

The Mathematical Association of Victoria
SOLUTIONS: Trial Exam 2013
SPECIALIST MATHEMATICS
Written Examination 2

SECTION 1: Multiple Choice

ANSWERS

1. C 2. E 3. C 4. E 5. D 6. B
 7. D 8. E 9. C 10. E 11. D 12. A
 13. C 14. C 15. B 16. A 17. D 18. C
 19. A 20. B 21. B 22. D

Question 1 Answer: C

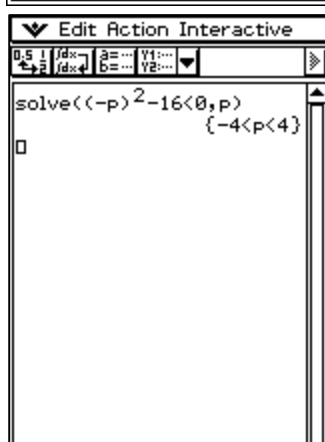
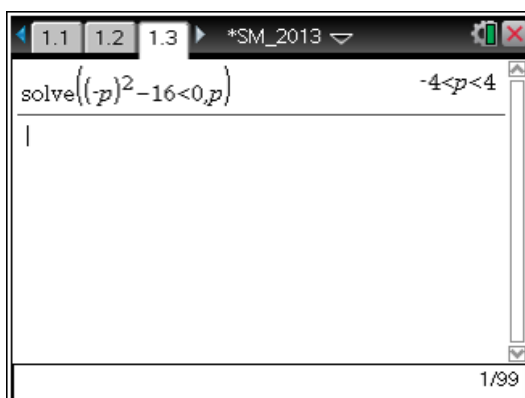
If the domain of g is R , the graph of g does **not** have vertical asymptotes. Therefore the graph of $y = 2x^2 - px + 2$ does **not** have x -axis intercepts, hence its discriminant ($\Delta = b^2 - 4ac$) is negative.

$$\Delta = (-p)^2 - 4 \times 2 \times 2 < 0$$

$$p^2 < 16$$

$$-4 < p < 4$$

$$p \in (-4, 4)$$



Question 2 Answer: E

Let $u = 2t$

$$x = \cos(2u) = 1 - 2\sin^2(u)$$

$$y = 6\sin^2(u)$$

Therefore

$$y = 3(1 - x), \text{ or}$$

$3x + y - 3 = 0$, which is the equation of a straight line.

Question 3 Answer: C

The domain of $y = \sin^{-1}(x)$ is $[-1, 1]$

For $\frac{x+1}{3} = -1$, $x = -4$

and $\frac{x+1}{3} = 1$, $x = 2$

Maximal domain of g is $[-4, 2]$

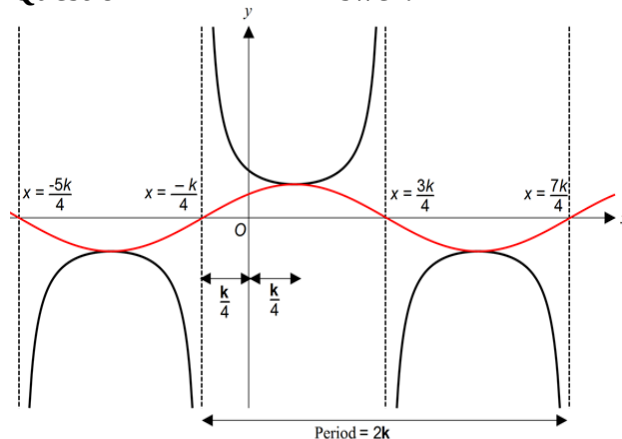
The range of $y = \sin^{-1}(\theta)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The range of $y = 2\sin^{-1}(\theta)$ is $[-\pi, \pi]$
 (multiply by a factor of 2)

The range of $y = 2\sin^{-1}(\theta) - \frac{\pi}{4}$ is

$$\left[-\frac{5\pi}{4}, \frac{3\pi}{4}\right] \text{ (subtract } \frac{\pi}{4})$$

Question 4 Answer: E



The graph could be the **reciprocal** of a cosine graph of the form $y = \cos(n(x-b))$, with:

- period = $2k$

$$\frac{2\pi}{n} = 2k$$

$$n = \frac{\pi}{k}$$

- Translated $\frac{k}{4}$ units to the right

$$b = \frac{k}{4}$$

$$f(x) = \frac{1}{\cos\left(\frac{\pi}{k}\left(x - \frac{k}{4}\right)\right)}$$

$$f(x) = \sec\left(\frac{\pi}{k}\left(x - \frac{k}{4}\right)\right)$$

Note: the graph could also be the reciprocal of

a sine graph: $f(x) = \frac{1}{\sin\left(\frac{\pi}{k}\left(x + \frac{k}{4}\right)\right)}$.

However, $f(x) = \operatorname{cosec}\left(\frac{\pi}{k}\left(x + \frac{k}{4}\right)\right)$ is **not** one of the options.

Question 5 **Answer: D**

If $(-3, 2)$ is the centre, the equation will be

$$\frac{(x+3)^2}{9} + \frac{(y-2)^2}{4} = 1$$

$$4(x+3)^2 + 9(y-2)^2 = 36$$

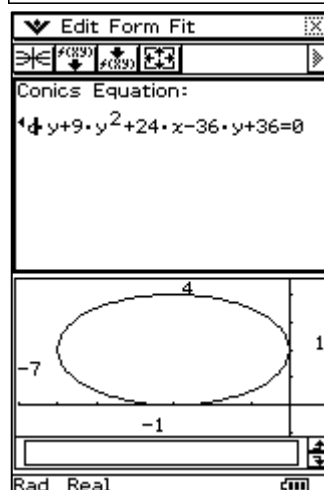
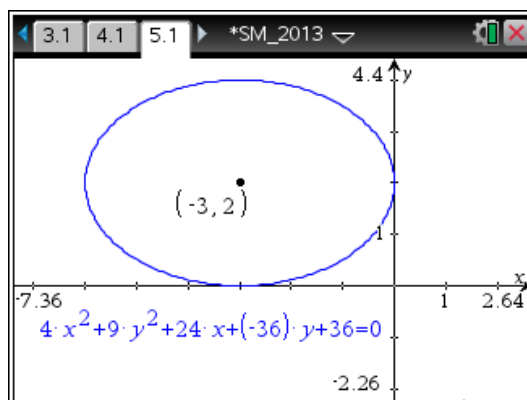
Expanding the brackets gives

$$4(x^2 + 6x + 9) + 9(y^2 - 4y + 4) = 36$$

$$4x^2 + 24x + 36 + 9y^2 - 36y + 36 = 36$$

$$4x^2 + 24x + 9y^2 - 36y + 36 = 0$$

Comparing with $4x^2 + mx + 9y^2 - ny + 36 = 0$
 $m = 24$ and $n = 36$



Question 6 **Answer: B**

Method 1: 'By hand'

Euler's method: $y_{n+1} = y_n + h f(x_n)$, where

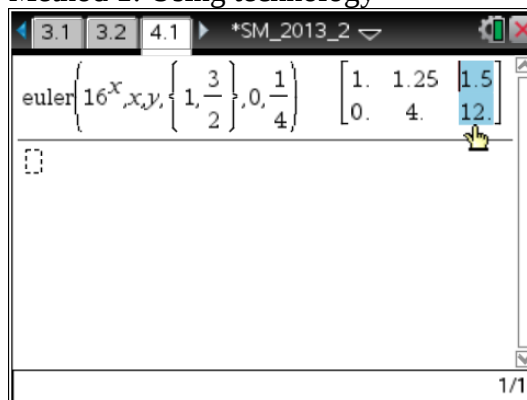
$$f(x_n) = 16^{x_n}, \quad x_0 = 1, \quad y_0 = 0 \quad \text{and} \quad h = \frac{1}{4}.$$

$$x_2 = x_0 + 2h = 1 + 2 \times \frac{1}{4} = \frac{3}{2}$$

$$y_1 = 0 + \frac{1}{4} \times 16^1 = 4 \quad \text{and} \quad x_1 = x_0 + h = 1 + \frac{1}{4} = \frac{5}{4}$$

$$y_2 = 4 + \frac{1}{4} \times 16^{\frac{5}{4}} = 12 \quad \text{and} \quad x_2 = \frac{5}{4} + \frac{1}{4} = \frac{3}{2}$$

Method 1: Using technology



	A	B	C
1	1	0	
2	1.25	4	
3	1.5	12	
4	1.75	28	
5	2	60	
6	2.25	124	
7	2.5	252	
8	2.75	508	
9	3	1020	
10	3.25	2044	
11			
12			
13			
14			
15			

=B2+16^A2/4

B3 12

Question 7 **Answer: D**

The shape of the graph could be of the form

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a, b are positive real constants.

