

NAME: _____

VCE SPECIALIST MATHEMATICS

Practice Written Examination 1

Reading time: 15 minutes

Writing time: 1 hour

QUESTION AND ANSWER BOOK

Structure of book

<i>Number of questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
8	8	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers.
- Students are **not** permitted to bring into the examination room: notes of any kind, a calculator of any type, blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 10 pages.
- Formula booklet of 5 pages.
- Working space is provided throughout the book.

Instructions

- All written responses must be in English.
- Write your student name in the space provided above on this page.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer **all** questions in the spaces provided.

Unless otherwise specified an **exact** answer is required to a question.

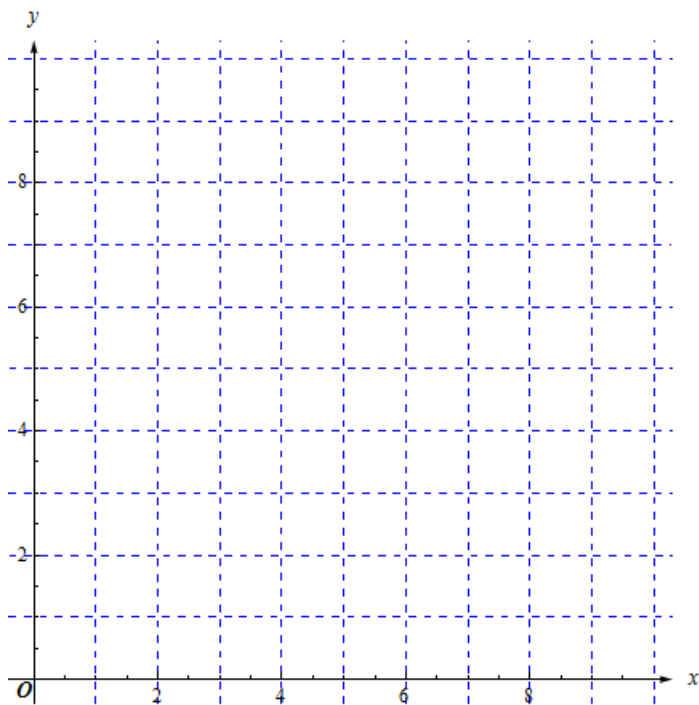
In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude $g \text{ m/s}^2$, where $g = 9.8$.

Question 1

- a. Sketch the graphs of $f(x) = x + \frac{3}{x}$ and $g(x) = 4$, on the axes below, showing any relevant asymptotes, turning points and/or points of intersection between $f(x)$ and $g(x)$.



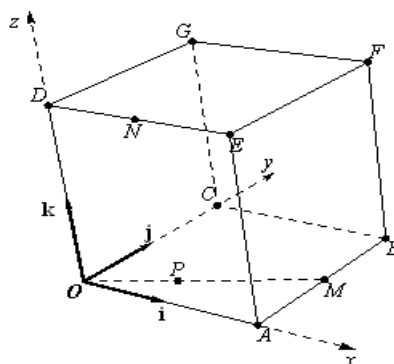
3 marks

- b. Find the volume when the area bounded between $f(x)$ and $g(x)$ is rotated about the x -axis.

2 marks

Question 2

In the diagram below, $OABCDEFG$ is a cube of side length 2 units.



Unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are parallel to $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OD}$ respectively. Points M and N are the midpoints of AB and DE respectively, and point P is the location on the OM that is closest to point N .

Express the following vectors in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

a. $\overrightarrow{OM}$ and $\overrightarrow{ON}$

b. $\overrightarrow{OP}$

2 marks

c. $\overrightarrow{PN}$

2 marks

1 mark

Question 3

Consider the polynomial $P(z) = z^3 - 6z - 9$, $z \in \mathbb{C}$.

- a. If $\omega = -\frac{3}{2} + \frac{i\sqrt{3}}{2}$, find $|\omega|$ and $\text{Arg}(\omega)$.

