

The Mathematical Association of Victoria

Trial Exam 2013

SPECIALIST MATHEMATICS

Written Examination 2

STUDENT NAME _____

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of Book

<i>Section</i>	<i>Number of questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
1	22	22	22
2	5	5	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 23 pages with a detachable sheet of miscellaneous formulas at the back.
- Answer sheet for multiple-choice questions.

Instructions

- Detach the formula sheet from the back of this book during reading time.
- Write your name in the space provided above on this page.
- All written responses must be in English.

At the end of the examination

- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1**Instructions for Section 1**

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Take the **acceleration due to gravity** to have magnitude $g \text{ m/s}^2$, where $g = 9.8$.

Question 1

Let g be a function with domain R , such that $g(x) = \frac{1}{2x^2 - px + 2}$, where p is a real constant. The set of possible values of p is

- A. $p \in \{-4, 4\}$
- B. $p \in R \setminus \{-4, 4\}$
- C. $p \in (-4, 4)$
- D. $p \in R \setminus [-4, 4]$
- E. $p \in [-4, 4]$

Question 2

The path of a particle has parametric equations $x = \cos(4t)$ and $y = 6\sin^2(2t)$, where $t \geq 0$. The particle moves on part of a curve defined by a

- A. parabola
- B. circle
- C. ellipse
- D. hyperbola
- E. straight line

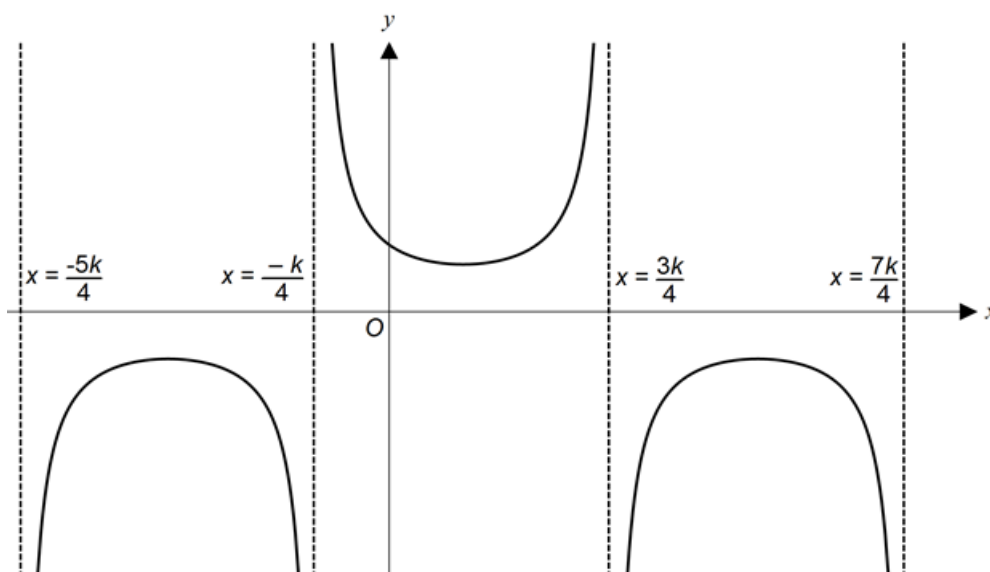
Question 3

Let g be a function with rule $g(x) = 2\sin^{-1}\left(\frac{x+1}{3}\right) - \frac{\pi}{4}$. The maximal domain and range of g are, respectively,

- A. $\left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right]$ and $[-2, 2]$
- B. $\left[-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ and $[-4, 2]$
- C. $[-4, 2]$ and $\left[-\frac{5\pi}{4}, \frac{3\pi}{4}\right]$
- D. $[-4, 2]$ and $\left[-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$
- E. $[-2, 2]$ and $\left[-\frac{5\pi}{4}, \frac{5\pi}{4}\right]$

Question 4

The graph of $y = f(x)$ is shown.



If k is a real number, the rule of f could be

- A. $f(x) = \sec\left(k\left(x - \frac{\pi}{4}\right)\right)$
- B. $f(x) = \operatorname{cosec}\left(k\left(x - \frac{\pi}{4}\right)\right)$
- C. $f(x) = \sec\left(\frac{2\pi}{k}\left(x - \frac{k}{2}\right)\right)$
- D. $f(x) = \operatorname{cosec}\left(\frac{\pi}{k}\left(x - \frac{k}{4}\right)\right)$
- E. $f(x) = \sec\left(\frac{\pi}{k}\left(x - \frac{k}{4}\right)\right)$

Question 5

Given that $(-3, 2)$ is the centre of the ellipse with equation $4x^2 + mx + 9y^2 - ny + 36 = 0$, the values of m and n are