By implicit differentiation and making $\frac{dy}{dx}$ the subject:

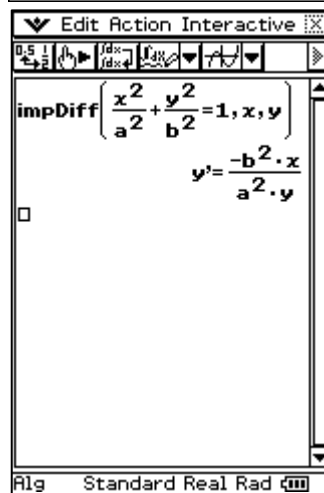
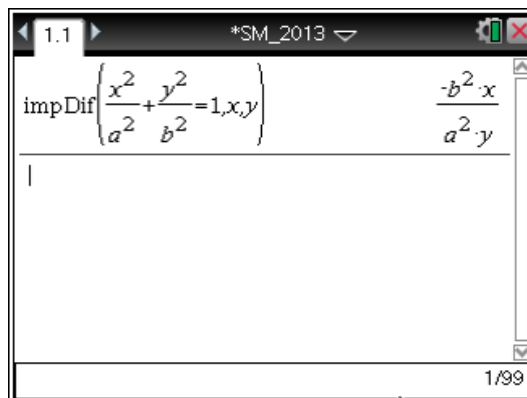
$$\frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{a^2} \times \frac{b^2}{2y}$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

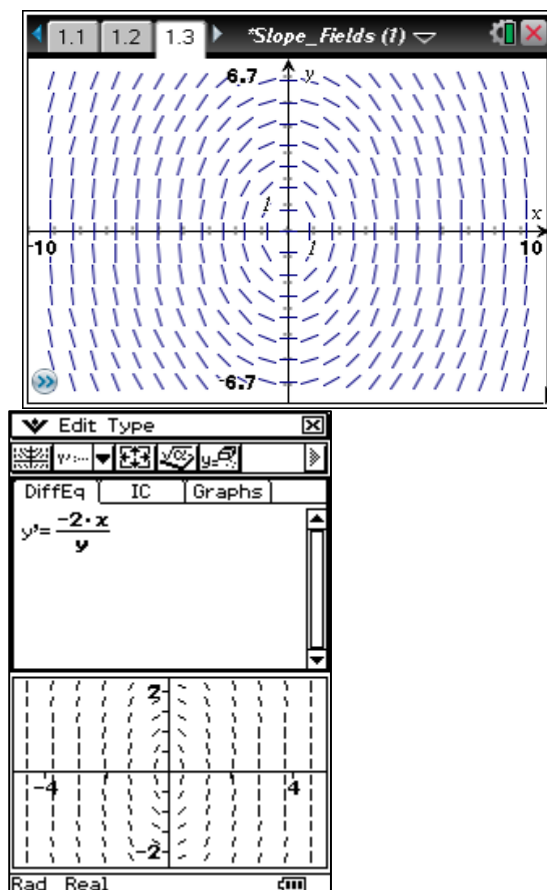
$$\frac{dy}{dx} = -\frac{mx}{y}, \text{ where } m = \frac{b^2}{a^2}.$$

Note that Option C cannot be correct because m must be a positive number.

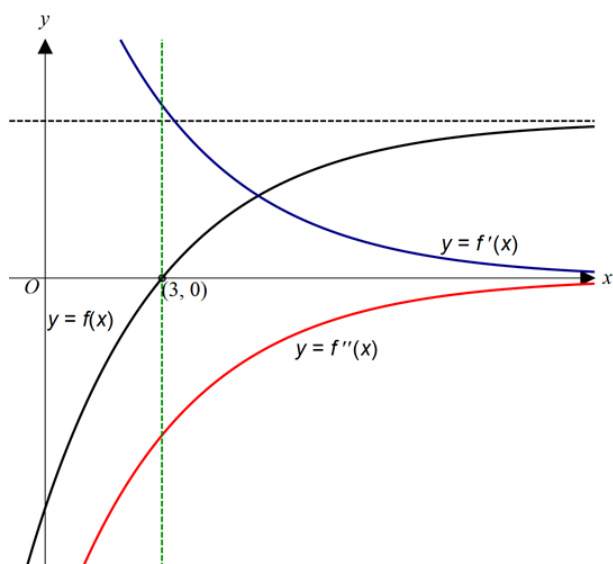


Test this by graphing the slope field for, say,

$$\frac{dy}{dx} = -\frac{2x}{y} \text{ (since } m > 1 \text{)}.$$

**Question 8** **Answer: E**

Consider the sketch graphs of $y = f'(x)$ and $y = f''(x)$.



The gradient of the graph of f is always positive, therefore $f'(3) > 0$.

The gradient of the graph of $y = f'(x)$ is always negative, therefore $f''(3) < 0$.

Since $f(3) = 0$, $f'(3) > f(3) > f''(3)$, and in particular, $f''(3) < f'(3)$.

Question 9 **Answer: C**

$1 - |z + i| = 0$ can be rewritten as $|z + i| = 1$,

which is a **circle** of radius 1, centred at $(0, -1)$.

The graphs for all other options are straight lines, with cartesian equations:

A. $y = 0$ B. $y = -\frac{1}{2}$

D. $y = 0$ E. $x + 2y = 0$ or $y = -\frac{x}{2}$

Question 10 **Answer: E**

$$u^{n-1} = u^n \times u^{-1} = \frac{ai}{1-i}$$

$$\begin{aligned} \frac{ai}{1-i} \times \frac{1+i}{1+i} &= \frac{ai(1+i)}{1-i^2} \\ &= \frac{a(-1+i)}{2} \\ &= \frac{-a}{2}(1-i) \end{aligned}$$

Question 11 **Answer: D**

If $(z + i)$ is a factor, then $(z - i)$ is also a factor (conjugate root theorem).

Since $P(z)$ has **integer** coefficients, if

$(z + 1 - \sqrt{2})$ is a linear factor, it would arise from a quadratic factor which is the product of $(z + 1 - \sqrt{2})$ and $(z + 1 + \sqrt{2})$ (otherwise, in the expansion of linear factors, irrational coefficients will occur as a consequence of multiplying by $\sqrt{2}$).

The *least factors* of $P(z)$ are therefore

$$(z + 1)(z + i)(z - i)(z + 1 - \sqrt{2})(z + 1 + \sqrt{2}).$$

Hence the degree of $P(z)$ is at least 5.