2 marks

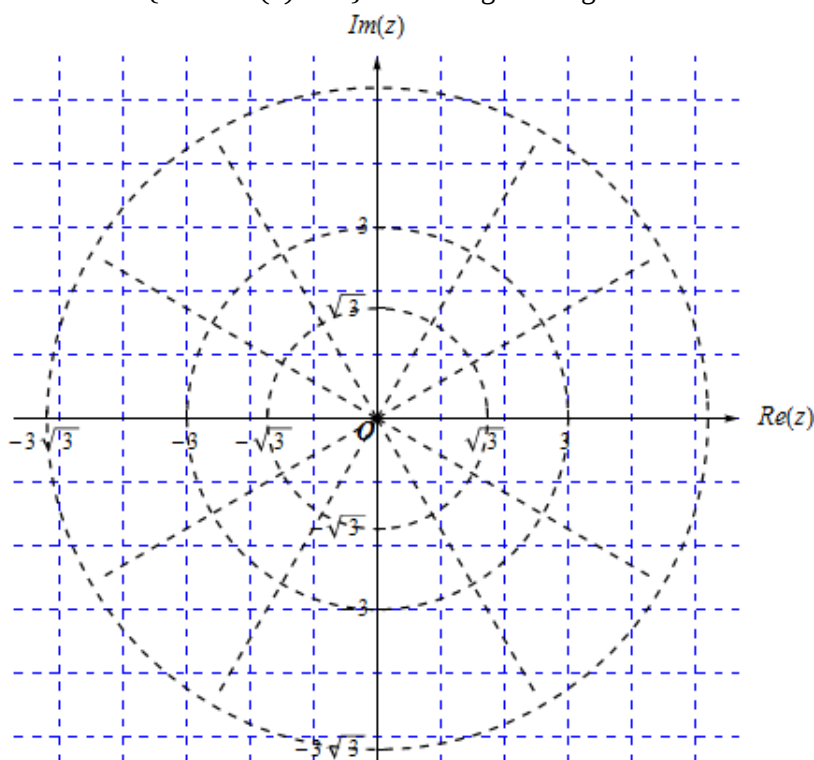
- b. Given that $P(\omega) = 0$, show that $(z^2 + 3z + 3)$ is also factor of $P(z)$.

1 mark

- c. Hence find a , in $P(z) = (z - a)(z^3 - 6z - 9)$.

1 mark

- d. Sketch $S = \{z \in \mathbb{C} : P(z) = 0\}$ on the Argand diagram below.



3 marks

Question 4

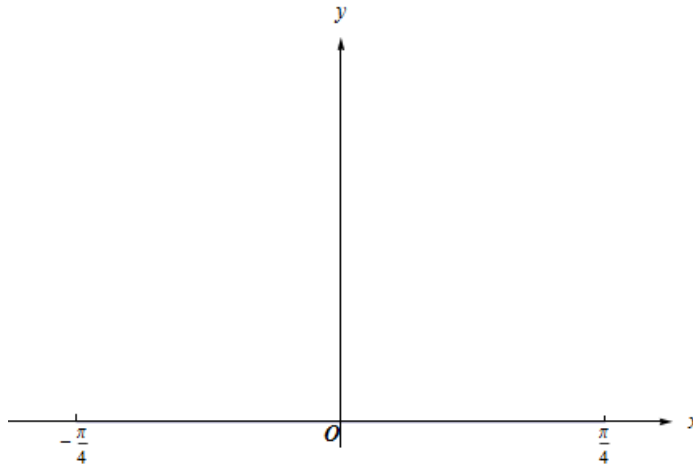
a. Show that $\frac{2}{1+\cos(4\theta)} = \sec^2(2\theta)$.

1 mark

b. Hence, find $\int_0^{\pi/12} \frac{2}{1+\cos(4\theta)} d\theta$.

2 marks

c. Sketch the graph of $y = \sec^2(2\theta)$, $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$



2 marks

d. Hence solve the equation $\frac{1}{1+\cos(4\theta)} = 2$ for $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

2 marks

Question 5

Find the gradient of the tangent to the curve $xy + y^2 + \arctan(x) = 6 + \frac{\pi}{4}$ at the point (1,2).

