



Units 3 and 4 Specialist Maths: Exam 2

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Section A – Multiple-choice questions

Question 1

The correct answer is A.

It is clear that -2 is the only real valued solution. Hence, the remaining two solutions must be complex conjugates of one another; it suffices to find one. We can express z^3 in polar form as $z^3 = 8 \operatorname{cis}(\pi)$. By De Moivre's theorem, $z = 2 \operatorname{cis}\left(\frac{\pi}{3}\right) = 1 + \sqrt{3}i$ is a solution. Therefore, $z = 1 - \sqrt{3}i$ is a solution.

Question 2

The correct answer is D.

Shaded area is contained between circles of radius 3 (inclusive) and 4 (exclusive), below the perpendicular bisector of the line connecting $z = -i$ and $z = 1$.

Question 3

The correct answer is B.

Asymptotes occur when $2x + \frac{\pi}{4} = k\pi$, for any integer k . Rearranging in terms of x gives $x = \frac{(4k-1)\pi}{8}$. Substituting appropriate values of k gives the desired result.

Question 4

The correct answer is D.

Question 5

The correct answer is C.

Radius is 50mm, so the height is 120mm ((5,12,13) is a Pythagorean triple). As radius and height are in equal proportion at any depth, $r = \frac{12}{5}h$. We can then express V in terms of h only as $V = \frac{1}{3}\pi \left(\frac{12}{5}h\right)^2 h = \frac{48}{25}\pi h^3$. Then, by using the chain rule, $\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} = \frac{144}{25}\pi h^2 \times \frac{5}{4\pi} = \frac{36h^2}{5}$.

Question 6

The correct answer is B.

$\frac{dy}{dx} = Ake^{kx}$ and $\frac{d^2y}{dx^2} = Ak^2e^{kx}$. Therefore we need to solve $Ak^2e^{kx} = -4Ae^{kx}(k+1)$. Dividing through by common terms and rearranging gives us the quadratic equation $k^2 + 4k + 4 = 0$, which has the unique solution $k = -2$.

Question 7

The correct answer is B.

The angle subtended by the circumference at any point on the circle (except A and C) is a right angle. So $\overline{CB} = 2r \sin \alpha$. Also $\angle BCA$ is $\frac{\pi}{2} - \alpha$. As $\overline{DB}$ is perpendicular to the circumference, $\sin\left(\frac{\pi}{2} - \alpha\right) = \frac{\overline{DB}}{\overline{CB}}$, so

$$\overline{DB} = 2\overline{CB} \sin\left(\frac{\pi}{2} - \alpha\right) = 2(2r \sin \alpha) \cos \alpha = 2r (2 \sin \alpha \cos \alpha) = 2r \sin(2\alpha)$$

Question 8

The correct answer is B.

$$|\mathbf{b}| = \sqrt{1^2 + 2^2 + 1^2}, \text{ so } \hat{\mathbf{b}} = \frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} - \mathbf{k}).$$

$$(\mathbf{a} \cdot \mathbf{b}) \frac{\hat{\mathbf{b}}}{|\mathbf{b}|} = (3 - 8 - 1) \frac{1}{6} \mathbf{b} = -\mathbf{b}$$

Question 9

The correct answer is C.

If $\cos \theta = \frac{2}{7}$, then $\sin \theta = \frac{\sqrt{7^2 - 2^2}}{7} = \frac{3\sqrt{5}}{7}$. Evaluate $\tan^{-1}(\sin \theta)$ using a calculator.

Question 10

The correct answer is A.

Make the observation that $f(x)$ can be expressed as $f(x) = \frac{\frac{d}{dx}(2x+3)}{(2x+3)^2+1}$ which looks very similar to the derivative of the inverse tangent function. Making the substitution $u = 2x + 3$ and yields the result $\int f(x)dx = \tan^{-1}(2x + 3) + c$

Question 11

The correct answer is D.

Acceleration down plane is $mg \sin \theta = 6g \sin 25^\circ = 24.9 \text{ N}$ down the plane. The maximum friction is $\mu mg \cos \theta = 3m \cos 25^\circ = 26.6 \text{ N}$ up the plane. $26.6 > 24.9$. Hence the friction is 24.9 N up the plane.

Question 12

The correct answer is A.

