

The Mathematical Association of Victoria
SOLUTIONS Trial Exam 2013

SPECIALIST MATHEMATICS

Written Examination 1

Question 1

$$80a = T - 80g$$

$$\Rightarrow T = 80a + 80g = 80(a + g). \quad [\text{M1}]$$

Require $T = 80(a + g) < 200g$:

$$80(a + g) < 200g$$

$$\Rightarrow a + g < \frac{5g}{2}$$

$$\Rightarrow a < \frac{5g}{2} - g = \frac{3g}{2} \text{ m/s}^2. \quad [\text{A1}]$$

Total 2 marks

Question 2

$$(2\hat{i} - \hat{j} + 3\hat{k}) \cdot (a\hat{i} - 6\hat{j} - 2\hat{k}) = 2a + 6 - 6 = 2a. \quad \dots (1)$$

$$\begin{aligned} (2\hat{i} - \hat{j} + 3\hat{k}) \cdot (a\hat{i} - 6\hat{j} - 2\hat{k}) &= |2\hat{i} - \hat{j} + 3\hat{k}| |a\hat{i} - 6\hat{j} - 2\hat{k}| \cos\left(\frac{2\pi}{3}\right) \\ &= \sqrt{14} \sqrt{a^2 + 40} \left(-\frac{1}{2}\right). \quad \dots (2) \quad [\text{M1}] \end{aligned}$$

Equate equations (1) and (2):

$$2a = -\frac{1}{2} \sqrt{14} \sqrt{a^2 + 40} \quad [\text{M1}]$$

$$\Rightarrow -4a = \sqrt{14} \sqrt{a^2 + 40}$$

$$\Rightarrow 16a^2 = 14(a^2 + 40)$$

$$\Rightarrow 2a^2 = 14 \times 40 \Rightarrow a^2 = 7 \times 40 \Rightarrow a = \pm \sqrt{280} = \pm 2\sqrt{70}.$$

Case 1: $a = \sqrt{280} \Rightarrow \cos(\theta) = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ where θ is the angle between $2\hat{i} - \hat{j} + 3\hat{k}$ and $a\hat{i} - 6\hat{j} - 2\hat{k}$.

Therefore $a = \sqrt{280}$ is rejected.

$$\text{Case 2: } a = -\sqrt{280} = -2\sqrt{70}. \quad [\text{A1}]$$

Total 3 marks

Justification for rejecting $a = \sqrt{280}$ is not required. Simplification of surd is not required.

Question 3**a.**

Use implicit differentiation:

$$\frac{1}{2} \frac{d}{dx} (y-1)^2 - \frac{2}{k} (x+2) = 0$$

$$\Rightarrow \frac{1}{2} \frac{d}{dy} (y-1)^2 \times \frac{dy}{dx} - \frac{2}{k} (x+2) = 0 \quad [\text{M1}]$$

$$\Rightarrow (y-1) \times \frac{dy}{dx} - \frac{2}{k} (x+2) = 0$$

$$\Rightarrow (y-1) \times \frac{dy}{dx} = \frac{2}{k} (x+2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(x+2)}{k(y-1)}. \quad [\text{A1}]$$

Total 2 marks**b.**The gradient of the line $7y - 3x = -11$ is $\frac{3}{7}$ therefore the gradient of the **normal** to the hyperbola at a point where $x = -1$ is $\frac{3}{7}$ therefore the gradient of the **tangent** to the hyperbola at a point where $x = -1$ is $-\frac{7}{3}$.Substitute $\frac{dy}{dx} = -\frac{7}{3}$ and $x = -1$ into $\frac{dy}{dx} = \frac{2(x+2)}{k(y-1)}$ (answer from **part a.**):

$$-\frac{7}{3} = \frac{2}{k(y-1)}. \quad \dots (1) \quad [\text{M1}]$$

Substitute $x = -1$ into the given normal $7y - 3x = -11$:

$$7y + 3 = -11$$

$$\Rightarrow y = -2. \quad [\text{A1}]$$

Therefore $7y - 3x = -11$ is normal to the hyperbola at the point $(-1, -2)$.Substitute $y = -2$ into equation (1):

$$-\frac{7}{3} = \frac{2}{-3k}$$

$$\Rightarrow k = \frac{2}{7}. \quad [\text{A1}]$$

Total 3 marks

Question 4

$$\tilde{a}(t) = \frac{d\tilde{v}}{dt} = \left(\frac{1}{1+t^2} \right) \tilde{i} - \frac{1}{(t+1)^2} \tilde{j} - \left(\frac{1}{t+1} \right) \tilde{k}$$

$$\Rightarrow \tilde{v}(t) = \int \left(\frac{1}{1+t^2} \right) \tilde{i} - \frac{1}{(t+1)^2} \tilde{j} - \left(\frac{1}{t+1} \right) \tilde{k} dt$$

$$= \tan^{-1}(t) \tilde{i} + \left(\frac{1}{t+1} \right) \tilde{j} - \log_e(t+1) \tilde{k} + \tilde{C} \quad (\text{Note that } |t+1| = t+1 \text{ for } t \geq 0)$$

[M1]

where $\tilde{C}$ is the arbitrary constant vector of integration.