- A. $m = 9$ and $n = 4$
- B. $m = 6$ and $n = 4$
- C. $m = 6$ and $n = -4$
- D. $m = 24$ and $n = 36$
- E. $m = -24$ and $n = 36$

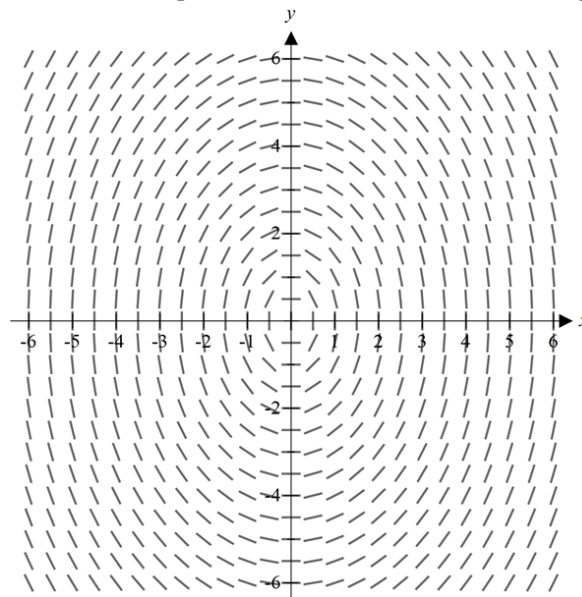
Question 6

Euler's method is used to solve the differential equation $\frac{dy}{dx} = 16^x$, with initial values $x = 1$ and $y = 0$, and a step size of $\frac{1}{4}$. The value of y when $x = \frac{3}{2}$ is

- A. 4
- B. 12
- C. 28
- D. $\frac{4}{\log_e(2)}$
- E. $\frac{12}{\log_e(2)}$

Question 7

The direction field shown below is that of a particular first order differential equation.

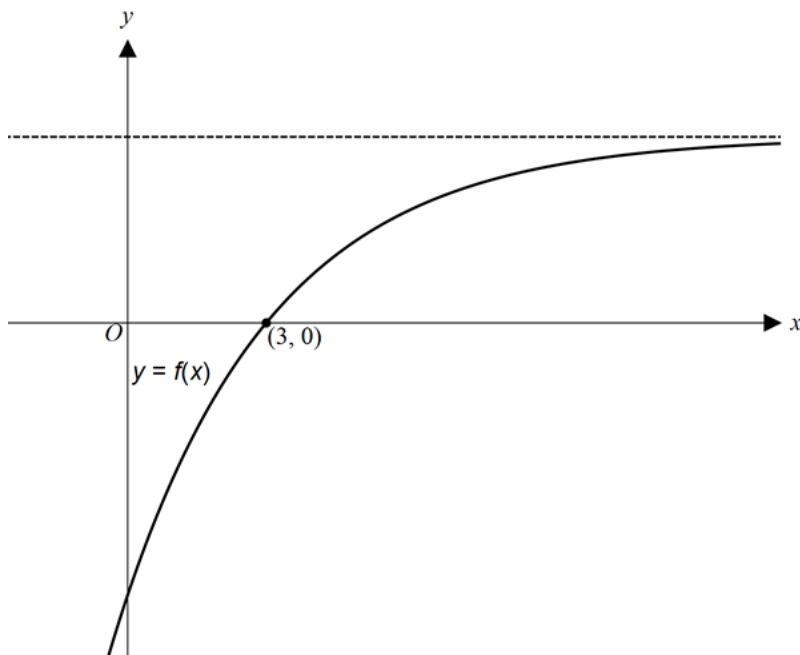


If $m > 1$, the differential equation could be

- A. $\frac{x^2}{m^2} + \frac{dy}{dx} = 1$
- B. $\frac{x^2}{m^2} - \frac{dy}{dx} = 1$
- C. $\frac{dy}{dx} = \frac{mx}{y}$
- D. $\frac{dy}{dx} = -\frac{mx}{y}$
- E. $\frac{dy}{dx} = -\frac{my}{x}$

Question 8

The graph of a smooth continuous function f is shown.



If $(3, 0)$ is a point on the graph, which one of the following is true?

- A. $f''(3) = f(3)$
- B. $f''(3) > f(3)$
- C. $f''(3) > f'(3)$
- D. $f''(3) = f'(3)$
- E. $f''(3) < f'(3)$

Question 9

If $z \in \mathbb{C}$, which one of the following relations does **not** have a graph that is a straight line?

