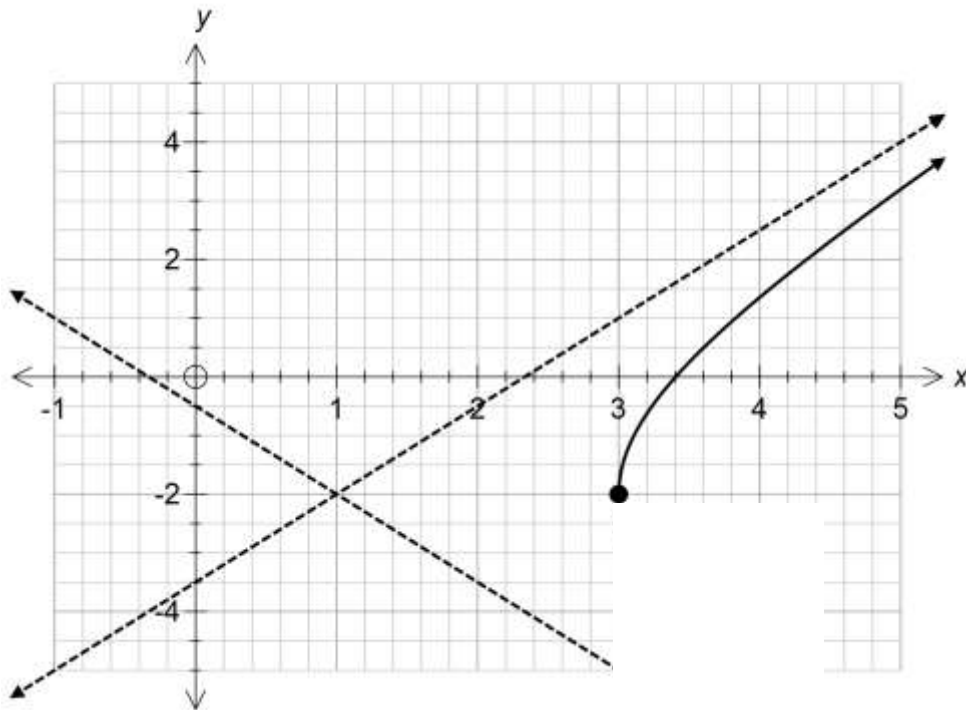
$$\left. \begin{aligned} \left(\frac{x-1}{2} \right)^2 - \left(\frac{y+2}{3} \right)^2 &= 1 \\ \frac{(x-1)^2}{4} - \frac{(y+2)^2}{9} &= 1 \end{aligned} \right\} \text{ [A1]}$$

$$\left. \begin{aligned} \text{Since } 0 \leq t < \frac{\pi}{2} \text{ then } 1 \leq \sec t < \infty \Rightarrow 3 \leq x < \infty \\ \text{Since } 0 \leq t < \frac{\pi}{2} \text{ then } 0 \leq \tan t < \infty \Rightarrow -2 \leq y < \infty \end{aligned} \right\} \text{ [A1]}$$

b.

$$\left. \begin{aligned} y+2 &= \pm \frac{3}{2}(x-1) \\ y &= \frac{3}{2}x - \frac{7}{2} \text{ and } y = -\frac{3}{2}x - \frac{1}{2} \end{aligned} \right\} \text{ [A1]}$$

c.



Shape and domain and range restrictions [A1]

Asymptotes and direction [A1]