Question 12 **Answer: A**

Method 1 – ‘by hand’

$$2 \frac{dx}{dt} - x^2 - 4 = 0$$

$$\frac{dt}{dx} = \frac{2}{4 + x^2}$$

$$t = \int \left(\frac{2}{4 + x^2} \right) dx$$

$$t = \tan^{-1} \left(\frac{x}{2} \right) + c$$

$$x = 2 \text{ at } t = 0$$

$$0 = \tan^{-1} \left(\frac{2}{2} \right) + c$$

$$c = -\frac{\pi}{4}$$

$$t = \tan^{-1} \left(\frac{x}{2} \right) - \frac{\pi}{4}$$

$$x = 2 \tan \left(t + \frac{\pi}{4} \right)$$

Method 2 – using technology,

$$2 \frac{dx}{dt} - x^2 - 4 = 0, \quad x = 2 \text{ at } t = 0$$

$$x = 2 \tan \left(t + \frac{\pi}{4} \right)$$

TI-Nspire calculator screen showing the solution to the differential equation $2x' - x^2 - 4 = 0$ with initial condition $x(0) = 2$. The screen displays the differential equation, the initial condition, and the solution $x = 2 \tan \left(t + \frac{\pi}{4} \right)$.

TI-Nspire calculator screen showing the solution to the differential equation $2x' - x^2 - 4 = 0$ with initial condition $x(0) = 2$. The screen displays the differential equation, the initial condition, and the solution $x = 2 \tan \left(t + \frac{\pi}{4} \right)$.

Question 13 **Answer: C**

$$\underline{p} \cdot \underline{q} = 2a^2 + 12 + 10a = 0$$

$$a^2 + 5a + 6 = 0$$

$$a = -2 \text{ or } a = -3$$

TI-Nspire calculator screen showing the solution to the dot product equation $(2a, 3, -5) \cdot (a, 4, -2a) = 0$. The screen displays the vectors, the dot product equation, and the solutions $a = -3$ or $a = -2$.

TI-Nspire calculator screen showing the solution to the dot product equation $(2a, 3, -5) \cdot (a, 4, -2a) = 0$. The screen displays the vectors, the dot product equation, and the solutions $a = -3$ or $a = -2$.

Question 14 **Answer: C**

$$\vec{BD} = \vec{BA} + \vec{AD}$$

$$= -\underline{b} + \underline{d}$$

The diagonals of a parallelogram bisect each other, therefore,

$$\vec{BM} = \frac{1}{2} \vec{BD}$$

$$= \frac{1}{2} (\underline{d} - \underline{b})$$

Question 15 **Answer: B**

$$\underline{r}(t) = \int \underline{v}(t) dt = (12t + c_1) \underline{i} + (18t - 3t^2 + c_2) \underline{j} + c_3 \underline{k}$$

However, $c_1 = c_2 = c_3 = 0$ since

$$\underline{r}(0) = 0 \underline{i} + 0 \underline{j} + 0 \underline{k} \text{ (the ball is hit at the origin).}$$

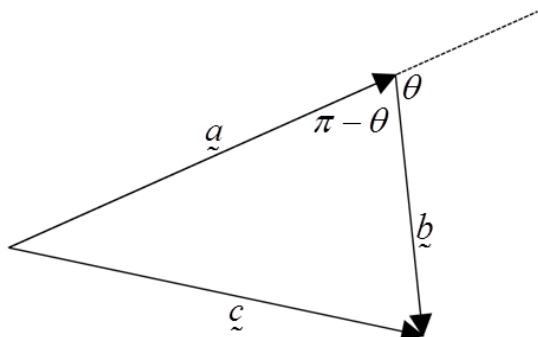
At its maximum height, the vertical component of the velocity is zero.

$$(18 - 6t)\hat{j} = 0\hat{j}, \text{ therefore } t = 3.$$

The position vector at the maximum height is

$$\mathbf{r}(3) = (12 \times 3)\hat{i} + (18 \times 3 - 3 \times (3^2))\hat{j}$$

$$\mathbf{r}(3) = 36\hat{i} + 27\hat{j}$$

**Question 16****Answer: A**

$$xy + y^2 - x^2 - 11 = 0$$

$$\frac{dy}{dx} = \frac{2x - y}{x + 2y}$$

$$\frac{2x - y}{x + 2y} = \frac{1}{8}$$

$$y = \frac{3x}{2}$$

Substitute $y = \frac{3x}{2}$ into $xy + y^2 - x^2 - 11 = 0$

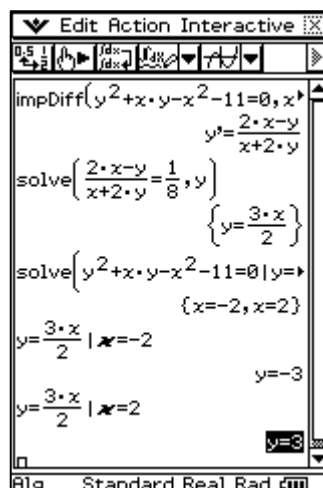
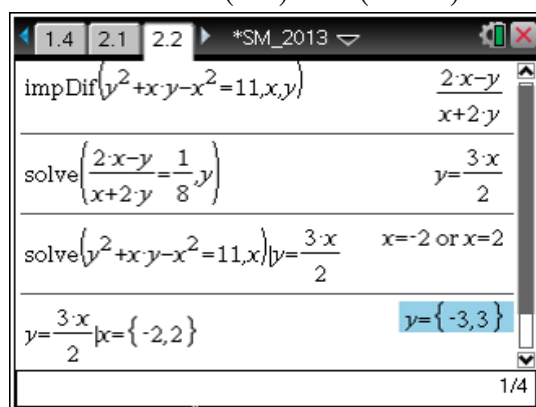
$$x^2 - 4 = 0$$

$$x = -2 \text{ or } x = 2$$

Substitute in $y = \frac{3x}{2}$

When $x = -2$ and $y = -3$ or $x = 2$ and $y = 3$

Coordinates are $(2, 3)$ and $(-2, -3)$



Alternatively, substitute the coordinates given

in each option into $\frac{2x - y}{x + 2y}$. Option A

coordinates give answers of $\frac{1}{8}$.

Question 17**Answer: D**

Let $u = \cos^{-1}(x)$.

$$\frac{du}{dx} = \frac{-1}{\sqrt{1 - x^2}}$$

When $x = 0$, $u = \cos^{-1}(0) = \frac{\pi}{2}$

When $x = \frac{1}{2}$, $u = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

$$\int_{x=0}^{\frac{1}{2}} \left(\frac{\sqrt{\cos^{-1}(x)}}{\sqrt{1 - x^2}} \right) dx = \int_{u=\frac{\pi}{2}}^{\frac{\pi}{3}} \left(\sqrt{u} \times -\frac{du}{dx} \right) dx$$

$$= -\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} (\sqrt{u}) du$$

Question 18**Answer: C****Method 1**

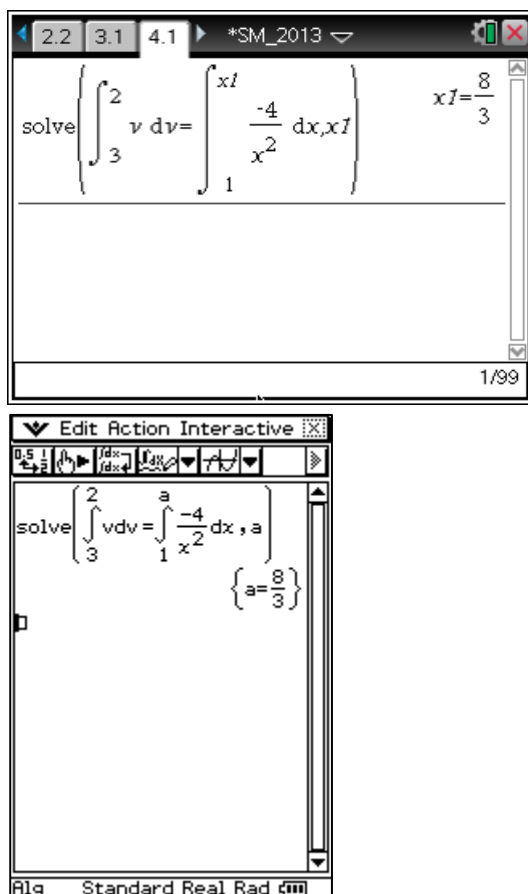
$$a = v \frac{d(v)}{dx} = -\frac{4}{x^2}$$

$$v = 3 \text{ at } x = 1$$

$$\int_3^2 v dv = -4 \int_1^{x_1} \frac{dx}{x^2}$$

Solve for x_1

$$x_1 = \frac{8}{3}, \text{ therefore } x = \frac{8}{3} \text{ m}$$

**Method 2**

$$a = \frac{d}{dx} \left(\frac{1}{2} v^2 \right) = -\frac{4}{x^2}$$

$$v^2 = -8 \int \frac{dx}{x^2}$$

$$v^2 = \frac{8}{x} + c$$

$$v = 3 \text{ at } x = 1, c = 1$$

$$v^2 = \frac{8}{x} + 1$$

$$\text{When } v = 2$$

$$x = \frac{8}{3} \text{ m}$$

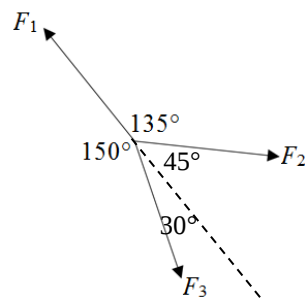
Question 19 **Answer: A**

The displacement gives the distance from the starting point.

