

VCE Specialist Mathematics Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E

12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E

SECTION 1**Question 1 C**

$$x^2 - y^2 = 4x - 2y$$

$$x^2 - 4x + 4 - (y^2 - 2y + 1) = 3$$

$$(x - 2)^2 - (y - 1)^2 = 3$$

$$\frac{(x - 2)^2}{3} - \frac{(y - 1)^2}{3} = 1$$

The equations of the asymptotes are $y - 1 = \pm(x - 2)$ i.e. $y = x - 1$ and $y = 3 - x$.

Question 2 D

$$x = a \cos(\theta) \text{ and } y = b \sin(\theta)$$

$$\frac{dx}{d\theta} = -a \sin(\theta) \text{ and } \frac{dy}{d\theta} = b \cos(\theta)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= -\frac{b \cos(\theta)}{a \sin(\theta)}, \sin(\theta) \neq 0 \end{aligned}$$

$$\text{The equation of the tangent is } y - b \sin(\theta) = -\frac{b \cos(\theta)}{a \sin(\theta)}(x - a \cos(\theta))$$

$$\text{So } y = \frac{b(a - \cos(\theta)x)}{a \sin(\theta)}.$$

Question 3 C

We require a vertical asymptote at $x = 1$.

So we can disregard **A**, **B** and **E**.

For $y = g(x)$:

$$\text{As } x \rightarrow 0^+, y \rightarrow -\infty$$

$$\text{For } y = \frac{1}{g(x)}:$$

$$\text{As } x \rightarrow 0^+, y \rightarrow 0^-$$

So we can disregard **D**.

Question 4 B

We require $-1 \leq 2 - x \leq 1$ and so the implied domain is $1 \leq x \leq 3$.

Question 5 **E**

From $\tan(x) = \frac{3}{4}$, we obtain $\sin(x) = \frac{3}{5}$ and $\cos(x) = \frac{4}{5}$.

From $\tan(y) = \frac{4}{3}$, we obtain $\sin(y) = \frac{4}{5}$ and $\cos(y) = \frac{3}{5}$.

$$\begin{aligned}\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ &= \frac{4}{5} \times \frac{3}{5} - \frac{3}{5} \times \frac{4}{5} \\ &= 0\end{aligned}$$

Question 6 **A**

$v \neq \bar{w}$ i.e. $\bar{w} = -\sqrt{3} + i$ and so **A** is incorrect

The other four alternatives are all correct.

Note that an equilateral triangle is a special case of an isosceles triangle having not just two, but all three sides and angles equal.

Question 7 **E**

Solving $(x-1)^2 + y^2 = 1$ and $x+y=1$ simultaneously we obtain $x = \frac{2-\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$ or $x = \frac{2+\sqrt{2}}{2}$

and $y = -\frac{\sqrt{2}}{2}$

So for $T \cap V$ we have $\frac{2-\sqrt{2}}{2} \leq \operatorname{Re}(z) \leq \frac{2+\sqrt{2}}{2}$.

Question 8 **C**

The cube roots of unity are $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

Let $u = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and let $v = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

$uv = 1$ and so **A** is incorrect.

$u \neq v$ so **B** is incorrect.

$v^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and so $u = v^2$ i.e. **C** is correct.

$u = \bar{v}$ and so **D** is incorrect.

$u+v = -1$ and so **E** is incorrect.

Question 9 A

If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$ then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$.

Here $x_0 = 0$, $y_0 = 2$ and $h = \frac{1}{5}$.

For $x_0 = 0$:

$$\begin{aligned}y_1 &= y_0 + hf(x_0) \\&= 2 + \frac{1}{5} \cos^{-1}(0) \\&= 2 + \frac{\pi}{10}\end{aligned}$$

For $x_1 = \frac{1}{5}$:

$$\begin{aligned}y_2 &= y_1 + hf(x_1) \\&= 2 + \frac{\pi}{10} + \frac{1}{5} \cos^{-1}\left(\frac{\pi}{10}\right)\end{aligned}$$

Question 10 A

There is a repeated linear factor in the denominator so the partial fraction form is $\frac{A}{(x-3)} + \frac{B}{(x-3)^2}$.

