

2013 VCAA Specialist Math Exam 2 Solutions

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SECTION 1

1	2	3	4	5	6	7	8	9	10	11
D	A	E	D	B	D	E	C	B	D	C
12	13	14	15	16	17	18	19	20	21	22
E	A	B	E	A	B	D	C	A	C	E

Q1 $-1 \leq 3x \leq 1, -\frac{1}{3} \leq x \leq \frac{1}{3}$ D

Q2 $x = 2 \operatorname{cosec}(t) + 1, \operatorname{cosec}(t) = \frac{x-1}{2}$

$y = 3 \cot(t) - 1, \cot(t) = \frac{y+1}{3}$

$\operatorname{cosec}^2(t) - \cot^2(t) = 1, \therefore \frac{(x-1)^2}{4} - \frac{(y+1)^2}{9} = 1$ A

Q3 $y = \frac{1}{ax^2 + bx + c} = \frac{1}{a(x+5)(x-3)} = \frac{1}{ax^2 + 2ax - 15a}$

Stationary point lies in the middle of $x = -5$ and $x = 3$, i.e.

$x = \frac{-5+3}{2} = -1$, and $y = -\frac{1}{8}, \therefore -\frac{1}{8} = \frac{1}{a-2a-15a}$

$\therefore a = \frac{1}{2}, b = 2a = 1$ and $c = -15a = -\frac{15}{2}$ E

Q4 $y = \tan^{-1}(bx), \frac{dy}{dx} = \frac{b}{1+(bx)^2} = b$ at $x = 0$

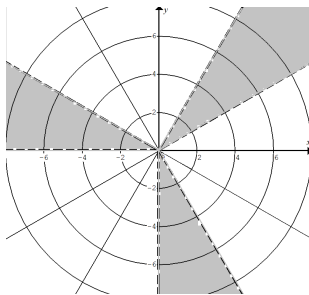
To intersect exactly 3 times, gradient of $y = \tan^{-1}(bx)$ at $x = 0$ must be greater than the gradient of $y = ax$ for positive gradients,

OR $y = \tan^{-1}(bx)$ at $x = 0$ must be less than the gradient of $y = ax$ for negative gradients.

$\therefore$ Either $b > a > 0$ or $b < a < 0$ D

Q5 $x^2 + y^2 > b^2, |z|^2 > b^2, |z| > b$ B

Q6 The complete set is $\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{6}, \frac{\pi}{3}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$.



The best choice is D

Q7 $z = r \operatorname{cis}(\theta), \frac{z^2}{\bar{z}} = \frac{r^2 \operatorname{cis}(2\theta)}{r \operatorname{cis}(-\theta)} = r \operatorname{cis}(3\theta)$ E

Q8 $z^2 = 1 + i, z^2 = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4} + 2k\pi\right), z = 2^{\frac{1}{4}} \operatorname{cis}\left(\frac{\pi}{8} + k\pi\right)$

Principal arguments are $-\frac{7\pi}{8}$ and $\frac{\pi}{8}$. C

Q9 $u = \log_e(x), \frac{du}{dx} = \frac{1}{x}, x = e^3, u = 3; x = e^4, u = 4$

$\int_{e^3}^{e^4} \frac{1}{x \log_e(x)} dx = \int_3^4 \frac{1}{u} du, \therefore a = 3$ and $b = 4$ B

Q10 $V = \int_0^3 \pi x^2 dy = \pi \int_0^3 y^{\frac{3}{2}} dy = \frac{18\pi 3^{\frac{3}{2}}}{5}$ D

Q11 $y_{n+1} \approx y_n + h \times \frac{dy}{dx}$

$x_0 = 0, y_0 = 1, \frac{dy}{dx} = \frac{1}{3}$

$x_1 = 0.1, y_1 \approx 1 + 0.1 \times \frac{1}{3} = \frac{31}{30}, \frac{dy}{dx} = 0.302115$

$x_2 = 0.2, y_2 \approx \frac{31}{30} + 0.1 \times 0.302115 \approx 1.064$ C

Q12 The solution curves are of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$.