The displacement can be found by calculating the signed area bounded by the graph and the horizontal axis.

$$\int_0^3 (4 - t^2) dt = \frac{5 \times (3 + 5)}{2} + \frac{4 \times 10}{2} = 3 - 20 + 20$$

$$\text{Distance} = 3 \text{ m}$$

Question 20**Answer: B**

Method 1: Lami's theorem

$$\frac{F_2}{\sin(150^\circ)} = \frac{100}{\sin(75^\circ)}$$

$$F_2 \approx 52 \text{ newtons}$$

$$\frac{F_3}{\sin(135^\circ)} = \frac{100}{\sin(75^\circ)}$$

$$F_3 \approx 73 \text{ newtons}$$

Method 2: resolving vectors

If the particle is in equilibrium,

Resolving forces perpendicular to F_1 :

$$F_2 \sin(45^\circ) = F_3 \sin(30^\circ)$$

$$F_2 \times \frac{\sqrt{2}}{2} = F_3 \times \frac{1}{2}$$

$$F_3 = \sqrt{2} F_2 \quad \dots \text{equation(1)}$$

Resolving forces parallel to F_1 :

$$100 = F_2 \cos(45^\circ) + F_3 \cos(30^\circ)$$

$$100 = F_2 \times \frac{\sqrt{2}}{2} + F_3 \times \frac{\sqrt{3}}{2}$$

Substitute equation(1)

$$100 = F_2 \times \frac{\sqrt{2}}{2} + F_2 \times \frac{\sqrt{6}}{2}$$

$$F_2 = \frac{200}{\sqrt{2} + \sqrt{6}} \approx 52 \text{ newtons}$$

$$F_3 = \sqrt{2} F_2 = \sqrt{2} \times \frac{200}{\sqrt{2} + \sqrt{6}} \approx 73 \text{ newtons}$$

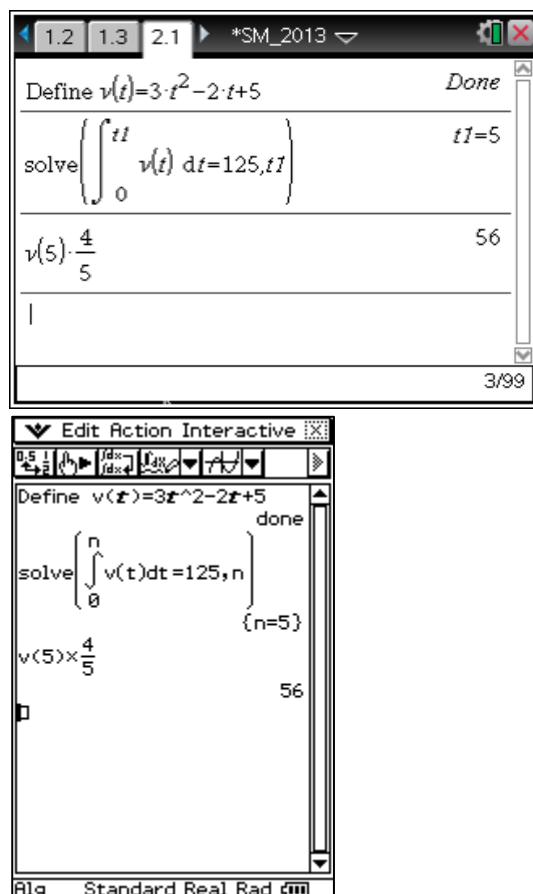
Question 21**Answer: B**

$$\text{Solve for } t_1, \int_0^{t_1} v(t) dt = 125$$

$$t_1 = 5 \text{ s}$$

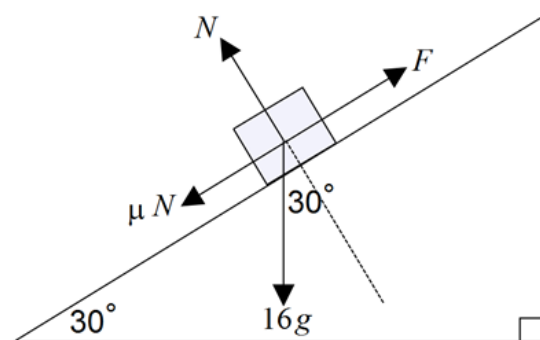
$$p = m \times v$$

$$p = \frac{4}{5} \times v(5) = 56 \text{ kg ms}^{-1}$$



Question 22

Answer: D



Resolving forces perpendicular to the plane

$$N - 16g \cos(30^\circ) = 0$$

$$N = 8\sqrt{3}g$$

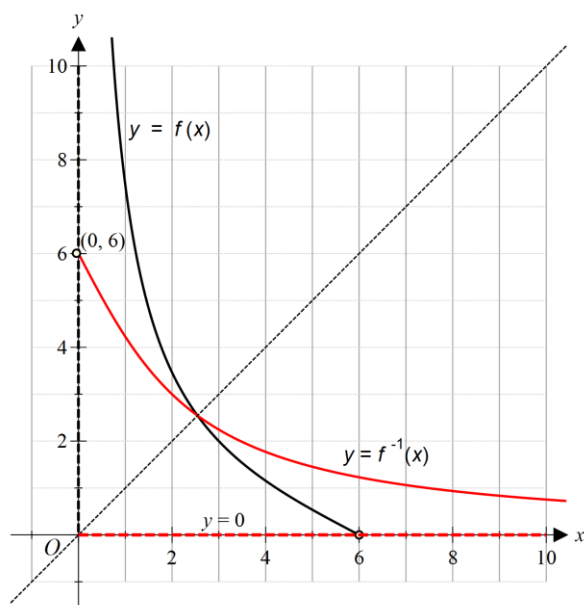
Let F be the minimum force required to pull the block up the plane at constant speed (zero acceleration).