Question 11 D

The volume, in litres, of the mixture at time t minutes is $100 + 3t$.

$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{dt_{\text{in}}} - \frac{dx}{dt_{\text{out}}} \\&= 30 \times 10 - \left(\frac{x}{100 + 3t}\right) \times 7\end{aligned}$$

$$\text{So } \frac{dx}{dt} = 300 - \frac{7x}{100 + 3t} \text{ (g/min)}$$

Question 12 C

Let V be the volume.

Using $V = \pi \int_a^b ((f(x))^2 - (g(x))^2) dx$ we obtain:

$$\begin{aligned} V &= \pi \int_0^{\sqrt{3}} \left(\left(\frac{4}{\sqrt{1+x^2}} \right)^2 - 2^2 \right) dx \\ &= \pi \int_0^{\sqrt{3}} \left(\frac{16}{1+x^2} - 4 \right) dx \end{aligned}$$

Question 13 E

Let $u = 1 + 2\tan(x)$.

$$\frac{du}{dx} = 2\sec^2(x)$$

$$\text{So } \int \frac{1}{(1 + 2\tan(x))^2 \cos^2(x)} dx = \int \frac{1}{2u^2} du.$$

Question 14 B

As $\vec{u} = 3\vec{v} - 2\vec{w}$ then each of these vectors can be expressed as a linear combination of the other two vectors. This is the definition for linearly dependent vectors.

Question 15 B

If $\vec{u}$ and $\vec{v}$ are perpendicular and $\vec{u}$ and $\vec{v}$ are non-zero vectors, then $\vec{u} \cdot \vec{v} = 0$.

$$\sin(2t)\cos(t) + \sin(t)\cos(2t) - 1 = 0$$

$$\sin(3t) - 1 = 0$$

$$\text{So } t = \frac{\pi}{6} \text{ as } 0 \leq t \leq \frac{\pi}{2}.$$

Question 16 D

The sum of these two vectors is $\vec{u}$

$$\begin{aligned} \vec{u} &= (2\vec{i} - 3\vec{j} + 2\vec{k}) + (3\vec{i} + 4\vec{j} + 2\vec{k}) \\ &= 5\vec{i} + \vec{j} + 4\vec{k} \end{aligned}$$

Question 17 A

$$\underline{\mathbf{a}}(t) = \sin(t)\underline{\mathbf{j}}$$

$$\underline{\mathbf{v}}(t) = -\cos(t)\underline{\mathbf{j}} + \underline{\mathbf{c}}$$

Using $\underline{\mathbf{v}}(\pi) = \underline{\mathbf{i}} + \underline{\mathbf{j}}$, we obtain $\underline{\mathbf{i}} + \underline{\mathbf{j}} = -\cos(\pi)\underline{\mathbf{j}} + \underline{\mathbf{c}}$

Hence $\underline{\mathbf{c}} = \underline{\mathbf{i}}$ and so $\underline{\mathbf{v}}(t) = \underline{\mathbf{i}} - \cos(t)\underline{\mathbf{j}}$

$$\begin{aligned}\underline{\mathbf{v}}(0) &= \underline{\mathbf{i}} - \cos(0)\underline{\mathbf{j}} \\ &= \underline{\mathbf{i}} - \underline{\mathbf{j}}\end{aligned}$$

Question 18 A

Given $y = e^{pt}$:

$$\frac{dy}{dt} = pe^{pt} \text{ and } \frac{d^2y}{dt^2} = p^2e^{pt}$$

Rearranging the differential equation we obtain $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0$

Substituting for y , $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ into the differential equation we obtain $p^2e^{pt} - 5pe^{pt} + 6e^{pt} = 0$.