By implicit differentiation, $\frac{2x}{a^2} - \frac{2y}{b^2} \times \frac{dy}{dx} = 0, \therefore \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$ E

Q13 At time t min., amount of salt = Q grams,
volume of solution = $50 + 4t$ litres,

$\therefore \text{concentration} = \frac{Q}{50 + 4t}$ grams per litre

Rate of inflow = 20 grams per min

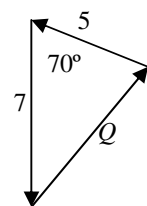
Rate of outflow = $\frac{6Q}{50 + 4t} = \frac{3Q}{25 + 2t}$ grams per min

$\therefore \frac{dQ}{dt} = 20 - \frac{3Q}{25 + 2t}$ A

Q14 $|\vec{OP}| = \sqrt{7^2 + (-1)^2 + (5\sqrt{2})^2} = 10$ B

Q15 $|\vec{u}| = \sqrt{18}, |\vec{v}| = \sqrt{18}, |-\vec{w}| = \sqrt{18}, \vec{u} \cdot \vec{v} = 0$
 $(\vec{u} + \vec{w}) \cdot \vec{v} = 6$ E

Q16 $Q = \sqrt{5^2 + 7^2 - 2(5)(7)\cos 70^\circ}$



Q17 Three vectors on the same plane are always linearly dependent. $\vec{a}, \vec{c}$ and $\vec{d}$ are on the same plane. B

$$Q18 \quad \frac{1}{2} \frac{d(v^2)}{dx} = \sqrt{(v^2)-1}, \quad \frac{dx}{d(v^2)} = \frac{1}{2\sqrt{(v^2)-1}},$$

$$x = \int \frac{1}{2\sqrt{(v^2)-1}} d(v^2) = \sqrt{v^2-1} + c$$

$$v = \sqrt{2} \text{ when } x = 0, \therefore c = -1$$

$$\therefore 1+x = \sqrt{v^2-1}, \quad v^2 = 1+(1+x)^2, \therefore v = \sqrt{1+(1+x)^2} \quad \text{D}$$

$$Q19 \quad u = +2, \quad a = -9.8, \quad s = -100, \quad s = ut + \frac{1}{2}at^2, \therefore t \approx 4.7 \text{ s} \quad \text{C}$$

$$Q20 \quad \text{Let } R \text{ be the reaction force of the floor on the 5kg parcel.} \\ \text{Resultant force} = 5g - R = 5 \times 3, \therefore R = -15 + 5g \quad \text{A}$$

$$Q21 \quad u = +20, \quad v = +2, \quad t = 4, \quad v = u + at, \quad a = -4.5 \text{ m s}^{-2} \\ |\text{Resultant force}| = m|a| = 9.0 \text{ N} \quad \text{C}$$

$$Q22 \quad ma = mg - kv^2, \quad mv \frac{dv}{dx} = mg - kv^2, \therefore \frac{dv}{dx} = \frac{g}{v} - \frac{kv}{m} \quad \text{E}$$

SECTION 2

$$Q1a \quad x = 1 + 3\cos(t), \quad \cos(t) = \frac{x-1}{3}$$

$$y = -2 + 2\sin(t), \quad \sin(t) = \frac{y+2}{2}$$

$$\sin^2(t) + \cos^2(t) = 1, \therefore \frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$$

$$Q1b \quad \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}, \quad 2\cos(t) = -\frac{2\sqrt{3}}{3} \times (-3\sin(t))$$