Substitute $\tilde{v}(0) = \tilde{i} - \tilde{j} + \tilde{k}$:

$$\tilde{i} - \tilde{j} + \tilde{k} = \tan^{-1}(0) \tilde{i} + \left(\frac{1}{0+1} \right) \tilde{j} - \log_e(0+1) \tilde{k} + \tilde{C}$$

$$\Rightarrow \tilde{i} - \tilde{j} + \tilde{k} = \tilde{j} + \tilde{C}$$

$$\Rightarrow \tilde{C} = \tilde{i} - 2\tilde{j} + \tilde{k}.$$

Therefore:

$$\tilde{v}(t) = \tan^{-1}(t) \tilde{i} + \left(\frac{1}{t+1} \right) \tilde{j} - \log_e(t+1) \tilde{k} + \tilde{i} - 2\tilde{j} + \tilde{k}$$

[A1]

$$= \left(1 + \tan^{-1}(t) \right) \tilde{i} + \left(\frac{1}{t+1} - 2 \right) \tilde{j} + \left(1 - \log_e(t+1) \right) \tilde{k}.$$

Components do not need to be collected.

Total 2 marks

Question 5

$$\left| \frac{z+1-2i}{z+2-i} \right| = 1$$

$$\Rightarrow \frac{|z+1-2i|}{|z+2-i|} = 1 \quad [\text{M1}]$$

$$\Rightarrow |z+1-2i| = |z+2-i|.$$

Method 1: Algebraic approach.Substitute $z = x + iy$:

$$|x + iy + 1 - 2i| = |x + iy + 2 - i|$$

$$\Rightarrow |(x+1) + i(y-2)| = |(x+2) + i(y-1)|$$

$$\Rightarrow \sqrt{(x+1)^2 + (y-2)^2} = \sqrt{(x+2)^2 + (y-1)^2}$$

$$\Rightarrow (x+1)^2 + (y-2)^2 = (x+2)^2 + (y-1)^2 \quad [\text{M1}]$$

$$\Rightarrow 2x + 1 - 4y + 4 = 4x + 4 - 2y + 1$$

$$\Rightarrow y = -x. \quad [\text{A1}]$$

Total 3 marks**Method 2:** Geometric approach.

$$|z+1-2i| = |z+2-i|$$

$$\Rightarrow |z - (-1+2i)| = |z - (-2+i)|.$$

This is recognised as the perpendicular bisector of the line segment joining $z = -1 + 2i$ and $z = -2 + i$.

$$\text{Coordinates of midpoint: } \left(-\frac{3}{2}, \frac{3}{2} \right).$$

$$\text{Gradient of line segment: } m_1 = 1.$$

$$\text{Gradient of perpendicular bisector: } \frac{-1}{m_1} = -1.$$

$$\text{Midpoint } \left(-\frac{3}{2}, \frac{3}{2} \right) \text{ and gradient } m = -1 \quad [\text{M1}]$$

Substitute into the standard form $y - y_1 = m(x - x_1)$ of a line:

$$y - \frac{3}{2} = -\left(x + \frac{3}{2} \right)$$

$$\Rightarrow y = -x. \quad [\text{A1}]$$

Total 3 marks

Question 6**a.**

$$\frac{1}{x^3 - 2x^2 + x} = \frac{1}{x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}. \quad [\text{M1}]$$

$$\Rightarrow 1 \equiv A(x-1)^2 + Bx(x-1) + Cx$$

$$\Rightarrow A = 1, \quad B = -1, \quad C = 1.$$

$$\text{Answer: } \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}. \quad [\text{A1}]$$

