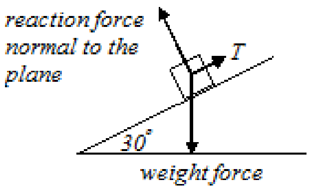
2013 VCAA Specialist Mathematics Exam 1 Solutions

Q1a



Q1b $weight = 10\times 9.8 = 98$ N, $T = 98\sin 30° = 49$ N

Q2 $\int_0^1 \frac{x-5}{x^2-5x+6}dx = \int_0^1 \frac{x-5}{(x-2)(x-3)}dx = \int_0^1\left(\frac{3}{x-2}-\frac{2}{x-3}\right)dx$

$= \left[3\log_e|x-2| - 2\log_e|x-3|\right]_0^1$

$= 3\log_e|-1| - 2\log_e|-2| - 3\log_e|-2| + 2\log_e|-3| = \log_e\left(\frac{9}{32}\right)$

Q3a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(\tilde{i}+5\tilde{k}\right) - \left(-\tilde{i}+2\tilde{j}+4\tilde{k}\right) = 2\tilde{i} - 2\tilde{j} + \tilde{k}$

Q3b

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(3\tilde{i}+5\tilde{j}+2\tilde{k}\right) - \left(-\tilde{i}+2\tilde{j}+4\tilde{k}\right) = 4\tilde{i}+3\tilde{j}-2\tilde{k}$

$\overrightarrow{AB}.\overrightarrow{AC} = 8-6-2 = 0$, .: $\overrightarrow{AB}\perp\overrightarrow{AC}$, $\angle BAC = 90°$

Q3c $\overrightarrow{BC} = 2\tilde{i}+5\tilde{j}-3\tilde{k}$, $\left|\overrightarrow{BC}\right| = \sqrt{2^2+5^2+(-3)^2} = \sqrt{38}$

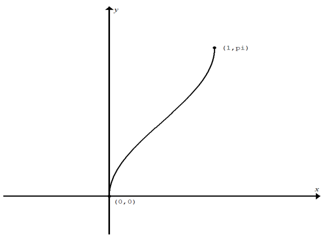
Q4a $-1\le 1-2x\le 1$, $-2\le -2x\le 0$, $0\le 2x\le 2$, $0\le x\le 1$

Maximal domain is $[0,1]$.

When $x=0$, $y=\cos^{-1}1 = 0$; when $x=1$, $y=\cos^{-1}(-1)=\pi$

Range is $[0,\pi]$.

Q4b



Q4c $y=\cos^{-1}(1-2x)$, $\frac{dy}{dx} = \frac{-1(-2)}{\sqrt{1-(1-2x)^2}} = \frac{2}{\sqrt{1-(1-2x)^2}}$

At $x=\frac{1}{4}$, $m_t = \frac{2}{\sqrt{1-\left(1-\frac{1}{2}\right)^2}} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$

Q5a $\frac{dT}{dt} = -k(T-20)$, $T>20$

$\frac{dt}{dT} = -\frac{1}{k}.\frac{1}{T-20}$, $t = -\frac{1}{k}\int\frac{1}{T-20}dT$, $-kt = \log_e(T-20)+c$

When $t=0$, $T=100$, .: $c=-\log_e 80$, .: $-kt = \log_e\left(\frac{T-20}{80}\right)$

When $t=5$, $T=80$, $-5k = \log_e\frac{3}{4}$, .: $e^{-5k} = \frac{3}{4}$

Q5b When $t=10$, $-10k = \log_e\left(\frac{T-20}{80}\right)$, $\frac{T-20}{80} = e^{-10k}$

$\frac{T-20}{80} = \left(e^{-5k}\right)^2$, $\frac{T-20}{80} = \left(\frac{3}{4}\right)^2$, $T = \left(\frac{3}{4}\right)^2\times 80+20 = 65$

Q6 $y^2 + \frac{3e^{x-1}}{x-2} = c$ …… (1)

By implicit differentiation, $2y\frac{dy}{dx} + \frac{(x-2)3e^{x-1} - 3e^{x-1}}{(x-2)^2} = 0$

.: $\frac{dy}{dx} = -\frac{3e^{x-1}(x-3)}{2y(x-2)^2}$, .: when $x=1$, $2=\frac{3}{y}$, .: $y=\frac{3}{2}$

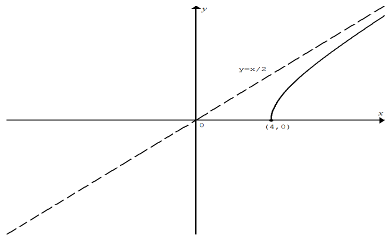
Substitute $x=1$ and $y=\frac{3}{2}$ in (1), $c=-\frac{3}{4}$

Q7a $\tilde{r} = 4\sec(t)\tilde{i} + 2\tan(t)\tilde{j}$. Let $\frac{x}{4} = \sec(t)$ and $\frac{y}{2} = \tan(t)$.

Since $\sec^2(t) - \tan^2(t) = 1$, .: $\frac{x^2}{16} - \frac{y^2}{4} = 1$, and $0\le t<\frac{\pi}{2}$

.: $x\ge 4$ and $y\ge 0$

Q7b



Q7c $\tilde{v}(t) = \frac{d\tilde{r}}{dt} = 4\sec(t)\tan(t)\tilde{i} + 2\sec^2(t)\tilde{j}$, .:

$\tilde{v}\left(\frac{\pi}{4}\right) = \left(4\sqrt{2}\right)\tilde{i} + 4\tilde{j}$, $speed = |\tilde{v}| = \sqrt{32+16} = \sqrt{48} = 4\sqrt{3}$ m s^{-1}

Q8 $z^4 - 2z^2 + 4 = 0$, $z^4 - 4z^2 + 4 + 2z^2 = 0$,

$\left(z^2-2\right)^2 - \left(i\sqrt{2}z\right)^2 = 0$, $\left(z^2 - i\sqrt{2}z - 2\right)\left(z^2 + i\sqrt{2}z - 2\right) = 0$

.: $z^2 - i\sqrt{2}z - 2 = 0$ or $z^2 + i\sqrt{2}z - 2 = 0$

By the quadratic formula: $z = \frac{1}{2}\left(\pm\sqrt{6} + i\sqrt{2}\right)$, $\frac{1}{2}\left(\pm\sqrt{6} - i\sqrt{2}\right)$

Q9 Outer cone: $V = \frac{1}{3}\pi\times\pi^2\times\frac{\pi}{3} = \frac{\pi^4}{9}$.

Inner void: $V = \int_0^{\frac{\pi}{3}}\pi\sin^2 x dx = \frac{\pi}{2}\int_0^{\frac{\pi}{3}}(1-\cos 2x)dx$

$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{3}} = \frac{\pi^2}{6} - \frac{\sqrt{3}\pi}{8}$

.: $solid\ volume = \frac{\pi^4}{9} - \frac{\pi^2}{6} + \frac{\sqrt{3}\pi}{8}$ unit cubes

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors