



INSIGHT

YEAR 12 Trial Exam Paper

2013

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocations
- tips on how to approach the questions

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Question 1a.**Worked solution**

$$z^4 + 2z^2 = 8$$

$$z^4 + 2z^2 - 8 = 0$$

$$(z^2 + 4)(z^2 - 2) = 0$$

$$(z^2 - 4i^2)(z^2 - 2) = 0$$

$$(z + 2i)(z - 2i)(z + \sqrt{2})(z - \sqrt{2}) = 0$$

$$\therefore z = \pm 2i \text{ or } z = \pm \sqrt{2}$$

Mark allocation: 2 marks

- 1 mark for factorising correctly.
- 1 mark for the correct answer.

**Tip**

- *The quadratic formula in z^2 could also have been used to solve the equation.*

Question 1b.**Worked solution**

$$z_1 = 2 \operatorname{cis}\left(\frac{\pi}{4}\right)$$

$$z_2 = \operatorname{cis}\left(\frac{-3\pi}{4}\right)$$

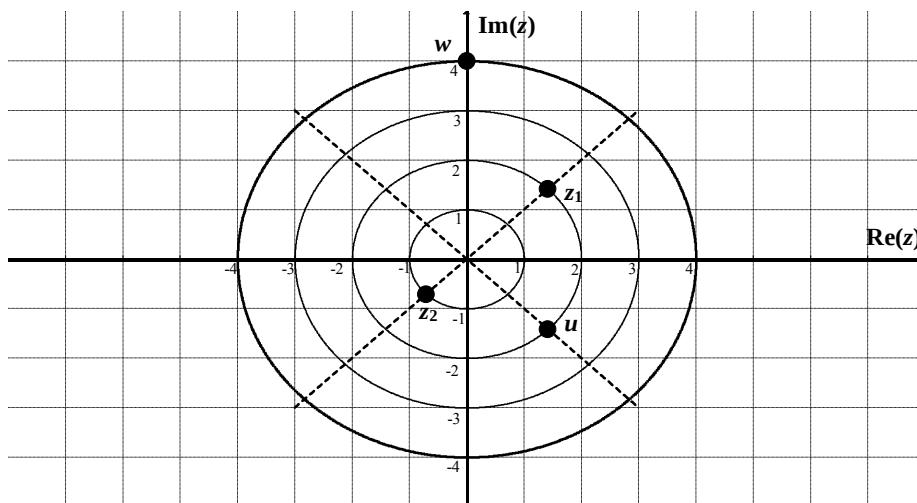
$$u = \frac{z_1}{z_2^2} = \frac{2 \operatorname{cis}\left(\frac{\pi}{4}\right)}{\left(\operatorname{cis}\left(\frac{-3\pi}{4}\right)\right)^2}$$

$$\Rightarrow u = 2 \operatorname{cis}\left(\frac{7\pi}{4}\right) = 2 \operatorname{cis}\left(\frac{-\pi}{4}\right)$$

$$w = i^2(\bar{z}_1)^2 = i^2 \times \left(2 \operatorname{cis}\left(\frac{-\pi}{4}\right)\right)^2$$

$$= \operatorname{cis}(\pi) \times 4 \operatorname{cis}\left(\frac{-\pi}{2}\right)$$

$$\therefore w = 4 \operatorname{cis}\left(\frac{\pi}{2}\right)$$

**Mark allocation: 2 marks**

- 1 mark for the correct position of u .
- 1 mark for the correct position of w .

**Tip**

- The point w can also be obtained by rotating the point $(\bar{z}_1)^2 = 4 \operatorname{cis}\left(\frac{-\pi}{2}\right)$, 180° anticlockwise.