- A. $\bar{z} - z = 0$
- B. $|z + i| - |z| = 0$
- C. $1 - |z + i| = 0$
- D. $i^2 z + \bar{z} = 0$
- E. $\operatorname{Re}(z) + 2\operatorname{Im}(z) = 0$

Question 10

Let $u = 1 - i$, $u \in \mathbb{C}$. If $u^n = ai$, where n is a positive integer and a is a non-zero real constant, then u^{n-1} is equal to

- A. $u = a(1+i)$
- B. $u = a(1-i)$
- C. $u = \frac{a}{2}(1+i)$
- D. $u = \frac{a}{2}(1-i)$
- E. $u = \frac{-a}{2}(1-i)$

Question 11

Let $P(z)$ be a polynomial with **integer** coefficients. If the linear factors of the polynomial include $(z+i)$, $(z-1)$ and $(z+1-\sqrt{2})$, the degree of $P(z)$ must be **at least**

- A. 2
- B. 3
- C. 4
- D. 5
- E. 6

Question 12

Given that $2\frac{dx}{dt} - x^2 - 4 = 0$, and that $x = 2$ at $t = 0$, then

- A. $x = 2 \tan\left(t + \frac{\pi}{4}\right)$
- B. $x = 2 \log_e\left(\frac{t^2 + 1}{2}\right)$
- C. $x = 2 \tan\left(t - \frac{\pi}{4}\right)$
- D. $x = \frac{1}{6}x^3 + 2x - 1$
- E. $x = \frac{1}{2}(t^2 + t + 1)$

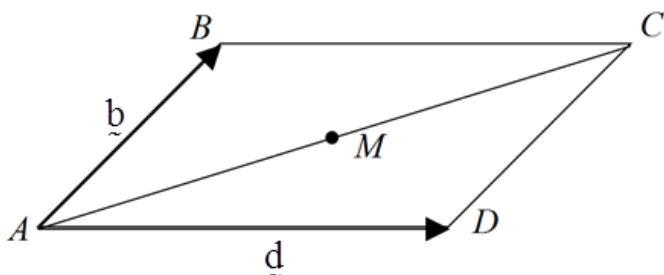
Question 13

If vector $\underline{p} = 2a\underline{i} + 3\underline{j} - 5\underline{k}$ is perpendicular to vector $\underline{q} = a\underline{i} + 4\underline{j} - 2a\underline{k}$, where a is a real constant, then the value of a could be

- A. $a = 2$ or $a = 3$
- B. $a = 4$ or $a = 3$
- C. $a = -2$ or $a = -3$
- D. $a = \frac{2}{3}$ or $a = 4$
- E. $a = -4$ or $a = \frac{2}{3}$

Question 14

Consider the parallelogram $ABCD$ below. M is the midpoint of AC .



If $\vec{AB} = \underline{\tilde{b}}$ and $\vec{AD} = \underline{\tilde{d}}$, then $\vec{BM}$ may be expressed as

- A. $\frac{1}{2}(\underline{\tilde{b}} + \underline{\tilde{d}})$
- B. $\frac{1}{2}(\underline{\tilde{b}} - \underline{\tilde{d}})$
- C. $\frac{1}{2}(\underline{\tilde{d}} - \underline{\tilde{b}})$
- D. $\underline{\tilde{b}} + \frac{1}{2}(\underline{\tilde{d}} - \underline{\tilde{b}})$
- E. $\underline{\tilde{b}} - \frac{1}{2}(\underline{\tilde{d}} - \underline{\tilde{b}})$

Question 15

A golf ball is hit from an origin at ground level. At time t the velocity vector of the ball is given by

$\underline{v} = 12\underline{i} + (18 - 6t)\underline{j}$, $t \geq 0$, where $\underline{i}$ is a unit vector in the horizontal forward direction and $\underline{j}$ is a unit vector in the upward vertical direction.

The position vector of the golf ball at its maximum height is given by

- A. $12\underline{i} + 0\underline{j}$
- B. $36\underline{i} + 27\underline{j}$
- C. $12\underline{i} + 27\underline{j}$
- D. $36\underline{i} + 54\underline{j}$
- E. $12\underline{i} - 27\underline{j}$

Question 16

Let P and Q be points on the graph of $xy + y^2 - x^2 - 11 = 0$, such that the gradient of the tangent to the curve at P and at Q is $\frac{1}{8}$. The coordinates of P and Q could be

- A. $(2, 3)$ and $(-2, -3)$
- B. $(2, 3)$ and $(2, -5)$
- C. $(2, -5)$ and $(-2, 5)$
- D. $(-2, 5)$ and $(-2, -3)$
- E. $(2, 5)$ and $(-2, 3)$

Question 17

Using a suitable substitution $\int_0^{\frac{1}{2}} \left(\sqrt{\frac{\cos^{-1}(x)}{1-x^2}} \right) dx$ can be expressed completely in terms of u as