Resolving forces parallel to the plane

$$F - \mu N - 16g \sin(30^\circ) = 0$$

$$F - \frac{1}{4} \times 8\sqrt{3}g - 8g = 0$$

$$F = (8 + 2\sqrt{3})g$$

END OF SECTION 1 SOLUTION**SECTION 2: Extended Response SOLUTIONS****Question 1****1a.**

Correct shape 1A, endpoint and asymptote correctly labelled 1A

1b. As $x \rightarrow 0$, $\tan^{-1}\left(\frac{2}{x}\right) \rightarrow \frac{\pi}{2}$, and $k \tan^{-1}\left(\frac{2}{x}\right) \rightarrow 6$. Therefore, $k = 6 \times \frac{2}{\pi} = \frac{12}{\pi}$

1A

1c.

$$\frac{d}{dx}(\log_e(x^2 + 4)) = \frac{2x}{x^2 + 4} \quad 1A$$

Using the product rule,

$$\frac{d}{dx}\left(x \tan^{-1}\left(\frac{2}{x}\right)\right) = x \times \frac{-2}{x^2 + 4} + \tan^{-1}\left(\frac{2}{x}\right) \quad 1M$$

$$\frac{d}{dx}(\log_e(x^2 + 4)) = \frac{2x}{x^2 + 4}$$

Therefore,

$$\begin{aligned} \frac{d}{dx}\left(\log_e(x^2 + 4) + x \tan^{-1}\left(\frac{2}{x}\right)\right) &= \frac{2x}{x^2 + 4} - \frac{2x}{x^2 + 4} + \tan^{-1}\left(\frac{2}{x}\right) \quad 1A \\ &= \tan^{-1}\left(\frac{2}{x}\right), \text{ as required} \end{aligned}$$

1d.

$$\text{Area} = \lim_{a \rightarrow 0} \frac{12}{\pi} \int_a^2 \left(\tan^{-1}\left(\frac{2}{x}\right) \right) dx$$

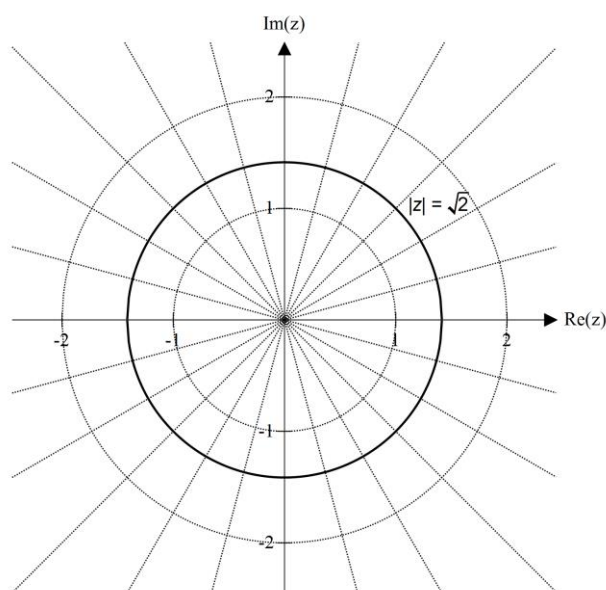
$$= \lim_{a \rightarrow 0} \frac{12}{\pi} \left[\log_e(x^2 + 4) + \tan^{-1}\left(\frac{2}{x}\right) \right]_a^2 \quad 1M$$

$$= \frac{12}{\pi} \left[(\log_e(8) + 2 \tan^{-1}(1)) - \left(\log_e(4) + 0 \times \frac{\pi}{2} \right) \right] \quad 1A$$

$$= \frac{12}{\pi} \left[3 \log_e(2) + 2 \times \frac{\pi}{4} - 2 \log_e(2) \right]$$

$$= \frac{12}{\pi} \times \left(\log_e(2) + \frac{\pi}{2} \right) \quad 1A$$

$$= \frac{12 \log_e(2) + 6\pi}{\pi}, \text{ as required}$$

Question 2**2a.**

Correct centre and shape 1A, correct radius 1A

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2b.

$$|z+i| = \left| z + \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \right|$$

$$\sqrt{x^2 + (y+1)^2} = \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}\right)^2} \quad 1M$$

$$x^2 + y^2 + 2y + 1 = x^2 + x + \frac{1}{4} + y^2 - \sqrt{3}y + \frac{3}{4}$$

$$y(2 + \sqrt{3}) = x$$

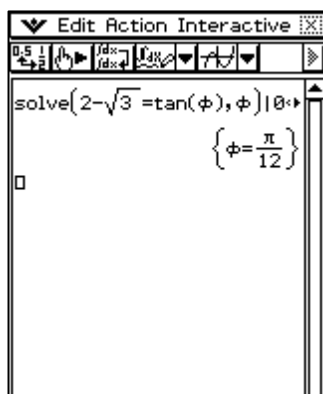
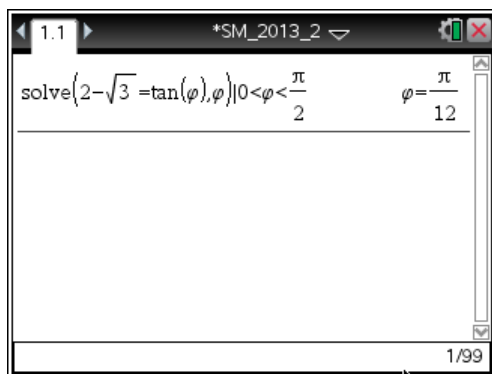
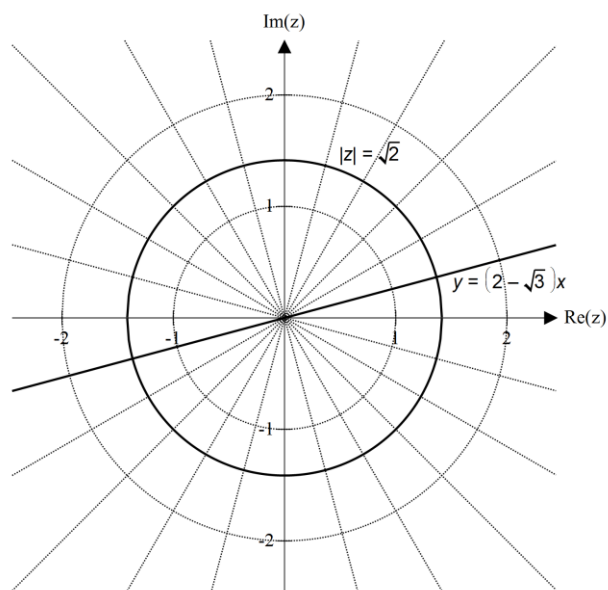
$$y = \frac{2 - \sqrt{3}}{2 + \sqrt{3}} \times \frac{x}{2 + \sqrt{3}} \quad 1A$$

$$y = (2 - \sqrt{3})x, \text{ as required}$$

2c.

$$m = \tan(\phi), \quad 0 < \phi < \frac{\pi}{2}$$

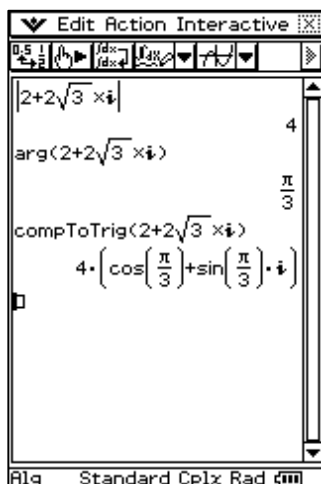
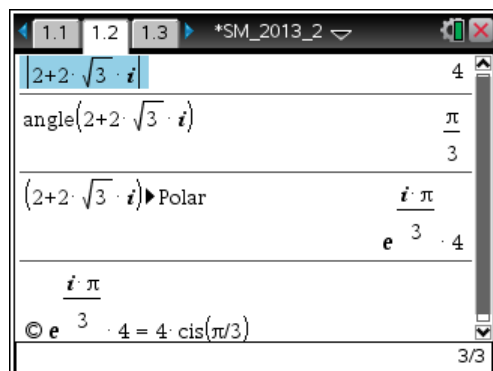
$$\phi = \tan^{-1}(2 - \sqrt{3}) = \frac{\pi}{12} \quad 1A$$

**2d.**

1A

2e.i.