Taking out e^{pt} as a common factor we obtain $e^{pt}(p^2 - 5p + 6) = 0$

$$p^2 - 5p + 6 = 0 \quad (\text{as } e^{pt} \neq 0)$$

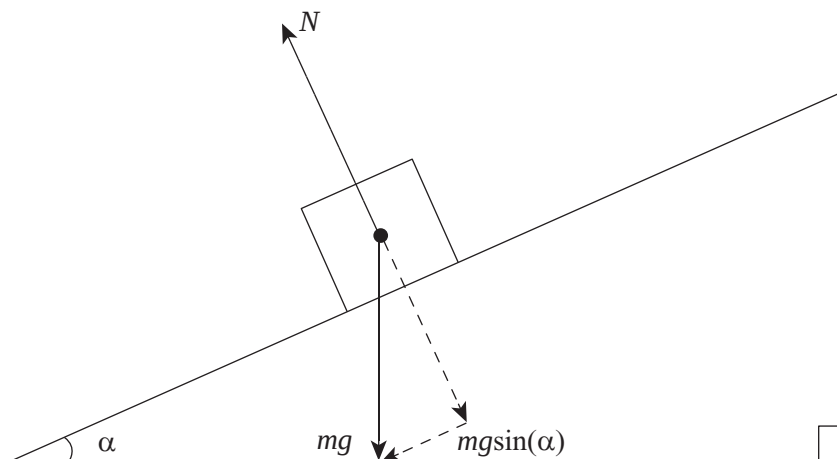
Hence $p = 2$ and 3 .

Question 19 C

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} \times \frac{dv}{dx}$$

$$= (x+2) \times \frac{d}{dx}(x+2)$$

$$= x+2$$

Question 20 E

The equation of motion parallel to the plane is $mg \sin(\alpha) = ma$.

So $a = g \sin(\alpha)$.

Question 21 B

The initial momentum (p_i) is 5×6 i.e. 30 (kg m/s) .

To calculate the final momentum (p_f), we need to find the particle's final velocity.

Given $u = 6$, $s = 20$ and $t = 2$ we can find v by using $s = \frac{1}{2}(u + v)t$.

Solving $20 = \frac{2}{2}(6 + v)$ for v we obtain $v = 14$.

The final momentum (p_f) is 5×14 i.e. 70 (kg m/s) .

$$\begin{aligned} \text{Change in momentum } (\Delta p) &= p_f - p_i \\ &= 40 \text{ (kg m/s)} \end{aligned}$$

Question 22 D

The resultant force of $\vec{S}$ and $\vec{T}$ is $10\sqrt{2}$ newtons in the southwest direction.

If the particle is in equilibrium then the third force must be the negative of this.

So $\vec{U}$ has magnitude $10\sqrt{2}$ newtons acting in the northeast direction.

SECTION 2

Question 1 (13 marks)

a. $|\mathbf{r}(t)| = \sqrt{(1-t^2)^2 + (1-t)^2 + t^4}$ A1

b. $\frac{d}{dt}|\mathbf{r}(t)|^2 = 8t^3 - 2t - 2$ M1 A1

Solving $8t^3 - 2t - 2 = 0$ for t with $t \geq 0$ gives $t = 0.76$ (s) (correct to two decimal places) M1 A1

c. $\frac{d^2}{dt^2}|\mathbf{r}(t)| = \frac{8t^6 - 6t^4 - 16t^3 + 24t^2 - 3}{(2t^4 - t^2 - 2t + 2)^{\frac{3}{2}}}$ M1 A1

When $t = 0.76069\dots$, $\frac{d^2}{dt^2}|\mathbf{r}(t)| = 7.875\dots (> 0)$, and so $t = 0.76$ (s) is when the particle is closest to the origin. A1

d. The coordinates, correct to two decimal places, are $(0.42, 0.24, 0.58)$ A1

e. $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$, $\mathbf{r}(1) = \mathbf{k}$ and $\mathbf{r}(2) = -3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ A1

If the particle travels in a straight line, then $\mathbf{r}(1) - \mathbf{r}(0) = m(\mathbf{r}(2) - \mathbf{r}(1))$ M1

$-\mathbf{i} - \mathbf{j} + \mathbf{k} = m(-3\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ M1