Question 2a.**Worked solution**

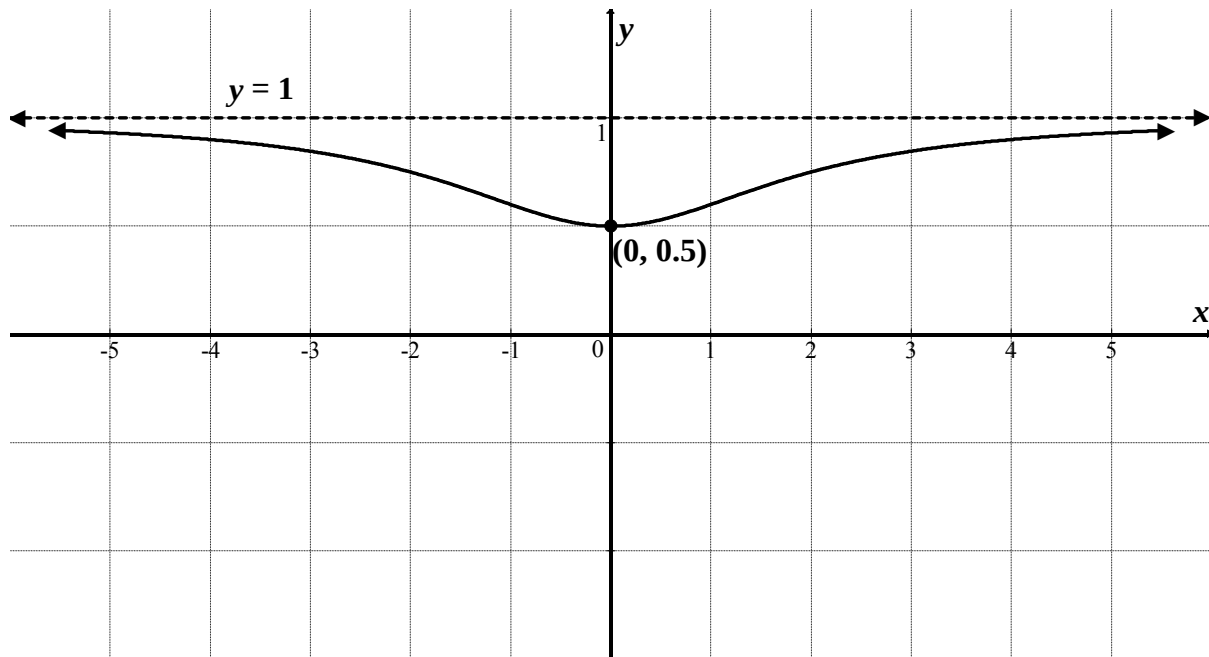
$f(x) = 1 - \frac{2}{x^2 + 4}$ has a horizontal asymptote $y = 1$.

$$f(0) = 1 - 0.5 = 0.5$$

The point $(0, 0.5)$ is also a local minimum.

As $x \rightarrow \infty$, $y \rightarrow 1$.

As $x \rightarrow -\infty$, $y \rightarrow 1$.

**Mark allocation: 3 marks**

- 1 mark for the correct asymptote.
- 1 mark for the correct y -intercept.
- 1 mark for the correct curve.

**Tip**

- The graph of $y = f(x)$ can also be obtained by first sketching $y = x^2 + 4$, then sketching its reciprocal $y = \frac{1}{x^2 + 4}$, then reflecting through the x -axis for $y = \frac{-1}{x^2 + 4}$, then dilating in the vertical by factor 2 for $y = \frac{-2}{x^2 + 4}$ and, finally, translating 1 unit up for $y = f(x)$.

Question 2b.**Worked solution**

$$\begin{aligned}
 \text{Area} &= \int_0^2 \left(1 - \frac{2}{x^2 + 4} \right) dx \\
 &= \left[x - \tan^{-1} \left(\frac{x}{2} \right) \right]_0^2 \\
 &= (2 - \tan^{-1}(1)) - (0 - \tan^{-1}(0)) \\
 &= 2 - \frac{\pi}{4} \\
 &= \frac{8 - \pi}{4}
 \end{aligned}$$

$\therefore$ The area enclosed is $\left(\frac{8 - \pi}{4} \right)$ square units.