Given the shape and that the intersection of the asymptotes of the hyperbola is $(3,1)$, the equation must be of the form $\frac{(x-3)^2}{a^2} - \frac{(y-1)^2}{b^2} = 1$. As $\theta = \frac{\pi}{3}$, we can find the acute angle made by each asymptote and the x-axis which also turns out to be $\frac{\pi}{3}$. Therefore, the gradients of the asymptotes are $\pm \frac{b}{a} = \pm \tan \frac{\pi}{3} = \pm \sqrt{3}$. Hence possible values for a^2 and b^2 are 1 and 3, respectively.

Question 13

The correct answer is E.

$$\text{We need to evaluate an integral of the form } \int_0^2 \pi \cdot \left(\frac{\pi}{2}\right)^2 dx - \int_0^2 \pi x(y)^2 dx = \int_0^2 \pi \left(\left(\frac{\pi}{2}\right)^2 - (x(y))^2\right) dx$$

(n.b. $x(y)$ denotes x as a function of y , i.e. $x(y) = \frac{\cos^{-1}(1-y)}{2}$): we find the volume of a cylinder and then 'hollow it out' by subtracting the volume we don't need. Evaluate using a calculator.

Question 14

The correct answer is D.

Question 15

The correct answer is A.

Look at the intercepts.

Question 16

The correct answer is B.

$$\frac{dy}{dx} = 2 \cos(2x) + 2 \sin(2x) \text{ and } \frac{d^2y}{dx^2} = -4 \sin(2x) + 4 \cos(2x).$$

Question 17

The correct answer is C.

$$4 \cos 60^\circ = 2N, \text{ and } 4 \sin 60^\circ = 2\sqrt{3}N. \text{ Therefore } |F_{net}| = \sqrt{(5 - 2\sqrt{3})^2 + (3 - 2)^2} = 1.83N$$

Question 18

The correct answer is C.

Solve $\int_0^k 20dt = \int_0^k 5\sqrt{t}dt$ for k . Evaluating integrals and factorizing gives $10k \left(2 - \frac{1}{3}\sqrt{k}\right) = 0$, which has non-trivial solution $\sqrt{k} = 6 \Rightarrow k = 36$

Question 19

The correct answer is C.

Look at key features of the slope field; there are exactly two lines along which the gradient is zero (corresponds to a quadratic). More decisively, the gradient is independent of x ; along any line $y = c$, where c is a constant, the gradient is the same; C is the only option where $\frac{dy}{dx}$ is independent of x .

Question 20

The correct answer is D.

$$\dot{\mathbf{r}}(t) = 2 \cos(t) \mathbf{i} - 2 \sin(t) \mathbf{j} - \frac{\pi^2}{(t + \frac{\pi}{2})^2} \mathbf{k}. \text{ Evaluating } |\dot{\mathbf{r}}(0)| \text{ gives } \sqrt{20} = 2\sqrt{5}$$

Question 21

The correct answer is C.

Simple evaluation

Question 22

The correct answer is A.

Use Newton's 2nd Law, $F = ma$. Note speed is a positive quantity by definition (magnitude of velocity)

Section B – Short-answer questions

Marks allocated are indicated by a number in square brackets, for example, [1] indicates that the line is worth one mark.

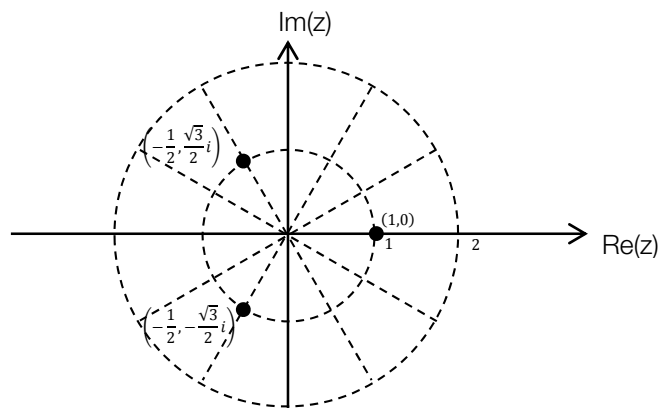
Question 1a i

Find the complex polar representation of 1, i.e. $1 = \text{cis}(0 + 2n\pi)$, where n is an integer.