3 marks

Question 6

Let $y = x \arcsin(e^c x), x \geq 0$

a. Assuming $x \geq 0$, show that $\tan(\arcsin(e^c x)) = \frac{e^c x}{\sqrt{1 - e^{2c} x^2}}$

1 mark

b. Hence, find the value of a given that $\frac{dy}{dx} = a \frac{y}{x} + \tan\left(\frac{y}{x}\right)$, where a is a real constant.

2 marks



- c. If $y(1) = \frac{\pi}{4}$, find the value of c in the form $\frac{\log_e\left(\frac{1}{a}\right)}{b}$ where a, b are positive integers.

1 mark

Question 7

The position of a particle at time t is given by

$$\mathbf{r}(t) = \frac{6t}{1+t^3} \mathbf{i} + \frac{6t^2}{1+t^3} \mathbf{j}, \quad t \in \mathbb{R}$$

- a. Find velocity vector $\mathbf{v}$.

2 marks

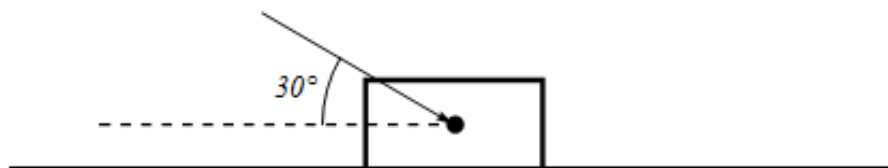
- b. For points on the curve, find the greatest value of x .

2 marks

Question 8

A box of mass 4 kg is resting on a rough horizontal plane. The box is pushed downwards with a force of magnitude P newtons acting at an angle of 30° to the horizontal. The box is in equilibrium. The coefficient of friction between the box and the plane is 0.5.

- a. On the diagram below, show all forces acting on the box and label them. (You may assume that all forces, including contact forces, act through the centre of mass).



1 mark

- b. Find, in terms of P , the normal component and frictional component of the contact force acting between the box and the plane.

2 marks

- c. Find the value of P when the box is just about to slide.

2 marks

SPECIALIST MATHEMATICS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
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Specialist Mathematics Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a + b)h$
curved surface area of a cylinder:	$2\pi rh$
volume of a cylinder:	$\pi r^2 h$
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc \sin A$
sine rule:	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
cosine rule:	$c^2 = a^2 + b^2 - 2ab \sin C$

Coordinate geometry

ellipse: $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$	hyperbola: $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$
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Circular (trigonometric) functions

$\cos^2(x) + \sin^2(x) = 1$	$\sin(2x) = 2 \sin(x) \cos(x)$
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y)$	$\sin(x - y) = \sin(x) \cos(y) - \cos(x) \sin(y)$
$\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$	$\cos(x - y) = \cos(x) \cos(y) + \sin(x) \sin(y)$
$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \tan(y)}$	$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x) \tan(y)}$

$\cos(2x) = \cos^2(x) - \sin^2(x)$ $= 2\cos^2(x) - 1$ $= 1 - 2\sin^2(x)$	$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$
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function	$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
domain	$[-1, 1]$	$[-1, 1]$	$\mathbb{R}$
range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + iy = r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta$	
$ z = \sqrt{x^2 + y^2} = r$	$-\pi < \operatorname{Arg}(z) \leq \pi$
$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$
$z^n = r^n \operatorname{cis}(n\theta)$ (de Moivre's theorem)	

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$
$\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$
$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e x + c$
$\frac{d}{dx}(\sin(ax)) = a \cos(ax)$	$\int \sin(ax) dx = -\frac{1}{a} \cos(ax) + c$
$\frac{d}{dx}(\cos(ax)) = -a \sin(ax)$	$\int \cos(ax) dx = \frac{1}{a} \sin(ax) + c$
$\frac{d}{dx}(\tan(ax)) = a \sec^2(ax)$	$\int \sec^2(ax) dx = \frac{1}{a} \tan(ax) + c$