Mark allocation: 3 marks

- 1 mark for correctly setting up the integral representing the required area.
- 1 mark for anti-differentiating correctly.
- 1 mark for the correct answer.

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Question 3**Worked solution**

$$\int_0^{e-1} \frac{\sin(\pi \log_e(x+1))}{x+1} dx$$

Let $u = \log_e(x+1)$

When $x = e-1 \Rightarrow u = \log_e e = 1$

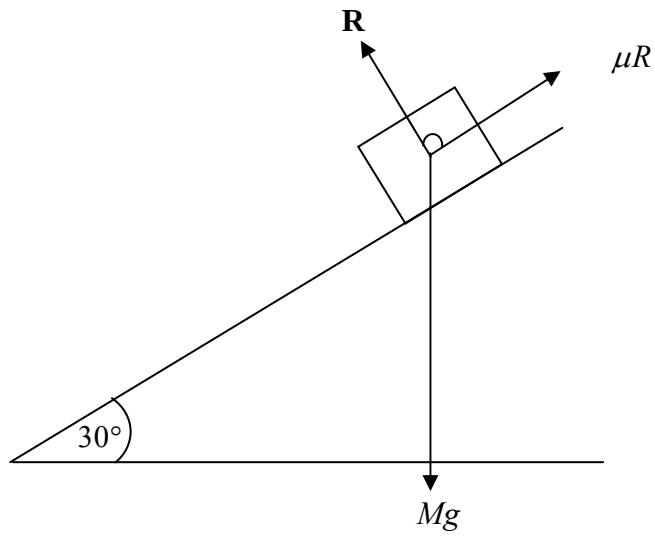
When $x = 0 \Rightarrow u = \log_e 1 = 0$

$$\frac{du}{dx} = \frac{1}{x+1}$$

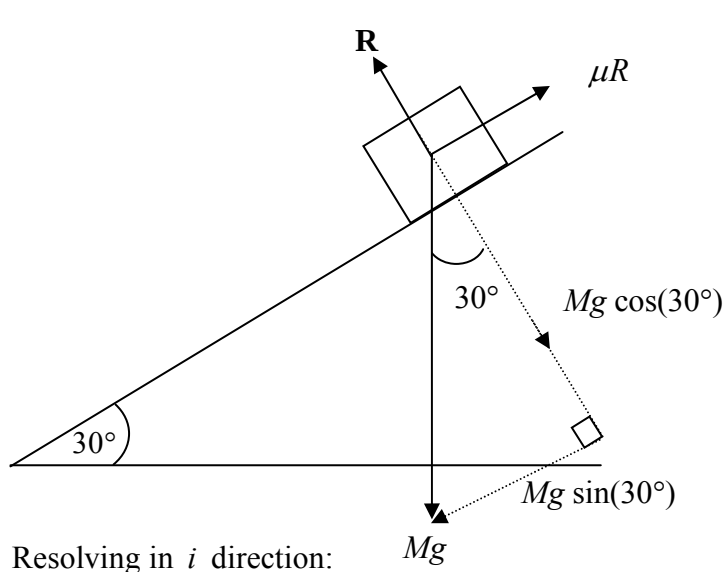
$$\begin{aligned} & \int_0^{e-1} \frac{\sin(\pi \log_e(x+1))}{x+1} dx \\ &= \int_0^1 \sin(\pi u) \frac{du}{dx} dx \\ &= \int_0^1 \sin(\pi u) du \\ &= \frac{-1}{\pi} [\cos(\pi u)]_0^1 \\ &= \frac{-1}{\pi} [\cos(\pi) - \cos(0)] \\ &= \frac{-1}{\pi} (-2) \\ &= \frac{2}{\pi} \end{aligned}$$

Mark allocation: 3 marks

- 1 mark for using the correct substitution.
- 1 mark for correctly expressing the integral in terms of u .
- 1 mark for the correct answer.