$$2 + 2\sqrt{3}i = 4\text{cis}\left(\frac{\pi}{3}\right) \quad 1A$$

**2e.ii.**

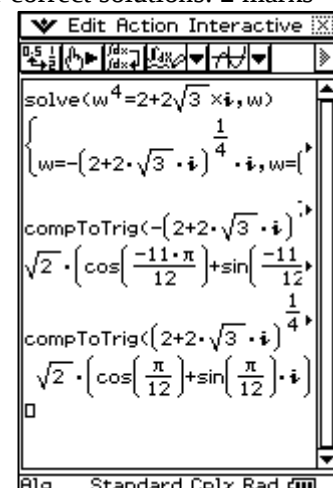
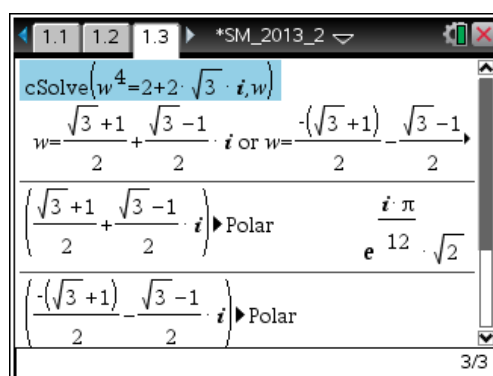
$$w^4 = 4\text{cis}\left(\frac{\pi}{3}\right)$$

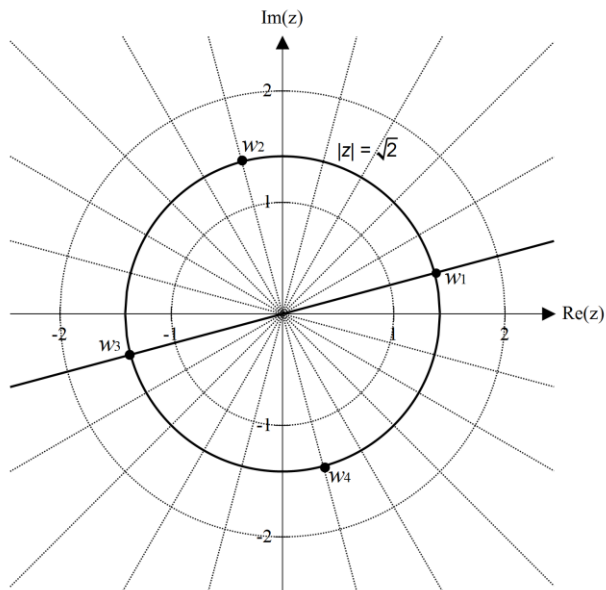
$$w = \sqrt[4]{4}\text{cis}\left(\frac{\frac{\pi}{3} + 2k\pi}{4}\right), \quad k = 0, 1, -1, -2$$

$$w_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{12}\right), \quad w_2 = \sqrt{2}\text{cis}\left(\frac{7\pi}{12}\right) \quad 1A$$

$$w_3 = \sqrt{2}\text{cis}\left(\frac{-5\pi}{12}\right), \quad w_4 = \sqrt{2}\text{cis}\left(\frac{-11\pi}{12}\right) \quad 1A$$

At least 1 correct solution: 1mark. All four correct solutions: 2 marks



2e.iii.

1A

Question 3**3a.i.**

$$|\vec{OP}| = |\vec{OQ}| = \sqrt{16 + 5 + 4} = 5 \quad 1M$$

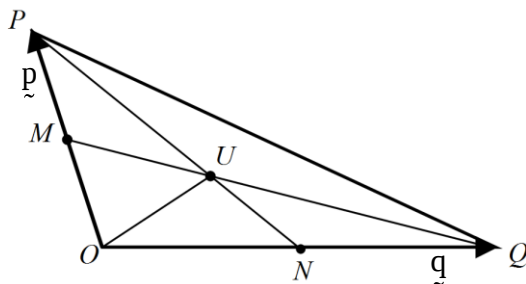
Therefore, $h = 5$, as required**3a.ii.**

$$\begin{aligned} \vec{PR} &= \vec{PO} + \vec{OR} = -\underline{\underline{p}} - \underline{\underline{q}} \\ &= 4\underline{\underline{i}} + \sqrt{5}\underline{\underline{j}} - 2\underline{\underline{k}} - 5\underline{\underline{i}} \\ &= -\underline{\underline{i}} + \sqrt{5}\underline{\underline{j}} - 2\underline{\underline{k}} \end{aligned} \quad 1A$$

3a.iii.If $\angle QPR$ is a right angle, then $\vec{QP} \cdot \vec{PR} = 0$. 1M

$$\vec{QP} = \underline{\underline{p}} - \underline{\underline{q}} = -9\underline{\underline{i}} - \sqrt{5}\underline{\underline{j}} + 2\underline{\underline{k}}$$

$$\begin{aligned} \vec{QP} \cdot \vec{PR} &= (-9 \times -1) + (-\sqrt{5} \times \sqrt{5}) + (2 \times -2) \\ &= 0, \text{ as required} \end{aligned} \quad 1A$$

3b.

3b.i.

$$\vec{PN} = \frac{1}{2}\vec{q} - \vec{p} \quad 1A$$

3b.ii.

$$\vec{QM} = \frac{1}{2}\vec{p} - \vec{q} \quad 1A$$

3b.iii.

$$\begin{aligned} \vec{OU} &= \vec{OP} + \vec{PU} = \vec{p} + \frac{a}{2}\vec{q} - a\vec{p} \\ &= (1-a)\vec{p} + \frac{a}{2}\vec{q} \quad \dots \text{equation 1} \end{aligned}$$

Also,

$$\begin{aligned} \vec{OU} &= \vec{OQ} + \vec{QU} = \vec{q} + \frac{b}{2}\vec{p} - b\vec{q} \\ &= \frac{b}{2}\vec{p} + (1-b)\vec{q} \quad \dots \text{equation 2} \end{aligned} \quad 1M$$

Equating coefficients for equations 1 and 2

$$1-a = \frac{b}{2} \quad \dots \text{equation 3, and } 1-b = \frac{a}{2} \quad \dots \text{equation 4}$$

Solving equations 3 and 4 simultaneously,

$$a = b = \frac{2}{3} \quad 1A$$

Note that this result shows that the centroid U of a triangle divides the medians, PN and QM , into parts in the ratio 2:1.

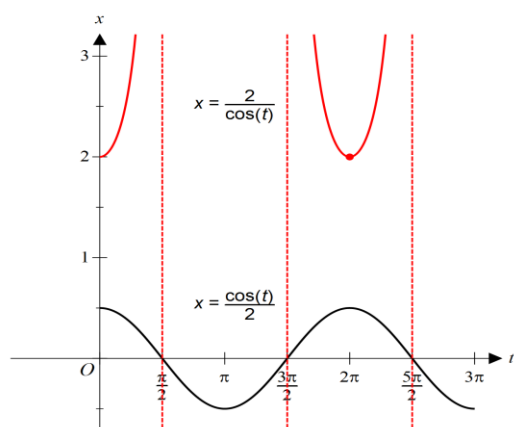
Question 4

4a.

For $\frac{3\pi}{2} < t < \frac{5\pi}{2}$, $0 \leq \frac{1}{2}\cos(t) \leq \frac{1}{2}$ and hence $2 < \frac{2}{\cos(t)} < \infty$

Since $x = \frac{2}{\cos(t)}$, $2 < x < \infty$, or $x \in [2, \infty)$, as required. 1M

A sketch graph of $x = \frac{2}{\cos(t)}$ could be used to show this.

**4b.**

$$x^2 = \left(\frac{2}{\cos(t)} \right)^2 = 4\sec^2(t) \text{ and } (y-3)^2 = (\sqrt{3}\tan(t))^2 = 3\tan^2(t) \quad 1M$$

Substituting into the identity $\tan^2(t) + 1 = \sec^2(t)$,

$$\frac{(y-3)^2}{3} + 1 = \frac{x^2}{4} \quad 1M$$

$$\frac{x^2}{4} - \frac{(y-3)^2}{3} = 1, \text{ as required.}$$

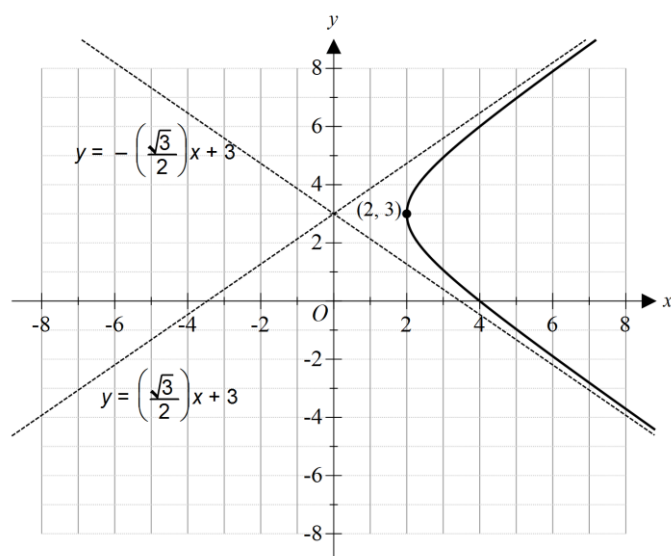
The coordinates of the vertices, $(h+a, k)$, $(h-a, k)$, could be $(2, 3)$, $(-2, 3)$.