$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}(ax) + c, a > 0$
$\frac{d}{dx}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1}(ax) + c, a > 0$
$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$	$\int \frac{a}{a^2+x^2} dx = \sin^{-1}(ax) + c$

product rule:	$\frac{d}{dx}(uv) = u \frac{d}{dx}(v) + v \frac{d}{dx}(u)$
quotient rule:	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{d}{dx}(u) - u \frac{d}{dx}(v)}{v^2}$
chain rule:	$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

Euler's method:	$\frac{dy}{dx} = f(x), x_0 = a \text{ and } y_0 = b$ $\Rightarrow x_{n+1} = x_n + h, y_{n+1} = y_n + hf(x_n)$
acceleration:	$a = \frac{d^2x}{dx^2} = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
constant (uniform) acceleration:	$v = u + at, s = ut + \frac{1}{2}at^2, v^2 = u^2 + 2as, s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

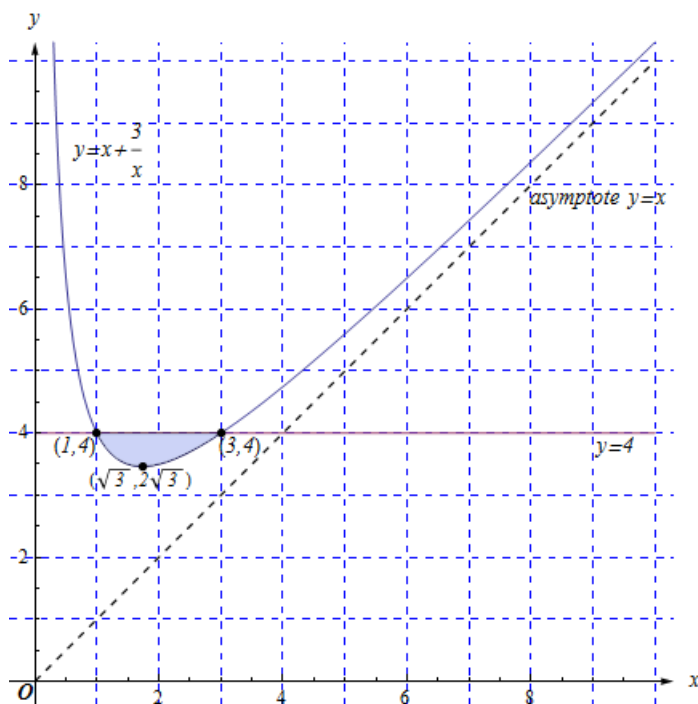
$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$	
$ \mathbf{r} = \sqrt{x^2 + y^2 + z^2} = r$	$\mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos(\theta) = x_1 x_2 + y_1 y_2 + z_1 z_2$
$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$	

Mechanics

momentum:	$\mathbf{p} = m\mathbf{v}$
equation of motion:	$\mathbf{R} = m\mathbf{a}$
friction:	$F \leq \mu N$

Solution Pathway

Below are sample answers. Please consider the merit of alternative responses.

Specialist Mathematics Exam 1: SOLUTIONS**Question 1****a.**

turning point (A1)

Asymptote (A1)

points of intersection (A1)

3 marks

b.

$$\begin{aligned}
 V &= \pi \int_1^3 g^2(x) - f^2(x) dx \\
 &= \pi \int_1^3 10 - \frac{9}{x^2} - x^2 dx \quad (M1) \\
 &= \frac{16\pi}{3} \text{ cubic units} \quad (A1)
 \end{aligned}$$

2 marks

Question 2**a.**

$$\overrightarrow{OM} = 2\mathbf{i} + \mathbf{j}, \quad (A1)$$

$$\overrightarrow{ON} = \mathbf{i} + 2\mathbf{k} \quad (A1)$$

2 marks

b.