Question 4a.**Worked solution****Mark allocation**

- 1 mark for showing all the forces correctly

Question 4b.**Worked solution**

Resolving in $\hat{i}$ direction:

$$Mg \sin(30^\circ) - \mu R = Ma$$

Resolving in $\hat{j}$ direction:

$$R = Mg \cos(30^\circ)$$

$$\therefore \frac{Mg}{2} - \mu Mg \frac{\sqrt{3}}{2} = Ma$$

$$\frac{g}{2} - \mu g \frac{\sqrt{3}}{2} = a$$

$$\text{So } a = \frac{g}{2}(1 - \mu\sqrt{3})$$

Mark allocation: 2 marks

- 1 mark for resolving in both $\hat{i}$ and $\hat{j}$ direction
- 1 mark for correct answer

Question 4c.**Worked solution**

$$v^2 = u^2 + 2as$$

$$9 = 0 + 2 \times \frac{g}{2} (1 - \mu\sqrt{3}) \times 3$$

$$\text{So } 9 = 3g(1 - \mu\sqrt{3})$$

$$3 = g - \mu g\sqrt{3}$$

$$\mu g\sqrt{3} = g - 3$$

$$\therefore \mu = \frac{g-3}{\sqrt{3}g}$$

Mark allocation: 2 marks

- 1 mark for determining $9 = 0 + 2 \times \frac{g}{2} (1 - \mu\sqrt{3}) \times 3$
- 1 mark for correct answer

Question 5a.**Worked solution**

$$a = \frac{d}{dx} \left(\frac{v^2}{2} \right) = x + 1$$

$$\frac{v^2}{2} = \int (x+1) dx$$

$$= \frac{x^2}{2} + x + c$$

$$x = 0, v = 1$$

$$\therefore c = \frac{1^2}{2} = \frac{1}{2}$$

$$\frac{v^2}{2} = \frac{x^2}{2} + x + \frac{1}{2}$$

$$v^2 = x^2 + 2x + 1$$

$$v^2 = (x+1)^2$$

$$\Rightarrow v = |x+1|, \text{ but as } v = 1 \text{ when } x = 0$$

$$\therefore v = x + 1$$

Mark allocation: 3 marks

- 1 mark for using the appropriate expression for acceleration.
- 1 mark for correctly integrating and substituting to obtain the expression for v^2 .
- 1 mark for justifying that $v = |x+1| = x+1$.

Question 5b.**Worked solution**

$$v = \frac{dx}{dt} = x + 1$$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{x+1}$$

$$t = \int \frac{1}{x+1} dx$$

$$t = \log_e |x+1| + c$$

$$t = 0, \quad x = 0$$

$$\Rightarrow c = 0$$

$$\Rightarrow t = \log_e |x+1|$$

$$|x+1| = e^t$$

$$\Rightarrow x+1 = \pm e^t$$

$$x = \pm e^t - 1$$

$$\therefore x = e^t - 1 \text{ when } t = 0, \quad x = 0.$$

Mark allocation: 2 marks

- 1 mark for correctly expressing t as a function of x .
- 1 mark for the correct answer.

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Question 6**Worked solution**

$$y^2 = 4 - 4(x-1)^2$$

$$\frac{y^2}{4} = 1 - (x-1)^2$$

$$\Rightarrow \frac{(x-1)^2}{1} + \frac{y^2}{4} = 1$$

This relation represents an ellipse with centre $(1, 0)$, $a = 1$ and $b = 2$.