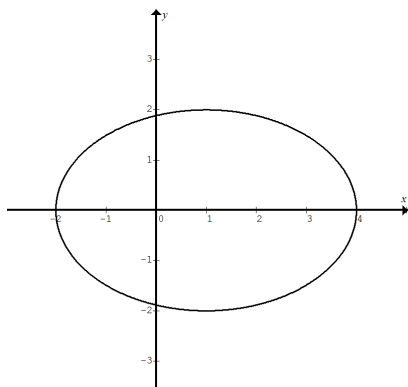
$$\therefore \tan(t) = \frac{1}{\sqrt{3}} \text{ and } 0 \leq t \leq 2\pi$$

$$\therefore t = \frac{\pi}{6}, \frac{7\pi}{6}$$

$$Q1c \quad \frac{(x-1)^2}{9} + \frac{y^2}{4} = 1$$

$$x\text{-intercepts: } y = 0, \quad (x-1)^2 = 9, \quad x-1 = \pm 3, \therefore x = -2, 4$$

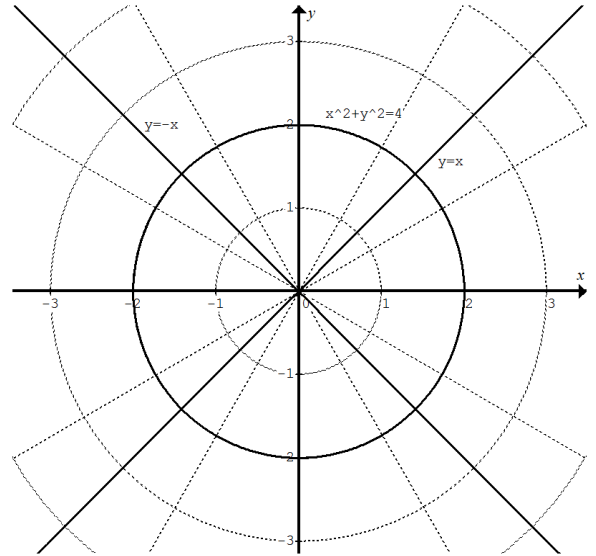
$$y\text{-intercepts: } x = 0, \quad y^2 = \frac{32}{9}, \quad y = \pm \frac{4\sqrt{2}}{3}$$



$$Q1di \quad \text{Volume} = \int_1^3 \pi y^2 dx = \int_1^3 4\pi \left(1 - \frac{(x-1)^2}{9}\right) dx$$

$$Q1dii \quad \text{Volume} = 4\pi \left[x - \frac{(x-1)^3}{27} \right]_1^3 = \frac{184\pi}{27} \text{ cubic units}$$

$$Q2a \quad z\bar{z} = 4, \quad x^2 + y^2 = 4; \quad |z - \bar{z}| = |z + \bar{z}|, \quad |y| = |x|$$



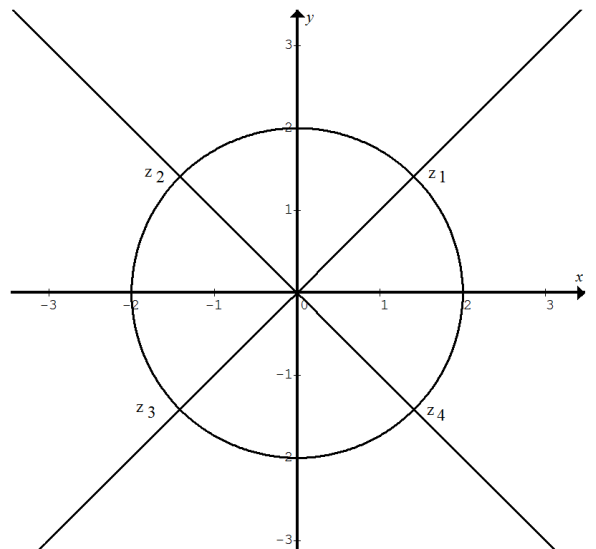
$$Q2b \quad \text{In the first quadrant: } x = 2\cos\frac{\pi}{4} = \sqrt{2}, \quad y = 2\sin\frac{\pi}{4} = \sqrt{2}$$