So $z^3 = \text{cis}(2n\pi)$, and by De Moivre's theorem, $z = \text{cis}\left(\frac{2n\pi}{3}\right)$ [1].

So the possible values of z are $\text{cis}(0) = 1$, $\text{cis}\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\text{cis}\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ [1].

Question 1a ii



[1 for correct markings, 1 for correct labels]

Question 1b i

Observe that $i^3 = -i$, so $(-i)^3 = i$. [1]

Now if $w^3 = 1$, then $(-iw)^3 = (-i^3)(w^3) = i$. Therefore, $k = -i$. [1]

Question 1b ii

The geometric interpretation of multiplication by i is a rotation of $\frac{\pi}{2}$ radians counter clockwise. So multiplication by k corresponds to rotation $\frac{3\pi}{2}$ counter clockwise [1].

Therefore, the solutions are $\text{cis}\left(\frac{3\pi}{2}\right)$, $\text{cis}\left(\frac{\pi}{6}\right)$ and $\text{cis}\left(\frac{5\pi}{6}\right)$ [1].

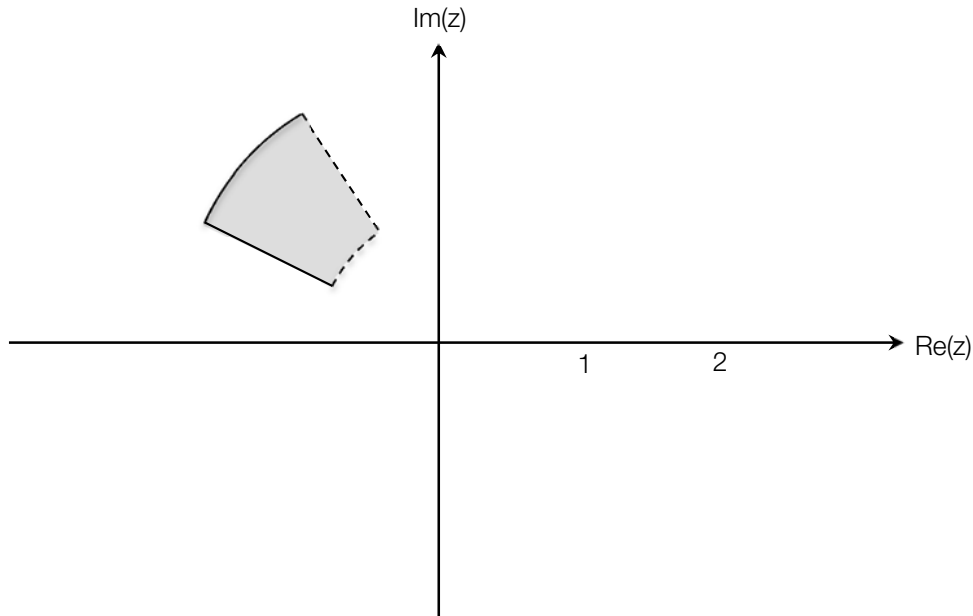
Question 1c

$z_1 = \text{cis}\left(\frac{2\pi}{3}\right)$ and $z_2 = \text{cis}\left(\frac{5\pi}{6}\right)$ [1].

$|z - z_1| = |z - z_2|$ describes the perpendicular bisector of the line joining these points. As both z_1 and z_2 have the same magnitude, the bisector must pass through the origin (n.b the bisector of a chord passes through the centre of the circle). The angle halfway between $\frac{4\pi}{6}$ and $\frac{5\pi}{6}$ is $\frac{9\pi}{12} = \frac{3\pi}{4}$, so the Cartesian equation is $y = -x$. [1]

Question 1d

[2 for correct shape, 1 for correct boundaries]



The first part of the set describes the set of points whose distance from iz_1 is strictly less than the distance from $-iz_1 = i^3 z_1$: i.e. the set of all points below the line joining z_1 to the origin. Similarly, the second part describes the set of points whose distance from iz_2 is greater than or equal to the distance from $i^3 z_2$: i.e. the set of points above the line joining z_2 to the origin. The third part requires that the magnitude of z is less than or equal to 2, but strictly greater than 1.