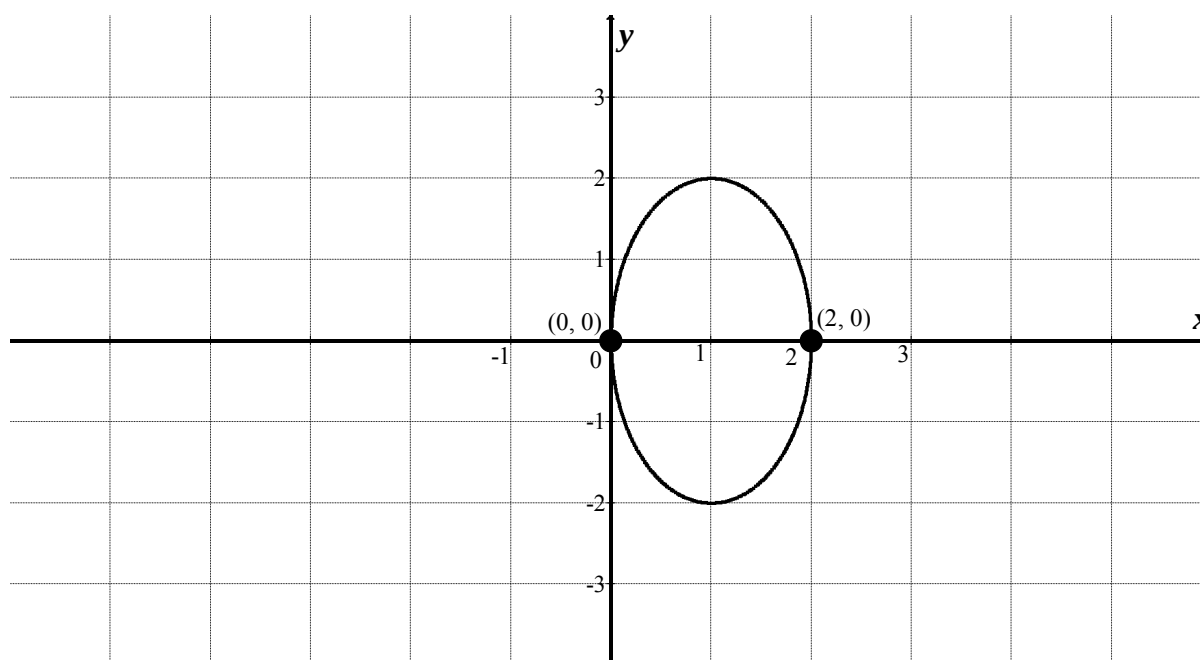
$$x = 0, y^2 = 4 - 4 = 0$$

$$\Rightarrow y = 0 \text{ only } (0, 0)$$

$$y = 0, (x-1)^2 = 1$$

$$\Rightarrow x-1 = \pm 1$$

$$\Rightarrow x = 0 \text{ or } x = 2 \quad (0, 0) \text{ and } (2, 0)$$

**Mark allocation: 2 marks**

- 1 mark for correctly expressing the relation in standard ellipse form.
- 1 mark for the correct graph.

Question 7a.**Worked solution**

$$\begin{aligned}
 \sec(t) &= 2(x - y) \\
 \Rightarrow \sec^2(t) &= 4(x - y)^2 \\
 \tan^2(t) &= y - 2 \\
 \Rightarrow 1 + \tan^2(t) &= y - 1 \\
 1 + \tan^2(t) &= \sec^2(t) \\
 \Rightarrow y - 1 &= 4(x - y)^2 \\
 y - 1 &= 4x^2 - 8xy + 4y^2 \\
 \therefore y &= 4x^2 - 8xy + 4y^2 + 1
 \end{aligned}$$

Mark allocation: 2 marks

- 1 mark for using the identity $1 + \tan^2(t) = \sec^2(t)$
- 1 mark for simplifying the cartesian equation.

**Tip**

- Use the identity $1 + \tan^2(t) = \sec^2(t)$.

Question 7b.**Worked solution**

$$t = \frac{\pi}{3}, \quad y = 2 + \tan^2\left(\frac{\pi}{3}\right)$$

$$y = 2 + (\sqrt{3})^2 = 5$$

$$x - y = 0.5 \sec(t)$$

$$\Rightarrow x - 5 = 0.5 \div \cos\left(\frac{\pi}{3}\right) = 0.5 \div 0.5 = 1$$

$$x = 6$$

$\therefore$ The coordinate of the particle is (6, 5).

Mark allocation: 2 marks

- 1 mark for evaluating x correctly.
- 1 mark for evaluating y correctly.