$$\therefore z = \sqrt{2} + i\sqrt{2}$$

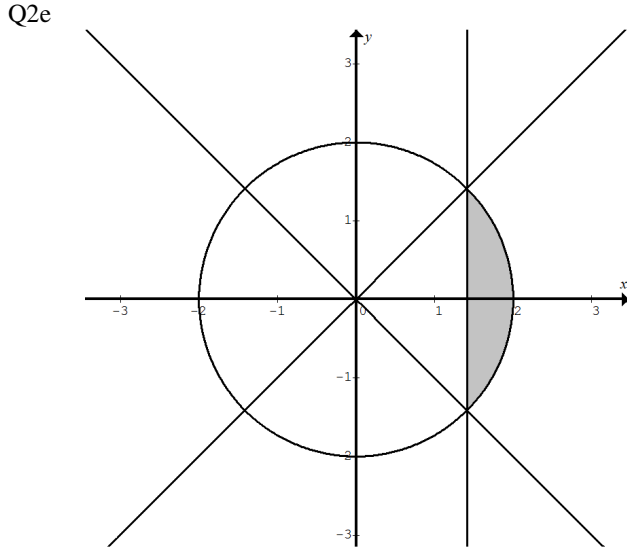
$$\text{The others are: } z = -\sqrt{2} + i\sqrt{2}, \quad z = -\sqrt{2} - i\sqrt{2}, \quad z = \sqrt{2} - i\sqrt{2}$$

$$Q2c \quad z_1 = \sqrt{2} + i\sqrt{2}$$

$$z_2 = -\sqrt{2} + i\sqrt{2}, \quad z_3 = -\sqrt{2} - i\sqrt{2}, \quad z_4 = \sqrt{2} - i\sqrt{2}$$



$$\begin{aligned} \text{Q2d } z^4 + 16 &= (z - z_1)(z - z_2)(z - z_3)(z - z_4) \\ &= (z - \sqrt{2} - i\sqrt{2})(z + \sqrt{2} - i\sqrt{2})(z + \sqrt{2} + i\sqrt{2})(z - \sqrt{2} + i\sqrt{2}) \end{aligned}$$



$$\begin{aligned} \text{Q2f Area of shaded segment} &= \text{area of sector} - \text{area of triangle} \\ &= \text{area of quarter circle} - \text{area of triangle} \\ &= \frac{1}{4} \times \pi \times 2^2 - \frac{1}{2} \times 2 \times 2 = \pi - 2 \text{ square units} \end{aligned}$$

$$\begin{aligned} \text{Q3a } \log_e(N) &= 6 - 3e^{-0.4t}, t \geq 0 \\ \frac{d}{dt} \log_e(N) &= \frac{d}{dt} (6 - 3e^{-0.4t}), \frac{d}{dN} \log_e(N) \times \frac{dN}{dt} = \frac{d}{dt} (6 - 3e^{-0.4t}) \\ \therefore \frac{1}{N} \frac{dN}{dt} &= 1.2e^{-0.4t} \\ \therefore \frac{1}{N} \frac{dN}{dt} + 0.4 \log_e(N) - 2.4 &= 1.2e^{-0.4t} + 0.4(6 - 3e^{-0.4t}) - 2.4 = 0 \end{aligned}$$

$$\text{Q3b When } t = 0, \log_e(N) = 3, N = e^3 \approx 20$$

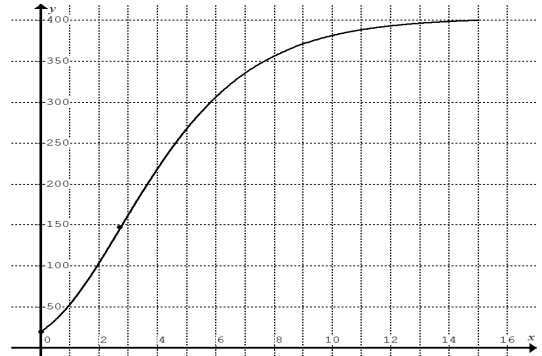
$$\text{Q3c When } t \rightarrow \infty, \log_e(N) \rightarrow 6, N \rightarrow e^6 \approx 403$$