Since $x \in [2, \infty)$, the vertex is $(2, 3)$. Hence right-hand branch of the hyperbola. 1M

4c.

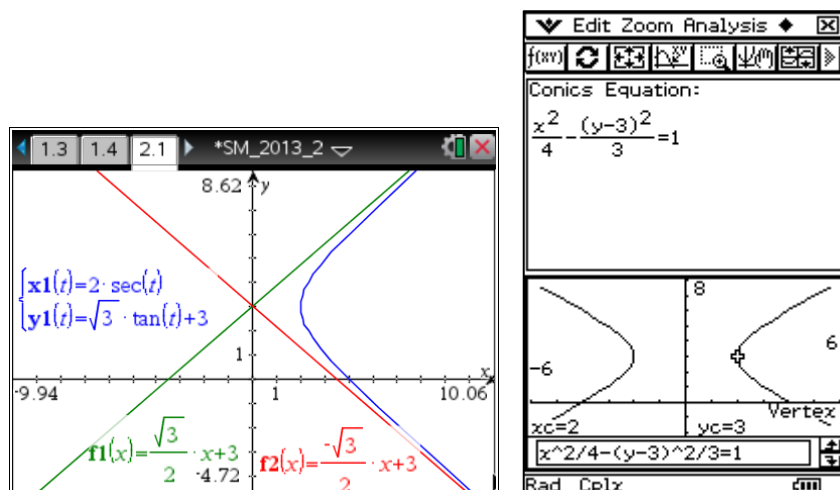
Coordinates of vertex: $(2, 3)$, by considering the translation from $\frac{x^2}{4} - \frac{y^2}{3} = 1$ to $\frac{x^2}{4} - \frac{(y-3)^2}{3} = 1$

Equation of asymptotes are of the form $y - k = \pm \frac{b}{a}(x - h)$, therefore $y = \pm \frac{\sqrt{3}}{2}x + 3$.



Correct shape, with vertex in the correct position: 1A. Asymptotes in correct position and labelled with their correct equations: 1A

A CAS can be used to graph the curve from either the parametric or cartesian equations.



4d.

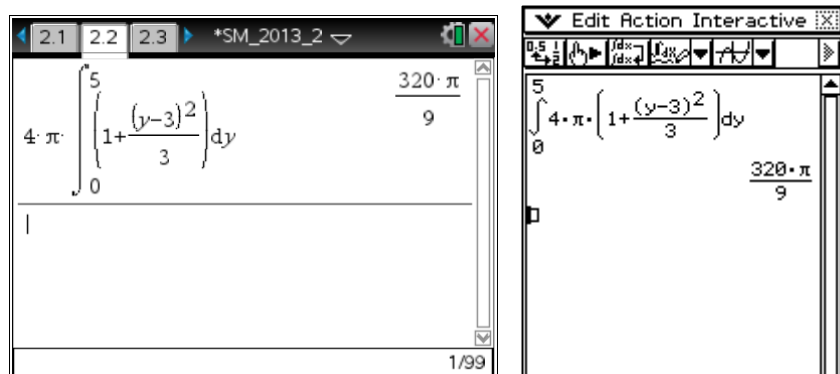
$$V = \pi \int_0^5 (x^2) dy$$

$$= 4\pi \int_0^5 \left(1 + \frac{(y-3)^2}{3} \right) dy$$

$$= \frac{320\pi}{9} \text{ m}^3.$$

1M

1A



4e.

$$\frac{dV}{dt} = 2 \text{ m}^3/\text{hour}, \text{ and } \frac{dV}{dy} = \frac{4\pi}{3} (y^2 - 6y + 12)$$

$$\frac{dy}{dt} = \frac{dy}{dV} \times \frac{dV}{dt}$$

$$= \frac{3}{4\pi (y^2 - 6y + 12)} \times 2 \quad 1A$$

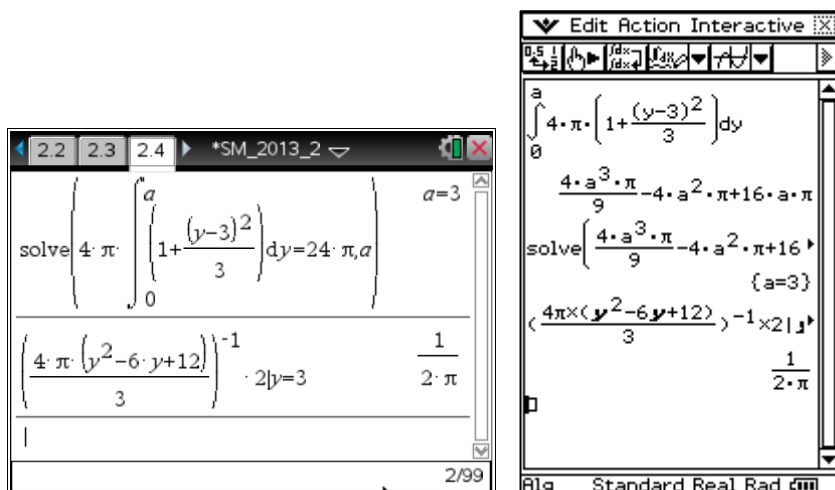
Let $y = a$ m when $V = 24\pi \text{ m}^3$.Solve for a ,

$$24\pi = 4\pi \int_0^a \left(1 + \frac{(y-3)^2}{3} \right) dy \quad 1M$$

$$a = 3$$

Substituting $y = 3$,

$$\frac{dy}{dt} = \frac{1}{2\pi} \text{ m/hour} \quad 1A$$



4f.i.

$$\frac{dN}{dt} = \text{rate of resin inflow} - \text{rate of resin outflow}$$

$$\frac{dN}{dt} = (0.02 \text{ tonne/m}^3 \times 2 \text{ m}^3/\text{hour}) - \left(\frac{N}{100} \text{ tonne/m}^3 \times 2 \text{ m}^3/\text{hour}\right) \quad 1\text{M}$$

$$\frac{dN}{dt} = 0.04 - 0.02N \quad \text{or} \quad \frac{dN}{dt} = \frac{2-N}{50} \quad \text{tonne/hour} \quad 1\text{A}$$

4f.ii.

Solve the differential equation $\frac{dN}{dt} = \frac{2-N}{50}$

$$t = 50 \int \frac{dN}{2-N}$$

$$= -50 \log_e \left(\frac{|2-N|}{c} \right), \text{ where } c \text{ is an integration constant} \quad 1\text{M}$$

When $t = 0$, $N = 10$, therefore

$$c = |2-10| = 8$$

Therefore,

$$t = 50 \log_e \left(\frac{8}{|2-N|} \right)$$

Pumping stops when the concentration of 100 m^3 of solution is 0.05 tonne/m^3 .