Total 2 marks**b.**

$$\frac{dy}{dx} - \frac{x+1}{\sqrt{1-3x^2}} = 0 \Rightarrow \frac{dy}{dx} = \frac{x+1}{\sqrt{1-3x^2}}$$

$$\Rightarrow y = \int \frac{x+1}{\sqrt{1-3x^2}} dx$$

$$= \int \frac{x}{\sqrt{1-3x^2}} dx + \int \frac{1}{\sqrt{1-3x^2}} dx. \quad [\text{M1}]$$

$$\bullet \text{ Let } I_1 = \int \frac{x}{\sqrt{1-3x^2}} dx.$$

$$\text{Substitute } u = 1 - 3x^2 \Rightarrow dx = \frac{du}{-6x}:$$

$$I_1 = \int \frac{x}{\sqrt{u}} \left(\frac{du}{-6x} \right) = \frac{-1}{6} \int \frac{1}{\sqrt{u}} du \quad [\text{M1}]$$

$$= \frac{-1}{6} \int u^{-1/2} du = \frac{-1}{3} u^{1/2}$$

$$= \frac{-1}{3} \sqrt{1-3x^2}.$$

• Let $I_2 = \int \frac{1}{\sqrt{1-3x^2}} dx$.

Substitute $u = \sqrt{3}x \Rightarrow dx = \frac{du}{\sqrt{3}}$.

$$I_2 = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{1-u^2}} du$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}(u) = \frac{1}{\sqrt{3}} \sin^{-1}(\sqrt{3}x).$$

Therefore $y = \frac{-1}{3} \sqrt{1-3x^2} + \frac{1}{\sqrt{3}} \sin^{-1}(\sqrt{3}x) + C$.

[A1]

Deduct 1 mark if the arbitrary constant of integration is not included in the final answer.

Total 3 marks

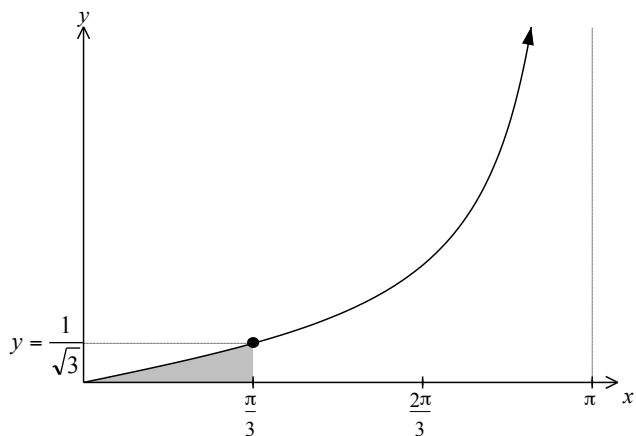
Question 7

$$y = \frac{1}{\sqrt{3}} \Rightarrow \frac{1}{\sqrt{3}} = \tan\left(\frac{x}{2}\right) \Rightarrow \frac{x}{2} = \frac{\pi}{6}$$

$$\Rightarrow x = \frac{\pi}{3}.$$

[M1]

$$V = \pi \int_a^b y^2 dx$$



$$= \pi \int_0^{\pi/3} \tan^2\left(\frac{x}{2}\right) dx$$

[M1]

Substitute from the identity $1 + \tan^2(A) = \sec^2(A)$:

$$= \pi \int_0^{\pi/3} \sec^2\left(\frac{x}{2}\right) - 1 dx$$

$$= \pi \left[2 \tan\left(\frac{x}{2}\right) - x \right]_0^{\pi/3}$$

[M1]

$$= \pi \left(2 \tan\left(\frac{\pi}{6}\right) - \frac{\pi}{3} \right)$$

$$= \frac{2\sqrt{3}\pi - \pi^2}{3}.$$

[M1]**Total 4 marks**

Question 8**a.**

The expression $x = v - v^2$ suggests using the form $a = v \frac{dv}{dx}$ for acceleration.

Method 1:

$$x = v - v^2 \Rightarrow \frac{dx}{dv} = 1 - 2v$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{1 - 2v} \quad [\text{M1}]$$

$$\Rightarrow a = v \frac{dv}{dx} = \frac{v}{1 - 2v}. \quad [\text{A1}]$$

Method 2:

Use implicit differentiation.

$$x = v - v^2$$

$$\Rightarrow 1 = \frac{dv}{dx} - 2v \frac{dv}{dx} \quad [\text{M1}]$$

$$\Rightarrow 1 = \frac{dv}{dx} (1 - 2v)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{1 - 2v}$$

$$\Rightarrow a = v \frac{dv}{dx} = \frac{v}{1 - 2v}. \quad [\text{A1}]$$

Method 3:

$$x = v - v^2$$

$$\Rightarrow \frac{dx}{dt} = \frac{dv}{dt} - 2v \frac{dv}{dt} \quad [\text{M1}]$$

$$\Rightarrow v = \frac{dv}{dt} (1 - 2v)$$

$$\Rightarrow \frac{v}{1 - 2v} = \frac{dv}{dt}$$

$$\Rightarrow a = \frac{dv}{dt} = \frac{v}{1 - 2v}. \quad [\text{A1}]$$

Total 2 marks

b.

$$\frac{dv}{dt} = \frac{v}{1-2v}$$

$$\Rightarrow \frac{dt}{dv} = \frac{1-2v}{v} = \frac{1}{v} - 2$$

$$\Rightarrow t = \int \frac{1}{v} - 2 \, dv \quad \text{[M1]}$$

$$= \log_e |v| - 2v + C.$$

Substitute $v = 1$ and $t = 0$: $C = 2$.