- A. $\int_0^{\frac{1}{2}} \left(\sqrt{\frac{u}{1-\cos^2(u)}} \right) du$
- B. $\int_0^{\frac{\pi}{6}} \left(\frac{\sqrt{u}}{\sin^2(u)} \right) du$
- C. $\int_0^{\frac{1}{2}} (\sqrt{u}) du$
- D. $-\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} (\sqrt{u}) du$
- E. $\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} (\sqrt{u}) du$

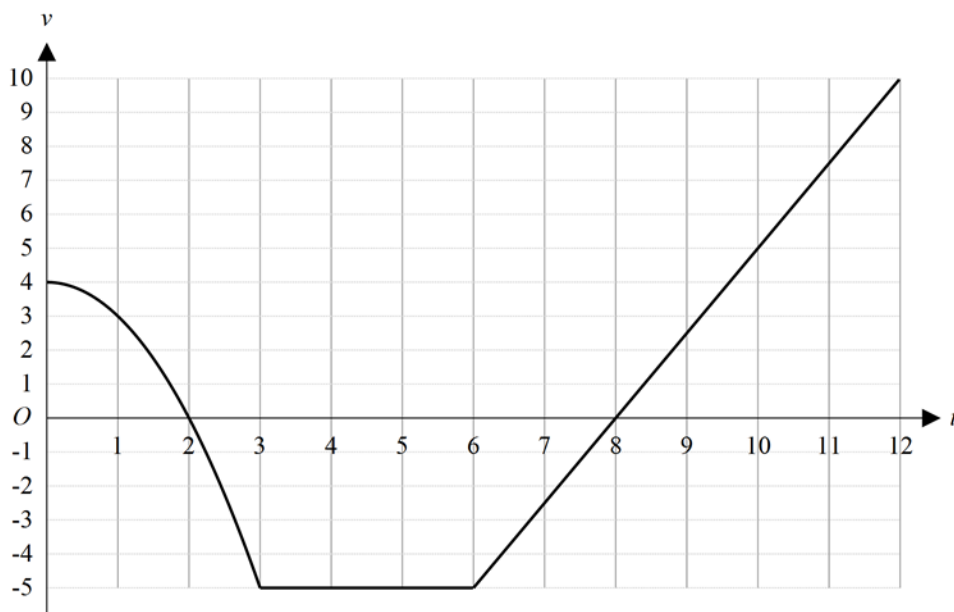
Question 18

A particle moves along the positive x -axis such that $a = -\frac{4}{x^2}$, $x > 0$, where $a \text{ ms}^{-2}$ is the acceleration of the particle when it is x metres from the origin. If the speed of the particle is 3 ms^{-1} at $x = 1$, the particle will have a speed of 2 ms^{-1} at x equal to

- A. 3 m
- B. 4 m
- C. $\frac{8}{3}$ m
- D. $\frac{3}{2}$ m
- E. $\frac{8}{13}$ m

Question 19

The velocity–time graph below shows the motion of a particle travelling in a straight line, where $v \text{ ms}^{-1}$ is its velocity at time t seconds.

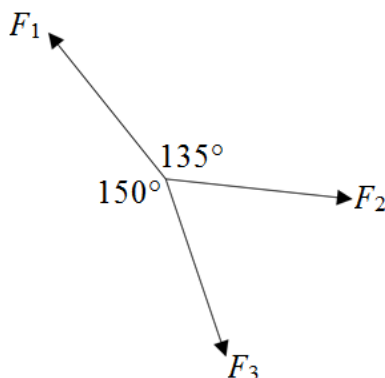


If $v = 4 - t^2$ for $0 \leq t < 3$, after 12 seconds the distance of the particle from its starting point is

- A. 3 m
- B. 10 m
- C. $\frac{1}{2}$ m
- D. $\frac{17}{6}$ m
- E. 43 m

Question 20

Three co-planar forces act on a particle which is in equilibrium, as shown.



The magnitude of F_1 is 100 newtons. The magnitudes of F_2 and F_3 , correct to the nearest integer, are

- A. $F_2 = 50$ newtons and $F_3 = 50$ newtons
- B. $F_2 = 52$ newtons and $F_3 = 73$ newtons
- C. $F_2 = 58$ newtons and $F_3 = 71$ newtons
- D. $F_2 = 87$ newtons and $F_3 = 71$ newtons
- E. $F_2 = 50$ newtons and $F_3 = 87$ newtons

Question 21

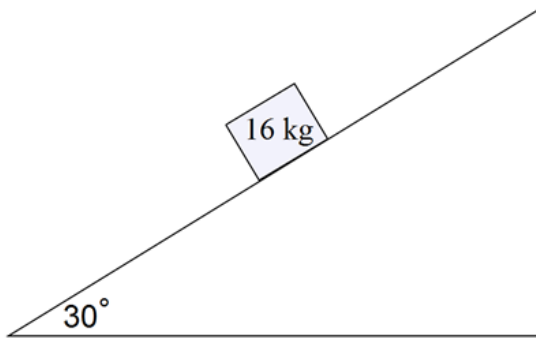
A particle of mass $\frac{4}{5}$ kg is moving in a straight line such that its speed, $v \text{ ms}^{-1}$ at time t seconds, is given by $v = 3t^2 - 2t + 5$, $t \geq 0$.