Question 2a

$$f'(x) = \frac{-3}{1+x^2} + \frac{(x^2+1)(2)-(2x)(2x)}{(x^2+1)^2} + x^2 + 1$$

applying appropriate rules [1]

$$f'(x) = -\frac{1}{x^2+1} - \frac{4x^2}{(x^2+1)^2} + \frac{x^4+2x^2+1}{x^2+1}$$

$$f'(x) = \frac{(x^4 + 2x^2)(x^2 + 1) - 4x^2}{(x^2 + 1)^2}$$

$$f'(x) = \frac{x^6 + 2x^4 + x^4 + 2x^2 - 4x^2}{(x^2 + 1)^2}$$

correct manipulation [1]

$$f'(x) = \frac{x^2(x^4 + 3x^2 - 2)}{(x^2 + 1)^2}$$

correct answer [1]

Question 2b

$$f'(x) = 0 \Leftrightarrow -x^2(x^4 + 3x^2 - 2) = 0$$

as the denominator of $f'(x)$ is always positive

$$\text{Let } u = x^2, \text{ then } u(u^2 + 3u - 2) = 0$$

observing that the function is a cubic in x^2 [1]

$$u = \frac{-3 \pm \sqrt{9+8}}{2}$$

solving the quadratic term of the cubic [1]

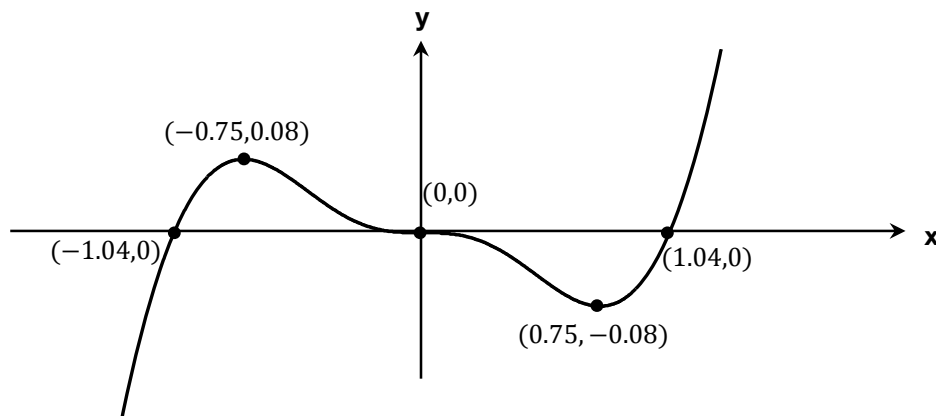
$$x = \sqrt{u} = \pm \sqrt{\frac{-3 \pm \sqrt{17}}{2}}$$

solving for x , requiring x to be real-valued [1]

$$x = 0, x = \sqrt{\frac{-3 + \sqrt{17}}{2}}, x = -\sqrt{\frac{-3 + \sqrt{17}}{2}}$$

listing solutions of $f'(x) = 0$ [1]

Question 2b



[2 for correct shape, 1 for correct intercepts, 1 for correct turning points]

Question 2c

$$\int f(x)dx = -3 \int \tan^{-1} x \, dx + \int \frac{2x}{x^2+1} dx + \int \frac{x^3}{3} + x \, dx$$

$$-3 \int \tan^{-1} x \, dx = -3(x \tan^{-1} x - \frac{1}{2} \log_e(x^2 + 1)) \quad [1]$$

$$\int \frac{2x}{x^2+1} dx = \int \frac{1}{u} du = \log_e u = \log_e(x^2 + 1) \quad \text{substituting } u = x^2 + 1 \quad [1]$$

$$\int \frac{x^3}{3} + x \, dx = \frac{x^4}{12} + \frac{x^2}{2}$$

Therefore,

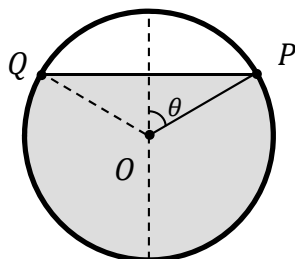
$$\int f(x)dx = -3 \left(x \tan^{-1} x - \frac{1}{2} \log_e(x^2 + 1) \right) + \log_e(x^2 + 1) + \frac{x^4}{12} + \frac{x^2}{2}$$

$$= -3x \tan^{-1} x + \frac{5}{2} \log_e(x^2 + 1) + \frac{x^4}{12} + \frac{x^2}{2} \quad [1]$$