$$\begin{aligned} \overrightarrow{OP} &= \text{proj}_{\overrightarrow{ON}}(\overrightarrow{OM}) \\ &= \left(\frac{\overrightarrow{ON} \cdot \overrightarrow{OM}}{\overrightarrow{OM} \cdot \overrightarrow{OM}} \right) \overrightarrow{OM} \quad \text{mostly correct}(M1) \\ &= \frac{4}{5}\mathbf{i} + \frac{2}{5}\mathbf{j} \quad (A1) \end{aligned}$$

2 marks

c.

$$\begin{aligned} \overrightarrow{PN} &= \overrightarrow{ON} - \overrightarrow{OP} \\ &= \frac{14}{5}\mathbf{i} - \frac{2}{5}\mathbf{j} + 2\mathbf{k} \quad (A1) \end{aligned}$$

1 mark

Question 3**a.**

$$|\omega| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{3}, \quad (A1)$$

$$\text{Arg}(\omega) = \arctan\left(\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(-\frac{3}{2}\right)}\right) + \pi = \frac{5\pi}{6} \quad (A1)$$

2 marks

b.

Coefficients of $P(z)$ are real, so conjugate root theorem applies $\Rightarrow P(\bar{\omega}) = 0$

$\therefore (z - \omega)(z - \bar{\omega}) = (z^2 + 3z + 3)$ is also a factor. (R1)

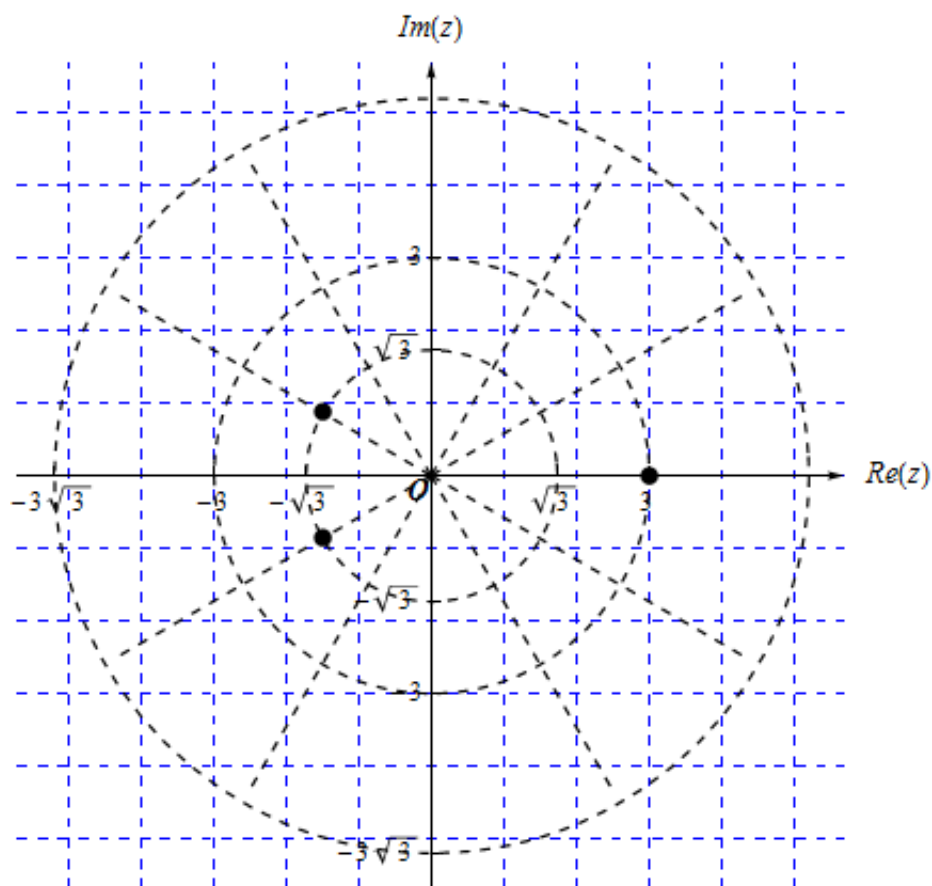
1 mark

c.

$$\begin{aligned} P(z) &= z^3 - 6z - 9 \\ &= (z - a)(z^2 + 3z + 3) \quad \text{expanding and equating like terms} \\ &= z^3 + (3 - a)z^2 + (3 - 3a)z - 3a \\ \therefore a &= 3 \quad (A1) \end{aligned}$$

1 mark

d.