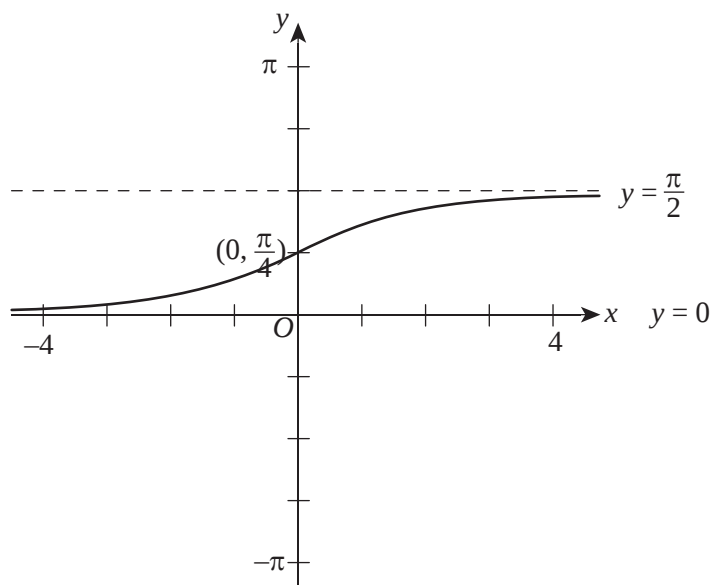
There is no solution for m , i.e. $\mathbf{r}(1) - \mathbf{r}(0)$ and $\mathbf{r}(2) - \mathbf{r}(1)$ are not parallel, and so $\mathbf{r}(0)$, $\mathbf{r}(1)$ and $\mathbf{r}(2)$ are not collinear. The particle does not travel in a straight line. A1

Question 2 (15 marks)

a. Solving $x = \log_e(2\theta)$ for θ gives $\theta = \frac{1}{2}e^x$. M1

Substituting $\theta = \frac{1}{2}e^x$ into $y = \tan^{-1}(2\theta)$ gives $y = \tan^{-1}(e^x)$. A1

b.



Correct shape

A1

Horizontal asymptotes $y = 0$ and $y = \frac{\pi}{2}$; y -axis intercept $(0, \frac{\pi}{4})$

A1

c. Method 1:

$$\frac{dx}{d\theta} = \frac{1}{\theta} \text{ and } \frac{dy}{d\theta} = \frac{2}{1+4\theta^2}$$

M1

$$\text{So } \frac{dy}{dx} = \frac{2\theta}{1+4\theta^2}$$

A1

When $y = k$, $2\theta = \tan(k)$ (or equivalent)

$$\frac{dy}{dx} = \frac{\tan(k)}{1+\tan^2(k)}$$

A1

$$= \sin(k)\cos(k)$$

$$\text{So } \frac{dy}{dx} = \frac{1}{2}\sin(2k).$$

A1

Only award the last A1 if the previous line is seen

Method 2:

$$\frac{dy}{dx} = \frac{1}{e^x + e^{-x}} \text{ (or equivalent)}$$

M1

$$\text{When } y = k, \theta = \frac{1}{2}\tan(k)$$

$$\text{So } x = \log_e(\tan(k)).$$

A1

$$\frac{dy}{dx} = \frac{\tan(k)}{1+\tan^2(k)}$$

A1

$$= \sin(k)\cos(k)$$

$$\text{So } \frac{dy}{dx} = \frac{1}{2}\sin(2k).$$

A1

Only award the last A1 if the previous line is seen

d. When $y = \frac{\pi}{4}$, $x = 0$. A1

The equation of the tangent at $\left(0, \frac{\pi}{4}\right)$ is $y = \frac{1}{2}x + \frac{\pi}{4}$ and the tangent cuts the x -axis at $\left(-\frac{\pi}{2}, 0\right)$ M1

The equation of the normal at $\left(0, \frac{\pi}{4}\right)$ is $y = -2x + \frac{\pi}{4}$ and the normal cuts the x -axis at $\left(\frac{\pi}{8}, 0\right)$. M1

Let the area of the triangle be A square units.

$$A = \frac{1}{2} \times \frac{\pi}{4} \times \left(\frac{\pi}{8} - \left(-\frac{\pi}{2} \right) \right) \quad \text{A1}$$

$$\text{So } A = \frac{5\pi^2}{64} \text{ (square units)} \quad \text{A1}$$

e. Let V be the volume of the solid.