Question 3a

$V = 20m^3$, and the surface area of the circle is πm^2 . Therefore, $L = \frac{20}{\pi} m \quad [1]$

Question 3b i



Area of the sector subtended by the angle $\angle POQ$ is given by $\frac{2\theta}{2\pi} \pi r^2 = \theta \quad [1/2]$. The area of the triangle $\triangle POQ$ is given by $\sin \theta \times \cos \theta = \frac{1}{2} \sin 2\theta \quad [1/2]$. So the area of the unshaded segment is given by $\theta - \frac{1}{2} \sin(2\theta) m^2 \quad [1]$.

Question 3b ii

The total volume is $20m^3$, and the unfilled volume is $\left(\theta - \frac{1}{2}\sin(2\theta) m^2\right) \times L$ [1]. So the amount of water in the tank as a function of θ is $V = 20 - \frac{L}{2}(2\theta - \sin(2\theta))$, where $0 \leq \theta \leq \pi$, as these are the only values θ can physically take. [1]

Question 3b iii

Take the derivative with respect to time of both sides of the equation in part ii.

$$\frac{dV}{dt} = \frac{d}{dt} \left(20 - \frac{L}{2}(2\theta - \sin(2\theta)) \right) = \frac{d\theta}{dt} \frac{d}{d\theta} \left(20 - \frac{10}{\pi}(2\theta - \sin(2\theta)) \right) = \frac{d\theta}{dt} \left(-\frac{20}{\pi}(1 - \cos(2\theta)) \right) [2]$$

Rearranging in terms of $\frac{d\theta}{dt}$, given that $\frac{dV}{dt} = -2$:

$$\frac{d\theta}{dt} = \frac{\pi}{10(1 - \cos(2\theta))} [1]$$

Question 3c

$$\theta(0.1) = \theta(0) + \Delta t \left(\frac{d\theta}{dt} \right)_{t=0} = 0 + 0.1 \times \frac{\pi}{10} = \frac{\pi}{100} [1]$$

$$\theta(0.2) = \theta(0.1) + \Delta t \left(\frac{d\theta}{dt} \right)_{t=0.1} = \frac{\pi}{100} + 0.1 \times \frac{\pi}{10(1 - \sin(0.2))} [1]$$

$$\theta(0.3) = \theta(0.2) + \Delta t \left(\frac{d\theta}{dt} \right)_{t=0.2} = \left(\frac{\pi}{100} + 0.1 \times \frac{\pi}{10(1 - \sin(0.2))} \right) + 0.1 \times \frac{\pi}{10(1 - \sin(0.4))} \approx 0.122 \text{ rad/s} [1]$$

Question 4a

$$|r(t)| = \sqrt{(2t)^2 + \left(2e^{-\frac{t^2}{10}} \cos \frac{\pi t}{5}\right)^2 + \left(2e^{-\frac{t^2}{10}} \sin \frac{\pi t}{5}\right)^2} \quad \text{displacement is given by the } |r(t)| [1]$$

$$= \sqrt{4t^2 + \left(4e^{-\frac{t^2}{5}}\right) \left(\cos^2 \left(\frac{\pi t}{5}\right) + \sin^2 \left(\frac{\pi t}{5}\right)\right)} \quad \text{rearranging}$$

$$= 2\sqrt{t^2 + e^{-\frac{t^2}{5}}} [1]$$

Question 4b

$$\dot{r}(t) = \frac{d}{dt}(2t)\mathbf{i} + \frac{d}{dt}\left(2e^{-\frac{t^2}{10}} \cos \frac{\pi t}{5}\right)\mathbf{j} + \frac{d}{dt}\left(2e^{-\frac{t^2}{10}} \sin \frac{\pi t}{5}\right)\mathbf{k}$$

$$\frac{d}{dt}(2t) = 2$$

$$\frac{d}{dt}\left(e^{-\frac{t^2}{10}} \cos \frac{\pi t}{5}\right) = \frac{d}{dt}\left(e^{-\frac{t^2}{10}}\right) \cos \frac{\pi t}{5} + \frac{d}{dt}\left(\cos \frac{\pi t}{5}\right) e^{-\frac{t^2}{10}} = -\frac{te^{-\frac{t^2}{10}}}{5} \cos \frac{\pi t}{5} - \frac{\pi}{5} \sin \frac{\pi t}{5} e^{-\frac{t^2}{10}} [1]$$