When the particle has travelled a distance of 125 m, its momentum, in kg ms^{-1} , is

- A. 5
- B. 56
- C. $\frac{16}{3}$
- D. 40
- E. 100

Question 22

The coefficient of friction between a block of mass 16 kg and a plane inclined at 30° to the horizontal is $\frac{1}{4}$.



The least force parallel to the inclined plane required to pull the block up the plane is given by

- A. $8g$
- B. $8\sqrt{3}g$
- C. $(8 + \sqrt{3})g$
- D. $(8 + 2\sqrt{3})g$
- E. $(8 + 8\sqrt{3})g$

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

Unless otherwise specified an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

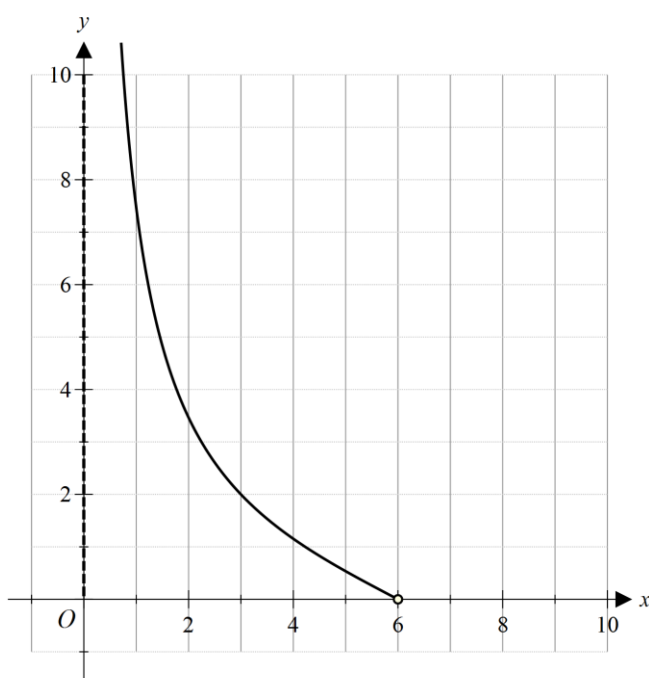
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude $g \text{ m/s}^2$, where $g = 9.8$.

Question 1 (9 marks)

Consider the function $f : \{x : 0 < x < 6\} \rightarrow \mathbb{R}$, $f(x) = 2 \cot\left(\frac{\pi}{12}x\right)$.

A part of the graph of f is shown below.



- a. On the set of axes above, sketch the graph of f^{-1} , the inverse function of f . Label the endpoint of f^{-1} with its coordinates and any asymptote with its equation.

2 marks

- b. The rule of f^{-1} can be expressed as $f^{-1}(x) = k \tan^{-1}\left(\frac{2}{x}\right)$, where k is a positive real constant.

Find the exact value of k .

1 mark

- c. Show, using calculus, that the derivative of $\log_e(x^2 + 4) + x \tan^{-1}\left(\frac{2}{x}\right)$ is $\tan^{-1}\left(\frac{2}{x}\right)$.

3 marks

- d. Hence, show using calculus that the area of the region bounded by the graph of f^{-1} and the x -axis for $0 < x \leq 2$ is $\frac{12\log_e(2) + 6\pi}{\pi}$.