3 marks

Question 4

a.

$$\begin{aligned}
 \cos(4\theta) &= 2\cos^2(2\theta) - 1 \\
 \Rightarrow \frac{\cos(4\theta) + 1}{2} &= \cos^2(2\theta) && \text{Attempt to use double angle cos(M1)} \\
 \therefore \frac{2}{1 + \cos(4\theta)} &= \sec^2(2\theta)
 \end{aligned}$$

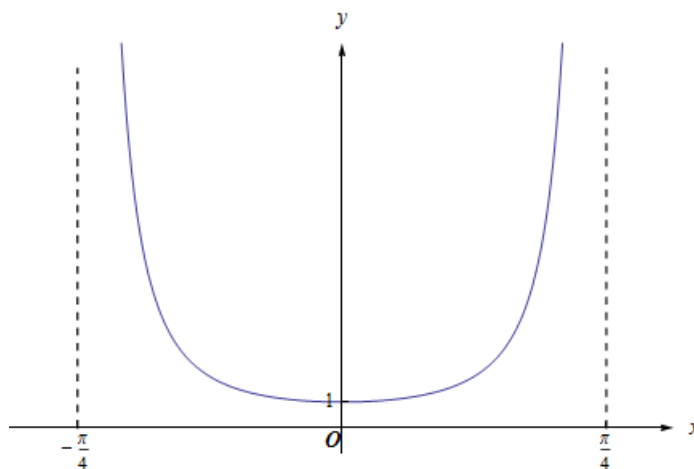
1 mark

b.

$$\begin{aligned}
 \int_0^{\pi/12} \frac{2}{1 + \cos(4\theta)} d\theta &= \int_0^{\pi/12} \sec^2(2\theta) d\theta \\
 \Rightarrow &= \frac{1}{2} \int_0^{\pi/12} \frac{d}{d\theta} (\tan(2\theta)) d\theta \quad \text{mostly correct (M1)} \\
 &= \frac{1}{2} [\tan(2\theta)]_0^{\pi/12} \\
 \therefore &= \frac{1}{2\sqrt{3}} \quad (A1)
 \end{aligned}$$

2 marks

c.



turning point (A1)

Asymptotes (A1)

2 marks

d.

$$\begin{aligned}
 \frac{1}{1 + \cos(4\theta)} &= 2 \\
 \Rightarrow \sec^2(2\theta) &= 4 \quad \text{mostly correct (M1)} \\
 \Rightarrow \cos(2\theta) &= \pm \frac{1}{2} \\
 \Rightarrow 2\theta &\in \left\{-\frac{\pi}{3}, \frac{\pi}{3}\right\} \\
 \therefore \theta &\in \left\{-\frac{\pi}{6}, \frac{\pi}{6}\right\} \quad (A1)
 \end{aligned}$$

2 marks

Question 5*differentiate both sides wrt x*

$$y + x \frac{d}{dx}(y) + 2y \frac{d}{dx}(y) + \frac{1}{1+x^2} = 0 \quad \text{mostly correct (M1)}$$

$$\frac{d}{dx}(y) = \frac{-y - \frac{1}{(1+x^2)}}{x+2y} \quad (A1)$$

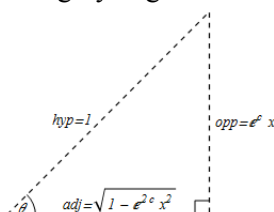
$$\left. \frac{d}{dx}(y) \right|_{(1,2)} = -\frac{1}{2} \quad (A1)$$

3 marks

Question 6

a. Let $\theta = \arcsin(e^c x) \Rightarrow \sin(\theta) = \frac{e^c x}{1} = \frac{\text{opp}}{\text{hyp}}$