$$\frac{d}{dt}\left(e^{-\frac{t^2}{10}} \sin \frac{\pi t}{5}\right) = \frac{d}{dt}\left(e^{-\frac{t^2}{10}}\right) \sin \frac{\pi t}{5} + \frac{d}{dt}\left(\sin \frac{\pi t}{5}\right) e^{-\frac{t^2}{10}} = -\frac{te^{-\frac{t^2}{10}}}{5} \sin \frac{\pi t}{5} + \frac{\pi}{5} \cos \frac{\pi t}{5} e^{-\frac{t^2}{10}} [1]$$

$$\dot{r}(t) = 2\mathbf{i} - \frac{2e^{-\frac{t^2}{10}}}{5} \left(t \cos \frac{\pi t}{5} + \pi \sin \frac{\pi t}{5} \right) \mathbf{j} + \frac{2e^{-\frac{t^2}{10}}}{5} \left(-t \sin \frac{\pi t}{5} + \pi \cos \frac{\pi t}{5} \right) \mathbf{k} [1]$$

Question 4c

$$|\mathbf{r}(5)| = 2\sqrt{25 + e^{-5}} \approx 10m \quad [1/2]$$

$$\dot{\mathbf{r}}(5) = 2\mathbf{i} - \frac{2e^{-\frac{5}{2}}}{5}(-5)\mathbf{j} + \frac{2e^{-\frac{5}{2}}}{5}(-\pi)\mathbf{k} \quad [1/2]$$

$$|\dot{\mathbf{r}}(5)| = \sqrt{4 + 4e^{-5} + \frac{4\pi}{25}e^{-5}} \approx 2 \text{ m/s} \quad [1]$$

Question 5a

The gradient of the ramp is given by $\frac{dy}{dx} = \frac{x}{2}$, so $\tan(\theta) = \frac{x}{2}$ [1]

Then, the magnitude of the normal is given by $|F_n| = mg \cos \theta = 49 \frac{2}{\sqrt{x^2+4}}$ [1]

We now need to resolve this into components parallel to the axes. A little geometric manipulation yields that the horizontal component is $-F_n \sin \theta$, and the vertical component is $F_n \cos \theta$ [1]

$$\text{Therefore, } F_{\text{normal}} = -49 \frac{4}{x^2+4} \mathbf{i} + 49 \frac{2x}{x^2+4} \mathbf{j} = \frac{196}{x^2+4} \mathbf{i} + \frac{98x}{x^2+4} \mathbf{j} \quad [1]$$

Question 5b i

As before, the gradient of the ramp at a point is given by $\frac{dy}{dx} = \frac{x}{2}$, and $\tan(\theta) = \frac{x}{2}$. The magnitude of the force tangent to the ramp (and therefore parallel to the direction of acceleration of the mass), is $|F| = mg \sin \theta = 49 \frac{2x}{\sqrt{x^2+4}}$ [1]

Then $a = \frac{19.6x}{\sqrt{x^2+4}}$ by Newton's second law [1]

Question 5b ii

As $a = \frac{d}{dx} \frac{1}{2} v^2$, we integrate both sides with respect to x from $x = k$ to $x = 0$ (we reverse the limits to account for the direction of acceleration) and solve for v [1]. We will have to do a u-substitution, so choose $u = x^2 + 4$, then $\frac{du}{dx} = 2x$ [1].

$$\frac{1}{2} v^2 = \int_k^0 \frac{19.6x}{\sqrt{x^2+4}} dx = \int_{k^2+4}^4 \frac{19.6x}{\sqrt{u}} \frac{1}{2x} du = 9.8 \left[\frac{1}{2} \sqrt{u} \right]_{k^2+4}^4 = 4.9(\sqrt{k^2+4} - 2) \quad [2]$$

Therefore,

$$v = \sqrt{9.8(\sqrt{k^2+4} - 2)} \quad [1]$$

(we ignore the negative root as speed must be positive)

Question 5b iii

Evaluating v at $a = 2$ gives $v = \sqrt{9.8(2\sqrt{2} - 2)} = 2.84 \text{ m/s}$ [1]