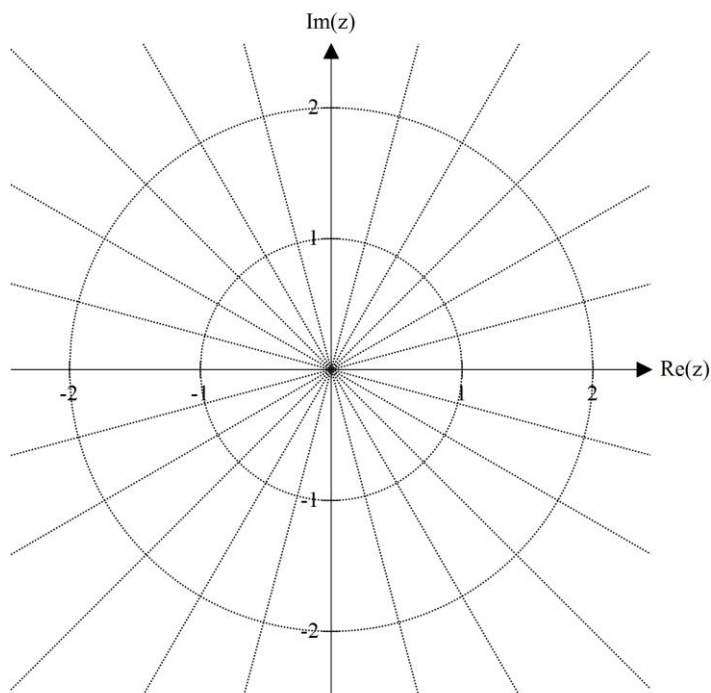
3 marks

Question 2 (10 marks)

In the complex plane, a set of points $S = \{z : |z| = \sqrt{2}\}$.

a. Sketch S on the argand diagram below.

2 marks



Let T be a subset of the complex plane with equation $|z + i| = \left| z + \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \right|$.

b. Show that the cartesian equation of T is given by $y = (2 - \sqrt{3})x$.

2 marks

c. If $\text{Arg}(z) = \phi$ is the equation of the part of T in the first quadrant, find the exact value of ϕ in radians.

1 mark

d. Sketch T on the argand diagram above. 1 mark

e. Let $w^4 = 2 + 2\sqrt{3}i$.

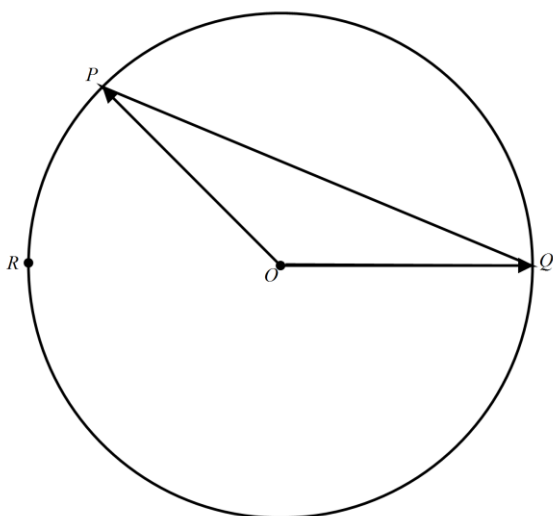
i. Express w^4 in polar form. 1 mark

ii. Find, in polar form, $\{w : w^4 = 2 + 2\sqrt{3}i\}$; that is, find all fourth roots of $2 + 2\sqrt{3}i$, in polar form. 2 marks

iii. On the argand diagram above, plot $\{w : w^4 = 2 + 2\sqrt{3}i\}$. Label the plotted points in the first, second, third and fourth quadrants as w_1, w_2, w_3 and w_4 , respectively. 1 mark

Question 3 (9 marks)

The diagram below shows a triangle OPQ which has O is at the centre of a circle and P and Q on the circumference of the circle. Point R , also on the circumference of the circle, is collinear with points O and Q .

**a.**

- i. If $\vec{OQ} = h\vec{i}$, where $h > 0$, and $\vec{OP} = -4\vec{i} - \sqrt{5}\vec{j} + 2\vec{k}$, show that $h = 5$.

1 mark

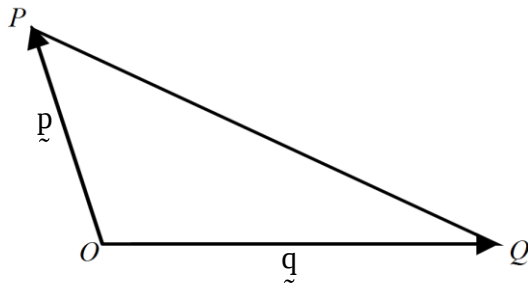
- ii. Find $\vec{PR}$.

1 mark

- iii. Use a vector method to show that angle QPR is a right angle.

2 marks

A different triangle OPQ , where O is the origin, is shown in the diagram below, with vectors $\vec{OP} = \underline{p}$ and $\vec{OQ} = \underline{q}$.



The points M and N are the midpoints of the line segments OP and OQ , respectively.

b. Let U be the point of intersection of the line segments PN and QM .

i. Express $\vec{PN}$ in terms of $\underline{p}$ and $\underline{q}$.

