

2013 Specialist Mathematics Trial Exam 1 Solutions

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Q1a $-\frac{\pi}{2} \leq \frac{1-y}{2} \leq \frac{\pi}{2}, -\pi \leq 1-y \leq \pi, -1-\pi \leq -y \leq -1+\pi$

$1+\pi \geq y \geq 1-\pi, 1-\pi \leq y \leq 1+\pi.$

The range is $[1-\pi, 1+\pi]$.

Q1b $\sin \frac{1-y}{2} = 1-2x, \frac{1-y}{2} = \sin^{-1}(1-2x),$

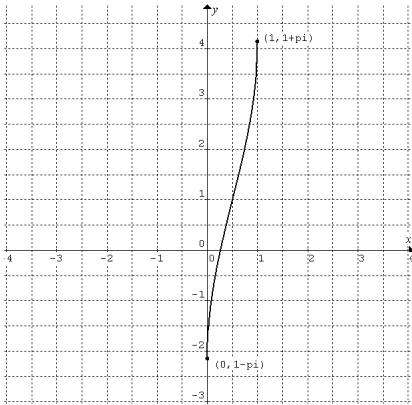
$y = 1 - 2\sin^{-1}(1-2x), \therefore -1 \leq 1-2x \leq 1, 0 \leq x \leq 1$

The maximal domain is $[0, 1]$.

Q1c When $x = \frac{1}{4}, y = 1 - 2\sin^{-1}\left(1 - 2\left(\frac{1}{4}\right)\right),$

$y = 1 - 2\sin^{-1}\left(\frac{1}{2}\right) = 1 - 2 \times \frac{\pi}{6} = 1 - \frac{\pi}{3}$

Q1d

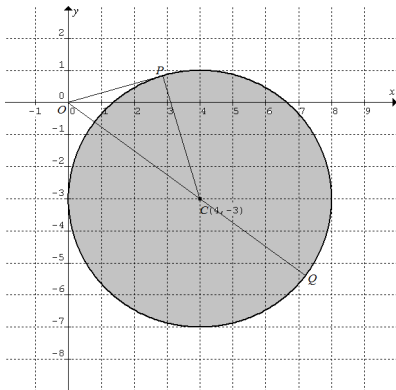


Q2 $2z^3 - iz^2 + 4z - 2i = 0, (2z^3 - iz^2) + (4z - 2i) = 0,$

$z^2(2z - i) + 2(2z - i) = 0, (z^2 + 2)(2z - i) = 0,$

$\therefore z = \pm\sqrt{2}i, \frac{i}{2}$

Q3a $S = \{z : 8 \geq |2z + 6i - 8|\}, \text{ i.e. } S = \{z : |z - (4 - 3i)| \leq 4\},$ a circle of radius 4 centred at $(4, -3)$. Set S is the shaded region.

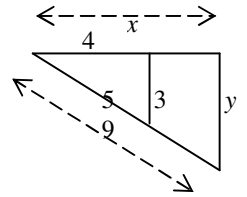


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Q3b Maximum $|z|$ occurs at point Q in the diagram.

$|z| = OQ = \sqrt{3^2 + 4^2} + 4 = 9$

Let $z = x - iy.$



$\frac{y}{9} = \frac{3}{5}, \therefore y = \frac{27}{5}$

$\frac{x}{9} = \frac{4}{5}, \therefore x = \frac{36}{5} \therefore z = \frac{36}{5} - \frac{27}{5}i$

Q3c Maximum value of $\text{Arg}(z)$ occurs at point P (refer to the diagram). OP is a tangent to the circle.

$\triangle POC$ is a 3, 4, 5 right-angled triangle.

$\angle POC = \tan^{-1}\left(\frac{4}{3}\right), \angle xOC = \tan^{-1}\left(\frac{3}{4}\right)$

$\text{Arg}(z) = \angle POx = \angle POC - \angle xOC$

$\tan \angle POx = \tan(\angle POC - \angle xOC) = \frac{\tan \angle POC - \tan \angle xOC}{1 + \tan \angle POC \times \tan \angle xOC}$
 $= \frac{\frac{4}{3} - \frac{3}{4}}{1 + \frac{4}{3} \times \frac{3}{4}} = \frac{7}{24}$

$\therefore \text{Arg}(z) = \angle POx = \tan^{-1}\left(\frac{7}{24}\right).$

Q4a $\overrightarrow{DG} = -\tilde{i} + 3\tilde{j} - 2\tilde{k}$

Q4bi Let θ be the angle between $\overrightarrow{DG}$ and $\overrightarrow{FC}.$

$\overrightarrow{DG} \cdot \overrightarrow{FC} = |\overrightarrow{DG}| |\overrightarrow{FC}| \cos \theta$

$\cos \theta = \frac{\overrightarrow{DG} \cdot \overrightarrow{FC}}{|\overrightarrow{DG}| |\overrightarrow{FC}|} = \frac{-1 + 9 - 4}{\sqrt{1+9+4} \sqrt{1+9+4}} = \frac{2}{7}$