$$\begin{aligned} \text{Q3di } \frac{dN}{dt} &= 0.4N(6 - \log_e(N)) \\ \frac{d^2N}{dt^2} &= 0.4 \frac{dN}{dt} (6 - \log_e(N)) + 0.4N \left(-\frac{1}{N} \right) \frac{dN}{dt} \\ &= \frac{dN}{dt} (2 - 0.4 \log_e(N)) = 0.4 \frac{dN}{dt} (5 - \log_e(N)) \\ &= 0.16N(6 - \log_e(N))(5 - \log_e(N)) \end{aligned}$$

$$\text{Q3dii Let } \frac{d^2N}{dt^2} = 0 \text{ to find the point of inflection.}$$

$$5 - \log_e(N) = 0, N = e^5 \approx 148 \text{ when } t \approx 2.7$$

Q3e



$$\begin{aligned} \text{Q4a } \tilde{b} &= \tilde{i} + \sqrt{3}\tilde{j} + 2\sqrt{3}\tilde{k}, |\tilde{b}| = \sqrt{1^2 + (\sqrt{3})^2 + (2\sqrt{3})^2} = 4 \\ \hat{b} &= \frac{\tilde{b}}{|\tilde{b}|} = \frac{1}{4} (\tilde{i} + \sqrt{3}\tilde{j} + 2\sqrt{3}\tilde{k}) \end{aligned}$$

Q4b Component of $\tilde{a}$ parallel to $\tilde{b}$:

$$\begin{aligned} (\tilde{a} \cdot \tilde{b}) \hat{b} &= \left(-\frac{4\sqrt{3}}{3} \right) \frac{1}{4} (\tilde{i} + \sqrt{3}\tilde{j} + 2\sqrt{3}\tilde{k}) \\ &= -\frac{\sqrt{3}}{3} (\tilde{i} + \sqrt{3}\tilde{j} + 2\sqrt{3}\tilde{k}) = -\left(\frac{\sqrt{3}}{3} \tilde{i} + \tilde{j} + 2\tilde{k} \right) \end{aligned}$$

Component of $\tilde{a}$ perpendicular to $\tilde{b}$:

$$\begin{aligned} \tilde{a} - (\tilde{a} \cdot \tilde{b}) \hat{b} &= \left(-\frac{7\sqrt{3}}{3} \tilde{i} + \tilde{j} - 2\tilde{k} \right) + \left(\frac{\sqrt{3}}{3} \tilde{i} + \tilde{j} + 2\tilde{k} \right) \\ &= -2\sqrt{3}\tilde{i} + 2\tilde{j} \end{aligned}$$

$$\text{Q4c } \tilde{c} \cdot \tilde{b} = |\tilde{c}| |\tilde{b}| \cos \frac{2\pi}{3}$$

$$m + \sqrt{3} - 4\sqrt{3} = \sqrt{m^2 + 1 + 4} \times 4 \times \left(-\frac{1}{2} \right)$$

$$\therefore 3m^2 + 6\sqrt{3}m - 7 = 0 \text{ and given } \tilde{c} \neq \tilde{a}, \text{ i.e. } m \neq -\frac{7\sqrt{3}}{3}$$

$$\therefore m = \frac{\sqrt{3}}{3}$$

$$\text{Q4d } \tilde{c} \cdot \tilde{a} = |\tilde{c}| |\tilde{a}| \cos \theta, \cos \theta = \frac{\tilde{c} \cdot \tilde{a}}{|\tilde{c}| |\tilde{a}|} = \frac{1}{4}, \therefore \theta \approx 75.5^\circ$$

$$\text{Q4ei } \overrightarrow{AN} = \tilde{u} + \frac{1}{2} \tilde{v}$$

$$\text{Q4eii } \overrightarrow{CM} = \overrightarrow{CA} + \frac{1}{2} \overrightarrow{AB} = -\tilde{u} + \frac{1}{2} (\tilde{u} + \tilde{v}) = \frac{1}{2} \tilde{v} - \frac{1}{2} \tilde{u}$$

$$\overrightarrow{BP} = -\frac{1}{2} \tilde{u} - \tilde{v}$$

Q4eiii

$$\overrightarrow{AN} + \overrightarrow{CN} + \overrightarrow{BP} = \left(\tilde{u} + \frac{1}{2} \tilde{v} \right) + \left(\frac{1}{2} \tilde{v} - \frac{1}{2} \tilde{u} \right) + \left(-\frac{1}{2} \tilde{u} - \tilde{v} \right) = \tilde{0}$$