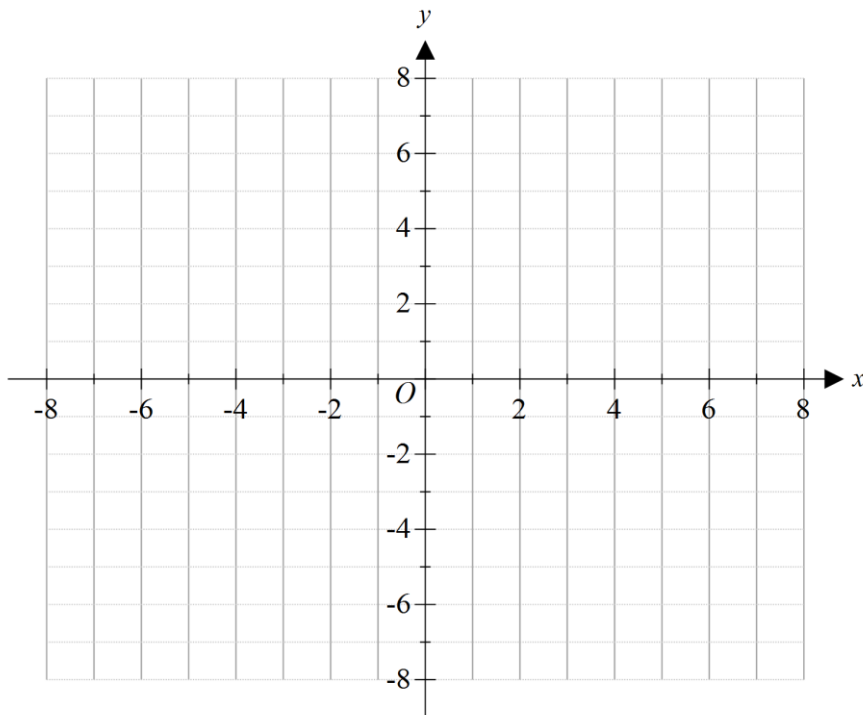
1 mark

ii. Express $\vec{QM}$ in terms of $\underline{p}$ and $\underline{q}$.

1 mark

iii. Given that $\vec{PU} = a\vec{PN}$ and $\vec{QU} = b\vec{QM}$, where a and b are scalars, use a vector method to find the values of a and b .

3 marks



A large storage tank at a paint factory is modelled by the volume of revolution formed when the region bounded by the curve, the coordinate axes and the line $y = 5$ is rotated about the y -axis, where the values on the axes are in metres.

The tank contains $V \text{ m}^3$ of a liquid solvent.

- d. Evaluate the exact maximum volume of solvent that the tank can hold, in m^3 .

2 marks

Solvent is pumped into the tank at a constant rate of $2 \text{ m}^3/\text{hour}$. It can be shown that the volume $V \text{ m}^3$ and depth $y \text{ m}$ of solvent in the tank are related by $\frac{dV}{dy} = \frac{4\pi}{3}(y^2 - 6y + 12)$

- e. Find the exact rate at which the depth of solvent is increasing at the instant when the volume of solvent in the tank is $24\pi \text{ m}^3$.

3 marks

A concentrated varnish is made in the tank by dissolving 10 tonne of a resin in the solvent. The volume of varnish solution in the tank is adjusted to exactly 100 m^3 . The varnish is stirred constantly while in the tank.

- f. To adjust the concentration of resin in the tank, a dilute solution containing 0.02 tonne of resin per m^3 of solution is pumped into the vat at a constant rate of $2 \text{ m}^3/\text{hour}$. Varnish is pumped out of the tank at the same rate.
- i. Let N tonne be the mass of resin dissolved in the tank t hours after the pumping started. Write down a differential equation relating N and t .

2 marks

- ii. Pumping was stopped when the varnish in the tank reached the desired concentration of 0.05 tonne of resin per m^3 of the varnish solution. When pumping was stopped, what was the value of t , correct to the nearest hour?

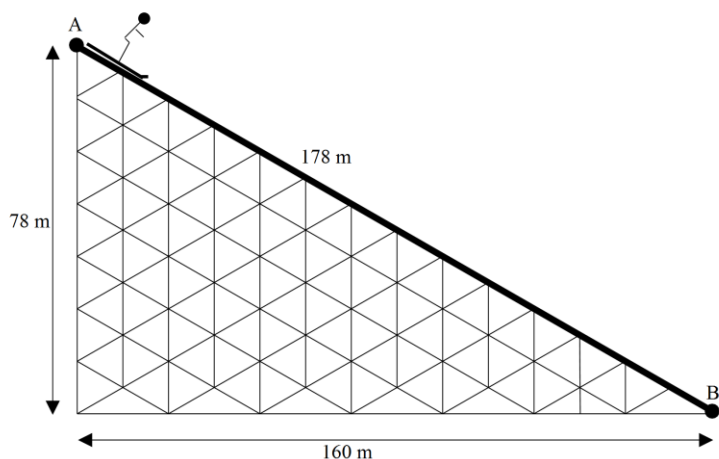
3 marks

Question 5 (14 marks)

At a training session for thrill-seekers, Xue slides down a ski ramp on a snowboard. A section of the ramp, from point A to point B, is shown in the diagram above. The distance AB is 178 metres.