Question 7c.**Worked solution**

$$y = 4x^2 - 8xy + 4y^2 + 1$$

Differentiate both sides with respect to x .

$$\frac{dy}{dx} = 8x - 8y - 8x \frac{dy}{dx} + 8y \frac{dy}{dx}$$

$$\frac{dy}{dx}(1 + 8x - 8y) = 8x - 8y$$

$$\frac{dy}{dx} = \frac{8x - 8y}{1 + 8x - 8y}$$

$$\Rightarrow \text{At } (6, 5), \quad \frac{dy}{dx} = \frac{48 - 40}{1 + 48 - 40} = \frac{8}{9}$$

Mark allocation: 2 marks

- 1 mark for differentiating correctly.
- 1 mark for the correct answer.

**Tip**

- Although the relation can be expressed explicitly as a function of y , it is easier to obtain $\frac{dy}{dx}$ using implicit differentiation.

Question 8a.**Worked solution**

$$\vec{OA} = 2\vec{i} + 2\vec{j} + \vec{k}$$

$$\vec{OB} = 3\vec{i} + 4\vec{k}$$

$$\vec{OC} = 5\vec{i} - 4\vec{j} + 10\vec{k}$$

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$\Rightarrow \vec{AB} = \vec{i} - 2\vec{j} + 3\vec{k}$$

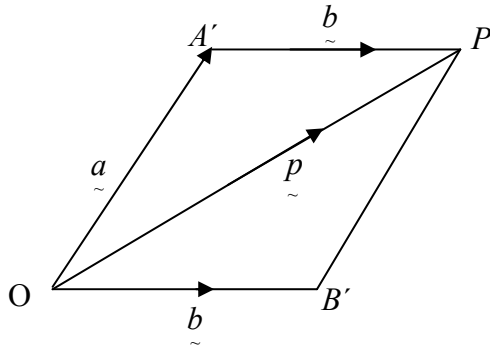
$$\Rightarrow \vec{BC} = 2\vec{i} - 4\vec{j} + 6\vec{k}$$

$$\Rightarrow \vec{BC} = 2\vec{AB}$$

$\therefore A, B$ and C are collinear, as B is common to $\vec{AB}$ and $\vec{BC}$.

Mark allocation: 2 marks

- 1 mark for correctly obtaining the vectors $\vec{AB}$ and $\vec{BC}$
- 1 mark for recognising that $\vec{AB}$ and $\vec{BC}$ are parallel.

Question 8b.**Worked solution**

$OA'PB'$ is a rhombus as $a = b$ because $\underline{a}$ and $\underline{b}$ are both unit vectors.

$\therefore \underline{p}$ bisects the angle $A'OB'$.

$$\underline{p} = \underline{a} + \underline{b}$$

$$\underline{p} = \frac{1}{15}(10\underline{i} + 10\underline{j} + 5\underline{k}) + \frac{1}{15}(9\underline{i} + 12\underline{k})$$

$\therefore \underline{p} = \frac{1}{15}(19\underline{i} + 10\underline{j} + 17\underline{k})$ is a vector that bisects the angle between $\vec{OA}$ and $\vec{OB}$.

or $\underline{p} = m(19\underline{i} + 10\underline{j} + 17\underline{k})$, where $m \in \mathbb{R} \setminus \{0\}$.

Mark allocation: 2 marks

- 1 mark for recognising that $\underline{a} + \underline{b}$ is a vector that bisects the angle between $\vec{OA}$ and $\vec{OB}$.
- 1 mark for the correct answer.