Using Pythagoras' Theorem



So,

$$\tan(\arcsin(e^c x)) = \tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{e^c x}{\sqrt{1-e^{2c}x^2}} \quad \text{mostly correct (M1)}$$

1 mark

b.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x \arcsin(e^c x))}{dx} = \arcsin(e^c x) + x \frac{e^c}{\sqrt{1-e^{2c}x^2}} \\ &= \arcsin(e^c x) + \tan(\arcsin(e^c x)) \quad (M1) \\ &= \frac{y}{x} + \tan\left(\frac{y}{x}\right) \\ \therefore a &= 1 \quad (A1) \end{aligned}$$

2 marks

c.

$$\begin{aligned} y(1) &= \frac{\pi}{4} \Rightarrow \arcsin(e^c) = \frac{\pi}{4} \\ e^c &= \sin\left(\frac{\pi}{4}\right) \\ \therefore c &= \ln\left(\sin\left(\frac{\pi}{4}\right)\right) = \frac{\log_e\left(\frac{1}{2}\right)}{2} \quad (A1) \end{aligned}$$

1 mark

Question 7**a.**

$$\mathbf{v} = \frac{d}{dt} \left(\frac{6t}{1+t^3} \right) \mathbf{i} + \frac{d}{dt} \left(\frac{6t^2}{1+t^3} \right) \mathbf{j} \quad (M1)$$

$$= \frac{6(1-2t^3)}{(1+t^3)^2} \mathbf{i} + \frac{6t(2-t^3)}{(1+t^3)^2} \mathbf{j} \quad (A1)$$

2 marks

b.

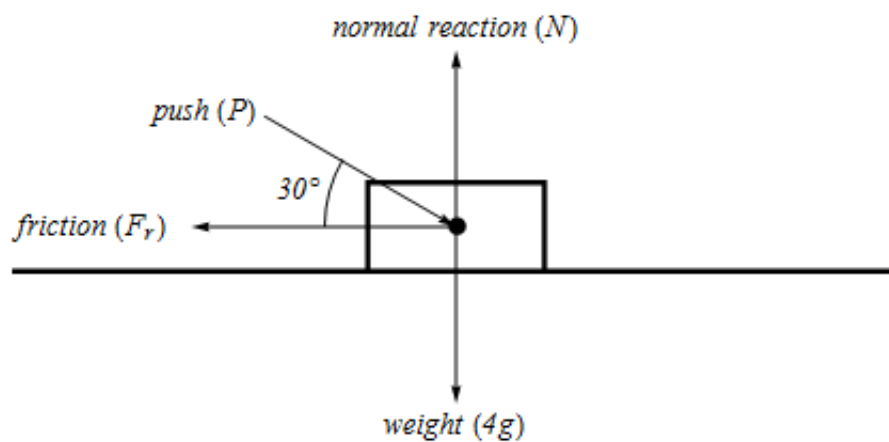
$$\text{maximum } x \Rightarrow \mathbf{i} \cdot \mathbf{v} = 0 \quad \frac{dx}{dt} = 0 \quad (M1)$$

$$\Rightarrow (1 - 2t^3) = 0$$

$$t = 1/\sqrt[3]{2}$$

$$\therefore x(1/\sqrt[3]{2}) = 2\sqrt[3]{4} \quad (A1)$$

2 marks

Question 8**a.**

1 mark

b.*Resolving along the plane*

$$\begin{aligned}F_r &= P \cos(30^\circ) \\&= \frac{\sqrt{3}}{2}P \quad (A1)\end{aligned}$$

Resolving perpendicular to the plane

$$\begin{aligned}N &= P \sin(30^\circ) + 4g \\&= \frac{P}{2} + 4g \quad (A1)\end{aligned}$$

2 marks

c.*Limiting Friction*

$$\begin{aligned}F_r &= \frac{1}{2}N \\ \frac{\sqrt{3}}{2}P &= \frac{1}{2}\left(\frac{P}{2} + 4g\right) \quad (M1) \\ \therefore P &= \frac{8g}{2\sqrt{3} - 1} \quad (A1)\end{aligned}$$

2 marks