The mass of Xue and her equipment is 75 kg, and she began sliding down ramp from rest at point A.

The coefficient of friction between the snowboard and the ramp is $\frac{1}{5}$. Assume that any other forces acting on Xue are negligible as she travels down the ramp.



- a. On the diagram above, clearly label the forces acting on Xue.

1 mark

- b. Find the magnitude of Xue's acceleration as she slides down the ramp. Express the answer in ms^{-2} , correct to two decimal places.

2 marks

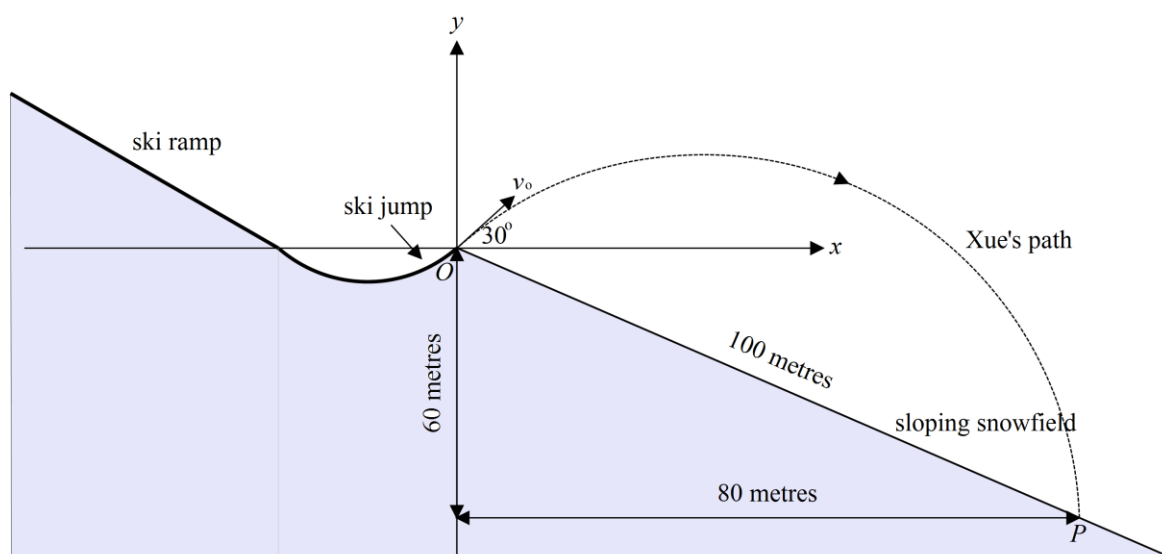
- c. As Xue passes point B, she is travelling at a speed of $k \text{ ms}^{-1}$. Determine the value of k correct to the nearest integer.

2 marks

- d. How long does it take Xue to reach point B ? Express the answer in seconds, correct to one decimal place.

1 mark

In another training session, Xue travels down a ski ramp on her snowboard and then travels up a short ski jump. Xue leaves the ski jump at O , the origin of a cartesian coordinate system, with a speed of $v_0 \text{ ms}^{-1}$ at an angle of 30° to the horizontal, as shown on the diagram below.



Xue lands at a point P on the snowfield of constant slope, 100 metres down-hill from O .

- e. The position vector of P is $80\mathbf{i} - 60\mathbf{j}$, where all measurements are in metres, and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the positive directions of the x and y axes, respectively. Assume that air resistance is negligible during Xue's jump from O to P .

- i. The position vector of Xue as she jumps from O to P is given by $\alpha\mathbf{i} + \beta\mathbf{j}$. Write expressions for α and β in terms v_0 , t seconds after leaving O .

3 marks

- ii. Find the value of v_0 , and the time taken for Xue to jump from point O to point P , in seconds. Express the answers correct to two decimal places.

2 marks

After landing at P , Xue continues to slide down the snowfield of constant slope. When she reaches a point Q her speed is 22ms^{-1} . At Q she uses the snowboard to brake with a retarding force of $b(t)$ newtons parallel to the slope, where $b(t) = 150t + 75, t \geq 0$, and t is the time in seconds from when she passed point Q . Assume that $b(t)$ is the only retarding force acting on Xue.

- f. How long after passing point Q does Xue come to rest? Express the answer in seconds, correct to one decimal place.

3 marks

END OF QUESTION AND ANSWER BOOKLET