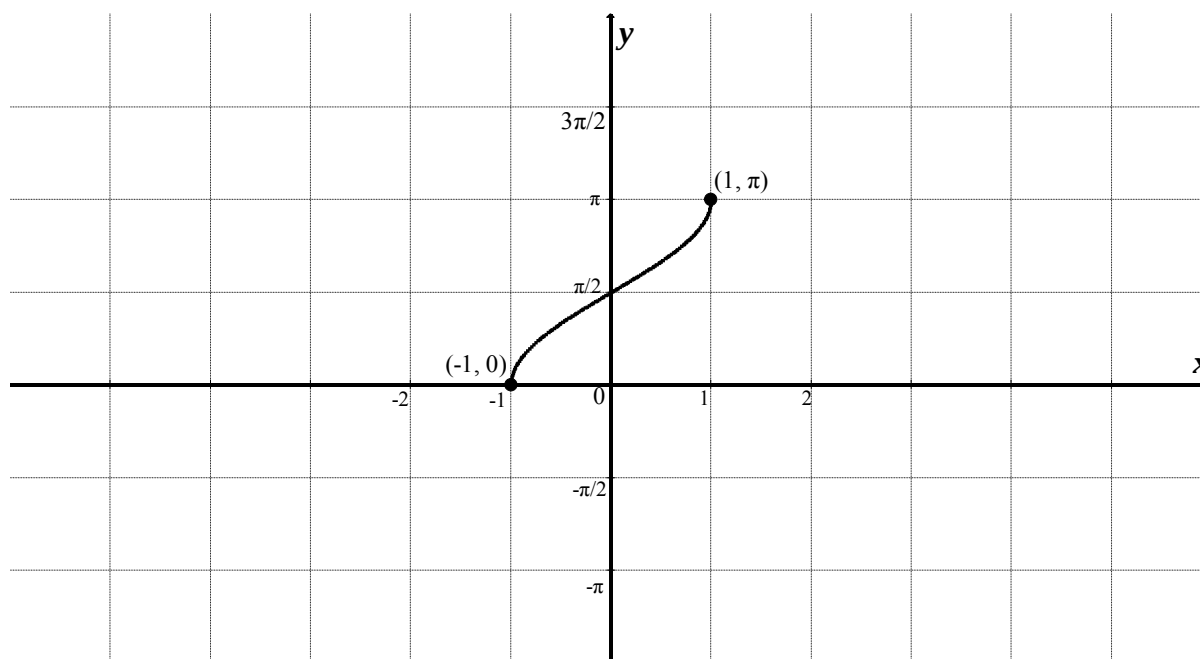
Question 9a.**Worked solution**

$$x = -1, y = \frac{\pi}{2} + \sin^{-1}(-1) = \frac{\pi}{2} - \frac{\pi}{2} = 0$$

One end point is $(-1, 0)$.

$$x = 1, y = \frac{\pi}{2} + \sin^{-1}(1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Other end point is $(1, \pi)$.

**Mark allocation: 2 marks**

- 1 mark for the correct end points.
- 1 mark for the correct graph.

Question 9b.**Worked solution**

$$y = \frac{\pi}{2} + \sin^{-1}(x)$$

$$y - \frac{\pi}{2} = \sin^{-1}(x)$$

$$\Rightarrow x = \sin\left(y - \frac{\pi}{2}\right)$$

Using symmetry:

$$\begin{aligned} \text{Volume} &= 2\pi \int_0^{\frac{\pi}{2}} \sin^2\left(y - \frac{\pi}{2}\right) dy \\ &= 2\pi \int_0^{\frac{\pi}{2}} \frac{1 - \cos(2y - \pi)}{2} dy \\ &= \pi \int_0^{\frac{\pi}{2}} (1 - \cos(2y - \pi)) dy \\ &= \pi \left[y - \frac{1}{2} \sin(2y - \pi) \right]_0^{\frac{\pi}{2}} \\ &= \pi \left[\left(\frac{\pi}{2} - \frac{1}{2} \sin(0) \right) - \left(0 - \frac{1}{2} \sin(-\pi) \right) \right] \\ &= \pi \left(\frac{\pi}{2} \right) \\ &= \frac{\pi^2}{2} \end{aligned}$$

$\therefore$ The volume is $\frac{\pi^2}{2}$ cubic units.

Mark allocation: 3 marks

- 1 mark for the correct definite integral representing the volume.
- 1 mark for correctly applying the double angle formula to the integrand.
- 1 mark for the correct answer.