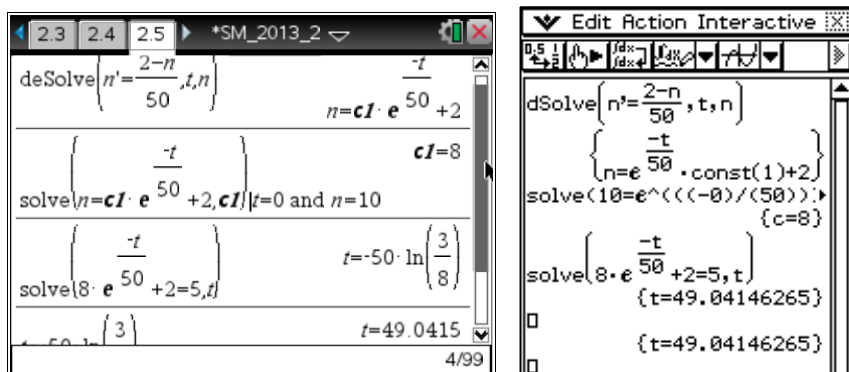
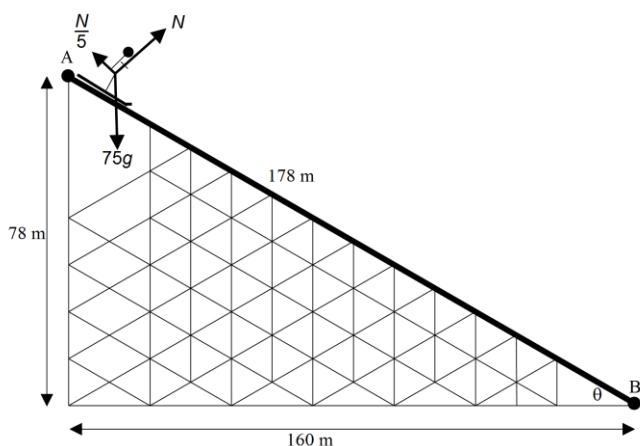
Therefore, when pumping stops, $N = 5$.

$$t = 50 \log_e \left(\frac{8}{|2-5|} \right) \quad 1\text{M}$$

$$t = 50 \log_e \left(\frac{8}{3} \right)$$

$$t = 49 \text{ hours, correct to the nearest hour.} \quad 1\text{A}$$

CAS differential equation solver could also be used.

**Question 5****5a.**

1A

5b.Let θ be the angle that the ramp makes with the horizontal.

$$75a = 75g \sin(\theta) - \frac{1}{5} \times 75g \cos(\theta) \quad 1M$$

$$75a = 75g \times \left(\frac{78}{178} - \frac{1}{5} \times \frac{160}{178} \right)$$

$$a = \frac{23g}{89}$$

$$a = 2.53 \text{ ms}^{-2}, \text{ correct to two decimal places.} \quad 1A$$

5c.

$$v^2 = u^2 + 2as, \text{ where } a = \frac{23g}{89}, s = 178 \text{ and } u = 0. \quad 1M$$

$$v = \sqrt{2 \times \frac{23}{89} \times 9.8 \times 178} = 30 \text{ ms}^{-1}, \text{ correct to the nearest integer.} \quad 1A$$

5d.

$$v = u + at$$

$$30.02 = 0 + 2.53t$$

$$t = \frac{30.02}{2.53} = 11.9 \text{ seconds, correct to one decimal place.} \quad 1A$$

Alternatively,

$$s = \frac{1}{2}(u+v)t, \text{ therefore } t = \frac{2s}{u+v}$$

$$t = \frac{2 \times 178}{30.0267} = 11.9 \text{ seconds, correct to one decimal place.} \quad 1A$$

5e.i.

Let u_x and u_y be the horizontal and vertical components, respectively, of the magnitude of Xue's velocity as she leaves O .

$$u_x = v_0 \cos(30^\circ) = \frac{\sqrt{3}v_0}{2}$$

$$u_y = v_0 \sin(30^\circ) = \frac{v_0}{2} \quad 1M$$

Let $\alpha \mathbf{i} + \beta \mathbf{j}$ be the position vector of Xue at time t seconds, as she jumps from O to P .

Using the formula $s = ut + \frac{1}{2}at^2$

$$\alpha = u_x t = \frac{\sqrt{3}v_0 t}{2} \text{ and} \quad 1A$$

$$\beta = u_y t - \frac{1}{2} \times g t^2 = \frac{v_0 t}{2} - 4.9 t^2 \quad 1A$$

5e.ii.

At point P , $\alpha = 80$ and $\beta = -60$, therefore

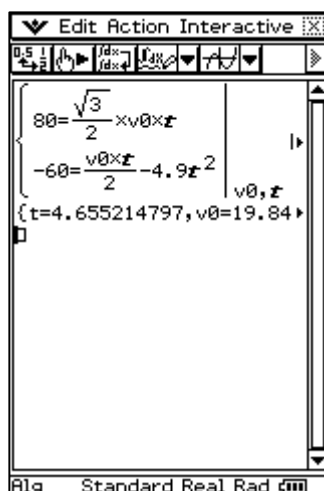
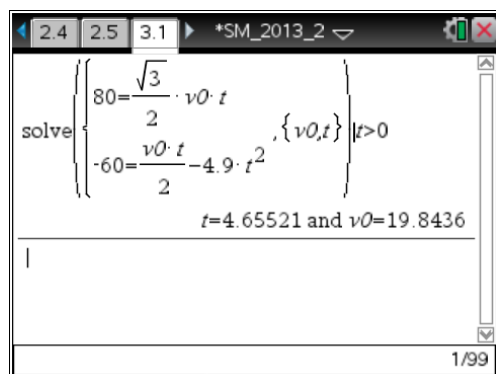
$$80 = \frac{\sqrt{3}v_0 t}{2} \quad \dots \text{ equation 1}$$

$$-60 = \frac{v_0 t}{2} - 4.9 t^2 \quad \dots \text{ equation 2}$$

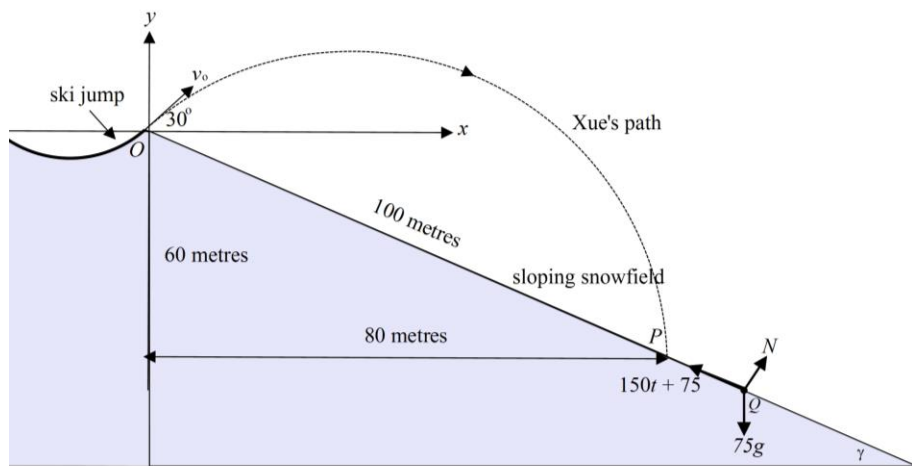
Solving equations 1 and 2 simultaneously,

$$v_0 = 19.84 \text{ ms}^{-1} \quad 1A$$

$$t = 4.66 \text{ s} \quad 1A$$



5f.



From point Q until Xue comes to rest,

$$75a = 75g \sin(\gamma) - (150t + 75) \quad 1M$$

$$a = \frac{3 \times 9.8}{5} - 2t - 1 = 4.88 - 2t$$

Therefore,

$$v = \int (4.88 - 2t) dt \quad 1M$$

$$v = 4.88t - t^2 + c$$

At $t = 0$, $v = 22$. Therefore $c = 22$

When Xue comes to rest,

$$t^2 - 4.88t - 22 = 0$$

$$t = 7.7 \text{ seconds, correct to one decimal place.} \quad 1A$$

END OF SECTION 2 SOLUTIONS