Q5a

$$\tilde{r}(t) = 7.5\tilde{i} + \left(50 - 10\sin\left(\frac{\pi}{6}\right)\right)\tilde{j}, \quad \dot{\tilde{r}}(t) = 7.5\tilde{i} - \frac{5\pi}{3}\cos\left(\frac{\pi}{6}\right)\tilde{j}$$

$$\text{Speed} = |\dot{\tilde{r}}(t)| = \sqrt{7.5^2 + \left(-\frac{5\pi}{3}\cos\left(\frac{\pi}{6}\right)\right)^2} \approx \sqrt{56.25 + 27.42\cos^2\left(\frac{\pi}{6}\right)}$$

$$\text{Minimum speed} = \sqrt{56.25} = 7.5 \text{ m s}^{-1}$$

$$\text{Maximum speed} \approx \sqrt{56.25 + 27.42} \approx 9.1 \text{ m s}^{-1}$$

Q5b Let $\ddot{r}(t) = \frac{5\pi^2}{18}\sin\left(\frac{\pi}{6}\right)\tilde{j} = \tilde{0}$, $\therefore \frac{\pi}{6} = 0, \pi, 2\pi, 3\pi, \dots$

$\therefore t = 0, 6, 12, 18, \dots$ i.e. $t = 6n$ where n is an integer ≥ 0 .

Q5c Vertical component of initial velocity $= 15\sin 30^\circ = 7.5$

Height of ramp $= 10\sin 30^\circ = 5$

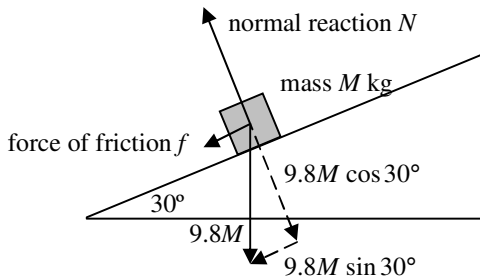
$$u = +7.5, \quad s = -5, \quad a = -9.8, \quad s = ut + \frac{1}{2}at^2$$

$$\therefore 4.9t^2 - 7.5t - 5 = 0, \quad t \approx 2.03 \text{ s}$$

Q5d Horizontal component of initial velocity $= 15\cos 30^\circ \approx 12.99$

Distance $\approx 12.99 \times 2.03 \approx 26 \text{ m}$

Q5e



$$N = 9.8M \cos 30^\circ, \quad f = \mu N = \frac{1}{8\sqrt{3}} \times 9.8M \cos 30^\circ = \frac{9.8M}{16}$$

$$a = \frac{\text{net. force}}{\text{mass}} = \frac{-9.8M \sin 30^\circ - \frac{9.8M}{16}}{M} = -5.5125$$

$$u = +10, \quad v = 0, \quad a = -5.5125, \quad v^2 = u^2 + 2as, \quad s \approx +9.1 \text{ m}$$

$$\therefore \text{distance up the plane} = 9.1 \text{ m}$$

Q5f Friction required $= 9.8M \sin 30^\circ = 4.9M$

$$f = \mu N, \quad 4.9M = \mu \times 9.8M \cos 30^\circ, \quad 4.9M = \mu \times 9.8M \times \frac{\sqrt{3}}{2}$$

$$\mu = \frac{1}{\sqrt{3}} \text{ which is } \mu = \tan 30^\circ.$$

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