$$V = \pi \int_0^3 (\tan^{-1}(e^x))^2 dx \quad \text{M1}$$

$$\text{So } V = 15.9 \text{ (m}^3\text{) (correct to the nearest tenth of a cubic metre)} \quad \text{A1}$$

Question 3 (8 marks)

a. $\text{Arg}(z - 2i) = \frac{\pi}{6}$

Substituting $z = x + yi$ we obtain $\text{Arg}(x + (y - 2)i) = \frac{\pi}{6}$

$$\frac{y-2}{x} = \tan\left(\frac{\pi}{6}\right) \quad \text{M1}$$

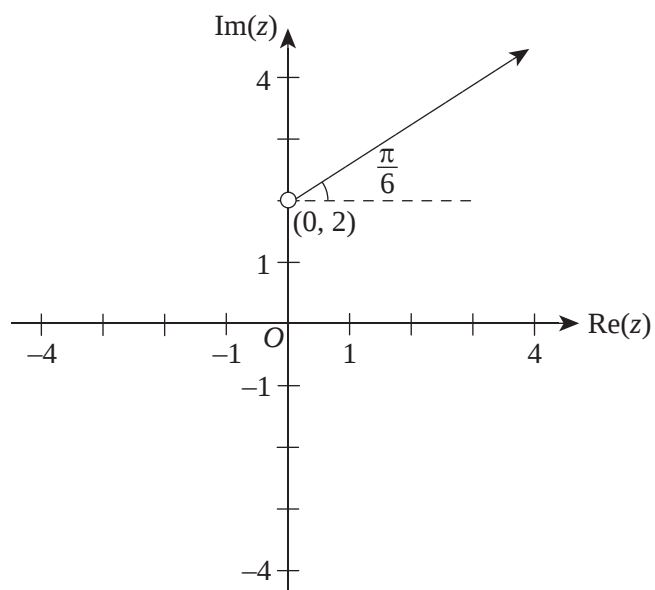
$$\frac{y-2}{x} = \frac{1}{\sqrt{3}}$$

$$y - 2 = \frac{1}{\sqrt{3}}x$$

$$\text{So } y = \frac{1}{\sqrt{3}}x + 2, x > 0. \quad \text{A1}$$

Only award the last A1 if one of the two previous lines are seen

b.



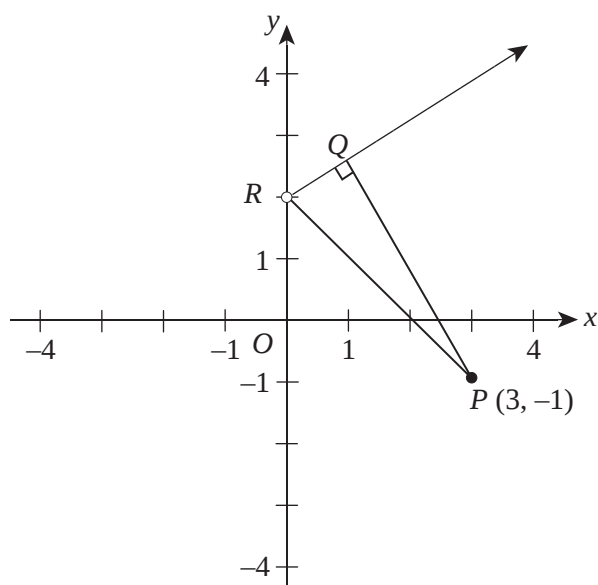
Correct shape

A1

Open circle at (0, 2)

A1

c.



$$RP = \sqrt{3^2 + 3^2}$$

$$= 3\sqrt{2}$$

A1

$$\angle QRP = \frac{\pi}{4} + \frac{\pi}{6}$$

$$= \frac{5\pi}{12}$$

A1

$$\frac{QP}{3\sqrt{2}} = \sin\left(\frac{5\pi}{12}\right)$$

M1

$$QP = \frac{3(\sqrt{3} + 1)}{2}$$

A1

Question 4 (10 marks)

a. $m \frac{dv}{dt} \propto -v^2$

So $m \frac{dv}{dt} = -kv^2$. A1

Only award the A1 if the first line is seen.