$\therefore \theta = \cos^{-1}\left(\frac{2}{7}\right)$

Q4bii Scalar resolute $= |\overrightarrow{DG}| \cos \theta = \sqrt{14} \times \frac{2}{7} = \frac{2\sqrt{14}}{7}$

Q5 $\tilde{p}, \tilde{q}$ and $\tilde{r}$ are linearly **dependent** when $a\tilde{p} + b\tilde{q} + \tilde{r} = \tilde{0}$ where $a \neq 0$ and $b \neq 0.$

$a(m\tilde{i} - \tilde{j}) + b(m\tilde{j} + \tilde{k}) + (\tilde{i} - 8m\tilde{k}) = \tilde{0}$

$\therefore (am+1)\tilde{i} + (bm-a)\tilde{j} + (b-8m)\tilde{k} = \tilde{0}$

$\therefore am+1=0, bm-a=0 \text{ and } b-8m=0$

Solve the three equations simultaneously, $8m^3 + 1 = 0$

$\therefore m = -\frac{1}{2}$

$\therefore \tilde{p}, \tilde{q}$ and $\tilde{r}$ are linearly **independent** when $m \neq -\frac{1}{2},$

i.e. $m \in \mathbb{R} \setminus \left\{-\frac{1}{2}\right\}$

Q6a $\tilde{r} = t\tilde{i} + \sqrt{3}t\tilde{j} - 4.9(1-t)^2\tilde{k}$

$\tilde{v} = \frac{d}{dt}\tilde{r} = \tilde{i} + \sqrt{3}\tilde{j} + 9.8(1-t)\tilde{k}$

Horizontal speed $= \sqrt{1^2 + (\sqrt{3})^2} = 2 \text{ m s}^{-1}$

Q6b Speed $= |\tilde{v}| = \sqrt{1^2 + (\sqrt{3})^2 + 9.8^2(1-t)^2}$

It is a minimum value at $t = 1$, $\therefore$ minimum speed $= 2 \text{ m s}^{-1}$

Q6c Acceleration $= \frac{d}{dt}\tilde{v} = -9.8\tilde{k}$ which is constant.

Q7a $f'(x) = \frac{16 \tan^{-1} x}{1+x^2}$, $f(x) = \int \frac{16 \tan^{-1} x}{1+x^2} dx$

Let $u = \tan^{-1} x$, $\frac{du}{dx} = \frac{1}{1+x^2}$

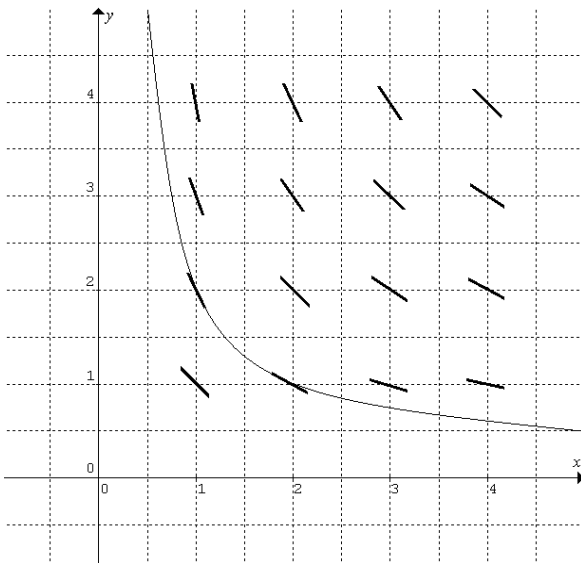
$\therefore f(x) = \int 16u \frac{du}{dx} dx = \int 16udu = 8u^2 + c = 8(\tan^{-1} x)^2 + c$

Q7b $\tan^{-1} x$ is an odd function, and $\frac{1}{1+x^2}$ is an even function,

$\therefore f'(x) = \frac{16 \tan^{-1} x}{1+x^2}$ is an odd function.

Area $= -\int_{-1}^0 f'(x) dx + \int_0^1 f'(x) dx = 2 \int_0^1 f'(x) dx = 2[f(x)]_0^1$
 $= 2[8(\tan^{-1} x)^2]_0^1 = \pi^2$

Q8a Q8b



Q9a $|reaction| = |weight| = mg = 5 \times 9.8 = 49 \text{ N}$
 (Newton's third law)

Q9b $|friction| = 49 \sin 30^\circ = 24.5 \text{ N}$

Q10a $u = -5$, $v = +10$, $t = 10$, $s = \frac{1}{2}(u+v)t = +25$

$\therefore$ position $x = -15 + +25 = +10 \text{ m}$

Q10b Displacement:

$t = 0$ to $t = 10$, $s = +25$

$t = 10$ to $t = 20$, $s = \frac{1}{2}(+10 + +15) \times 10 = +125$

$t = 20$ to $t = 30$, $s = \frac{1}{2}(+15) \times 10 = +75$

Total displacement $= +25 + +125 + +75 = +225 \text{ m}$

Average velocity $= \frac{\text{displacement}}{\text{time}} = \frac{+225}{30} = +7.5 \text{ m s}^{-1}$

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