Therefore $t = \log_e |v| - 2v + 2$.

Substitute $v = \frac{1}{2}$:

$$t = \log_e \left(\frac{1}{2} \right) + 1 = 1 - \log_e (2). \quad \text{[A1]}$$

Total 2 marks

Question 9

Substitute the double angle formulae into $1 + \cos(2\theta) = \sqrt{3} \sin(2\theta)$:

$$1 + \cos^2(\theta) - \sin^2(\theta) = 2\sqrt{3} \sin(\theta) \cos(\theta)$$

$$\Rightarrow 1 - \sin^2(\theta) + \cos^2(\theta) = 2\sqrt{3} \sin(\theta) \cos(\theta) \quad [\text{M1}]$$

$$\Rightarrow 2\cos^2(\theta) = 2\sqrt{3} \sin(\theta) \cos(\theta)$$

$$\Rightarrow \cos^2(\theta) = \sqrt{3} \sin(\theta) \cos(\theta)$$

$$\Rightarrow \cos^2(\theta) - \sqrt{3} \sin(\theta) \cos(\theta) = 0$$

$$\Rightarrow \cos(\theta)(\cos(\theta) - \sqrt{3} \sin(\theta)) = 0. \quad [\text{M1}]$$

Apply the Null Factor Law:

Case 1: $\cos(\theta) = 0$

$$\Rightarrow \theta = \frac{(2n-1)\pi}{2}, \quad n \in \mathbb{Z}.$$

Over $-\pi \leq \theta \leq \pi$: $\theta = -\frac{\pi}{2}, \frac{\pi}{2}. \quad [\text{A1}]$

Case 2: $\cos(\theta) - \sqrt{3} \sin(\theta) = 0$

$$\Rightarrow \cos(\theta) = \sqrt{3} \sin(\theta)$$

$$\Rightarrow \tan(\theta) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \frac{\pi}{6} + n\pi, \quad n \in \mathbb{Z}.$$

Over $-\pi \leq \theta \leq \pi$: $\theta = -\frac{5\pi}{6}, \frac{\pi}{6}. \quad [\text{A1}]$

Total 4 marks

Question 10

a.

i. $r = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2.$

$$\tan(\theta) = -\sqrt{3} \text{ and } \theta \text{ lies in the fourth quadrant: } \theta = -\frac{\pi}{3}.$$

$$1 - i\sqrt{3} = 2\text{cis}\left(-\frac{\pi}{3}\right). \quad [\text{A1}]$$

ii. **Part i.** suggests working in polar form.

$$\sqrt{3} + i = 2\operatorname{cis}\left(\frac{\pi}{6}\right). \quad [\text{M1}]$$

Therefore:

$$(\sqrt{3} + i)^m = (1 - i\sqrt{3})^m$$

$$\Rightarrow \left[2\operatorname{cis}\left(\frac{\pi}{6}\right)\right]^m = \left[2\operatorname{cis}\left(-\frac{\pi}{3}\right)\right]^m$$

$$\Rightarrow 2^m \operatorname{cis}\left(\frac{m\pi}{6}\right) = 2^m \operatorname{cis}\left(-\frac{m\pi}{3}\right). \quad [\text{M1}]$$

From here there are two options:

Option 1:

$$\operatorname{cis}\left(\frac{m\pi}{6}\right) = \operatorname{cis}\left(-\frac{m\pi}{3}\right)$$

$$\Rightarrow \frac{m\pi}{6} = -\frac{m\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$$

$$\Rightarrow m = 4k, \quad k \in \mathbb{Z}^+ \quad (\text{since } m \in \mathbb{Z}^+ \text{ is given in the question}). \quad [\text{A1}]$$

Option 2:

$$\frac{\left[2\operatorname{cis}\left(\frac{m\pi}{6}\right)\right]^m}{\left[2\operatorname{cis}\left(-\frac{m\pi}{3}\right)\right]^m} = 1 \quad \Rightarrow \left(\frac{2\operatorname{cis}\left(\frac{m\pi}{6}\right)}{2\operatorname{cis}\left(-\frac{m\pi}{3}\right)}\right)^m = 1$$

$$\Rightarrow \left[\operatorname{cis}\left(\frac{\pi}{2}\right)\right]^m = 1 = \operatorname{cis}(2k\pi), \quad k \in \mathbb{Z}$$

$$\Rightarrow \operatorname{cis}\left(\frac{m\pi}{2}\right) = \operatorname{cis}(2k\pi), \quad k \in \mathbb{Z}$$

$$\Rightarrow \frac{m\pi}{2} = 2k\pi$$

$$\Rightarrow m = 4k, \quad k \in \mathbb{Z}^+ \quad (\text{since } m \in \mathbb{Z}^+ \text{ is given in the question}). \quad [\text{A1}]$$

Total 3 marks

b.**Method 1:** Cartesian form approach.Substitute $z = x + iy \Rightarrow \bar{z} = x - iy$:

$$z^2 = i\bar{z}$$

$$\Rightarrow (x + iy)^2 = i(x - iy)$$

$$\Rightarrow x^2 + 2xyi - y^2 = ix + y$$

$$\Rightarrow x^2 - y^2 + 2xyi = y + ix.$$

[M1]

Equate real and imaginary parts:

$$x^2 - y^2 = y. \quad \dots (1)$$

$$2xy = x$$

$$\Rightarrow 2xy - x = 0$$

$$\Rightarrow x(2y - 1) = 0. \quad \dots (2)$$

Both equations [M1]

From equation (2) (using the null factor law):

Case 1: $x = 0$.Substitute $x = 0$ into equation (1): $-y^2 = y$

$$\Rightarrow y^2 + y = 0$$

$$\Rightarrow y(y + 1) = 0$$

$$\Rightarrow y = 0, -1.$$

Therefore $z = 0$ or $z = -i$.**[A1]****Case 2:** $2y - 1 = 0 \Rightarrow y = \frac{1}{2}$.Substitute $y = \frac{1}{2}$ into equation (1): $x^2 - \frac{1}{4} = \frac{1}{2}$

$$\Rightarrow x^2 = \frac{3}{4}$$

$$\Rightarrow x = \pm \frac{\sqrt{3}}{2}.$$

Therefore $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ or $z = -\frac{\sqrt{3}}{2} + \frac{i}{2}$.**[A1]**Solutions: $z = 0$, $z = -i$, $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$, $z = -\frac{\sqrt{3}}{2} + \frac{i}{2}$.**The solutions from the two cases do not need to be consolidated into a final answer.****Total 4 marks**

Method 2: Polar form approach.

Substitute $z = r\text{cis}(\theta) \Rightarrow \bar{z} = r\text{cis}(-\theta)$:

$$z^2 = i\bar{z}$$

$$\Rightarrow r^2\text{cis}(2\theta) = ir\text{cis}(-\theta)$$

$$\Rightarrow r^2\text{cis}(2\theta) = r\text{cis}\left(\frac{\pi}{2}\right)\text{cis}(-\theta)$$

$$\Rightarrow r^2\text{cis}(2\theta) = r\text{cis}\left(\frac{\pi}{2} - \theta\right).$$

Equate moduli:

$$r^2 = r$$

$$\Rightarrow r^2 - r = 0$$

$$\Rightarrow r(r - 1) = 0$$

$$\Rightarrow r = 0, 1.$$

[M1]

Equate arguments:

$$2\theta = \frac{\pi}{2} - \theta + 2k\pi, \quad k \in \mathbb{Z}$$

$$3\theta = \frac{\pi}{2} + 2k\pi$$

$$\Rightarrow \theta = \frac{\pi}{6} + \frac{2k\pi}{3}.$$

[M1]

Case 1: $r = 0$.

$$z = 0.$$

Case 2: $r = 1$.

$$z = \text{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right), \quad k \in \mathbb{Z}.$$

$$\underline{k = 0}: \quad z = \text{cis}\left(\frac{\pi}{6}\right).$$

$$\underline{k = 1}: \quad z = \text{cis}\left(\frac{5\pi}{6}\right).$$

$$\text{Convert into the form } a + ib: \quad z = \frac{\sqrt{3}}{2} + \frac{i}{2}, \quad z = -\frac{\sqrt{3}}{2} + \frac{i}{2}.$$

[A1]

$$\underline{k = -1}: \quad z = \text{cis}\left(-\frac{3\pi}{6}\right) = \text{cis}\left(-\frac{\pi}{2}\right) = -i.$$

$$z = 0, \quad z = -i.$$

[A1]