b. Using either integration or a differential equation solver with $v = u$ when $t = 0$. M1

$\frac{1}{u} - \frac{1}{v} = -\frac{kt}{m}$ (or equivalent) A1

Making v the subject we obtain $v = \frac{mu}{kut + m}$. A1

Only award the last A1 if the first line is seen.

c. Using either integration or a differential equation solver with $x = 0$ when $t = 0$. M1

$x = \frac{m}{k} \log_e \left(\frac{kut + m}{m} \right)$ (or equivalent) A1

So $x = \frac{m}{k} \log_e \left(1 + \frac{kut}{m} \right)$. A1

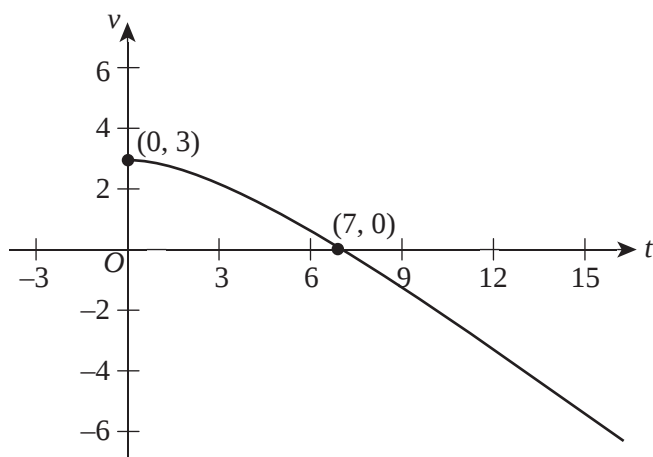
Only award the last A1 if the previous line is seen.

d. $a = \frac{dv}{dt}$ and $v = \frac{mu}{kut + m}$

$\frac{d}{dt} \left(\frac{mu}{kut + m} \right) = -\frac{kmu^2}{(kut + m)^2}$ A1

e. Over a long period of time:

- the particle's speed continually decreases A1
- it never comes to rest A1

Question 5 (12 marks)**a.**Correct shape (concavity) for $0 \leq t \leq 2$

A1

Correct shape (linear) for $t > 2$

A1

Axes intercepts: $(0, 3)$ and $(7, 0)$

A1

b.**i.** 3 m/s

A1

ii. $t = 7$ (s)

A1

c.Let d be the distance travelled by the particle, d_1 be the distance travelled for $0 \leq t \leq 2$ and d_2 be the distance travelled for $2 \leq t \leq 7$.Attempting to find d_1 and d_2

M1

For $0 \leq t \leq 2$:

$$d_1 = \int_0^2 \sqrt{9 - t^2} dt$$

$$= \frac{9}{2} \sin^{-1}\left(\frac{2}{3}\right) + \sqrt{5}$$

A1

For $2 \leq t \leq 7$:Either use of a definite integral $d_2 = \int_2^7 \frac{7-t}{\sqrt{5}} dt$ or calculating the area of the triangle.

$$d_2 = \frac{5\sqrt{5}}{2}$$

A1

From $d = d_1 + d_2$ we obtain $d = \frac{1}{2} \left(9 \sin^{-1}\left(\frac{2}{3}\right) + 7\sqrt{5} \right)$ (m)

A1

- d. Distance travelled before coming to rest is $d = \frac{1}{2} \left(9 \sin^{-1} \left(\frac{2}{3} \right) + 7\sqrt{5} \right)$ (m).

$$\int_7^T \left| \frac{7-t}{\sqrt{5}} \right| dt = \frac{1}{2} \left(9 \sin^{-1} \left(\frac{2}{3} \right) + 7\sqrt{5} \right) \quad \text{A1}$$

Attempting to solve $\int_7^T \left| \frac{7-t}{\sqrt{5}} \right| dt = \frac{1}{2} \left(9 \sin^{-1} \left(\frac{2}{3} \right) + 7\sqrt{5} \right)$ for T M1

$T = 14.0$ (s) A1