

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2



(TSSM's 2012 trial exam updated for the current study design)

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: A

Explanation

The gradient, $m = \pm \frac{2}{3}$, therefore $a = 3$ and $b = 2$ (multiples of 3 and 2 are excluded by given choices). **Options B and E are therefore eliminated.**

The x -intercepts for each asymptote respectively are $\left(\frac{11}{2}, 0\right)$ and $\left(-\frac{7}{2}, 0\right)$.

The x -coordinate of the centre of the hyperbola is $\frac{1}{2}\left(\frac{11}{2} - \frac{7}{2}\right) = 1$.

The y -intercepts for each asymptote respectively are $\left(0, -\frac{11}{3}\right)$ and $\left(0, -\frac{7}{3}\right)$.

The y -coordinate of the centre of the hyperbola is $\frac{1}{2}\left(-\frac{11}{3} - \frac{7}{3}\right) = -3$.

The centre of the hyperbola is therefore $(1, -3)$. **Options C and D are therefore eliminated,**

It follows that the equation of the hyperbola could be

$$\frac{(y+3)^2}{4} - \frac{(x-1)^2}{9} = 1 \Rightarrow 9(y+3)^2 - 4(x-1)^2 = 36$$

Question 2

Answer: D

Explanation:

For two vertical asymptotes, require two solutions to the equation $2x^2 + kx + 5 = 0$

The discriminant, $\Delta > 0 \Rightarrow k^2 - 40 > 0 \Rightarrow k \in (-\infty, -2\sqrt{10}) \cup (2\sqrt{10}, \infty)$

Question 3

Answer: C

Explanation:

$$3x - 6 \in [-1, 1]$$

$$3x \in [5, 7]$$

$$x \in \left[\frac{5}{3}, \frac{7}{3}\right]$$

$$\cos^{-1}(3x - 6) \in [0, \pi]$$

$$2\cos^{-1}(3x - 6) \in [0, 2\pi]$$

$$2\cos^{-1}(3x - 6) - 1 \in [-1, 2\pi - 1]$$

Question 4

Answer: B

Explanation:

$$z^4 = -8i$$

$$|z|^4 \operatorname{cis} 4\theta = 8 \operatorname{cis} \left(-\frac{\pi}{2}\right)$$

$$|z|^4 = 8$$

$$|z| = 2^{\frac{3}{4}}$$

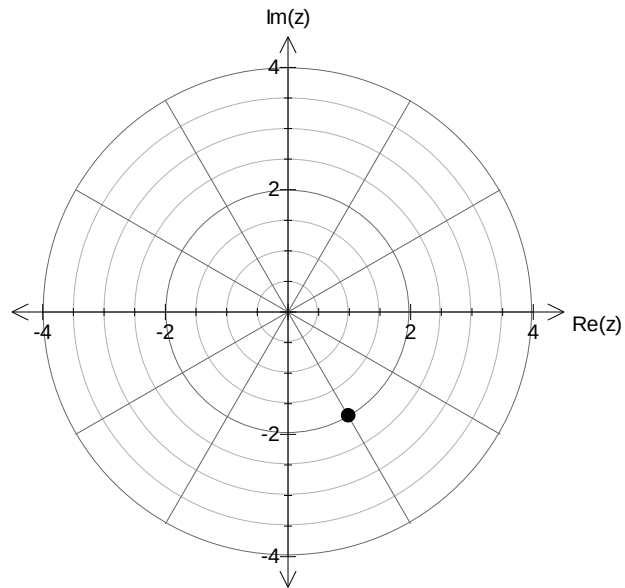
$$4\theta = -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$$

$$\begin{aligned} \theta &= -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8} \\ &= -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8} \end{aligned}$$

Therefore, the fourth roots are $2^{\frac{3}{4}} \operatorname{cis} \left(-\frac{5\pi}{8}\right)$, $2^{\frac{3}{4}} \operatorname{cis} \left(-\frac{\pi}{8}\right)$, $2^{\frac{3}{4}} \operatorname{cis} \left(\frac{3\pi}{8}\right)$ and $2^{\frac{3}{4}} \operatorname{cis} \left(\frac{7\pi}{8}\right)$

Question 5*Answer: C**Explanation:*

$$\begin{aligned}\bar{z} &= -2 - 2\sqrt{3}i \\ &= 4\text{cis}\left(-\frac{2\pi}{3}\right) \\ \sqrt{\bar{z}} &= \bar{z}^{\frac{1}{2}} \\ &= 2\text{cis}\left(-\frac{\pi}{3}\right)\end{aligned}$$

**Question 6***Answer: B**Explanation:*

$$\begin{aligned}\frac{(-1-i\sqrt{3})^2}{(\sqrt{3}+i)^3} &= \frac{4\text{cis}\left(-\frac{4\pi}{3}\right)}{8\text{cis}\left(\frac{\pi}{2}\right)} \\ &= \frac{1}{2}\text{cis}\left(-\frac{4\pi}{3}-\frac{\pi}{2}\right) \\ &= \frac{1}{2}\text{cis}\left(-\frac{11\pi}{6}\right)\end{aligned}$$

$$\text{Therefore, } \text{Im}\left(\frac{(-1-i\sqrt{3})^2}{(\sqrt{3}+i)^3}\right) = \frac{1}{2}\sin\left(-\frac{11\pi}{6}\right) = \frac{1}{4}$$

Question 7

Answer: E

Explanation:

$$\sec A = \frac{5}{2} \Rightarrow \cos A = \frac{2}{5} \Rightarrow \sin A = \frac{\sqrt{21}}{5}$$

$$\operatorname{cosec} B = -3 \Rightarrow \sin B = -\frac{1}{3} \Rightarrow \cos B = \frac{2\sqrt{2}}{3}$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\begin{aligned} &= \frac{\sqrt{21}}{5} \times \frac{2\sqrt{2}}{3} - \frac{2}{5} \times \frac{1}{3} \\ &= \frac{2}{15}(\sqrt{42} - 1) \end{aligned}$$

Question 8

Answer: C

Explanation:

$$\underline{b} - \underline{c} = 3\underline{j} - 2\underline{k}$$

The scalar resolute of $\underline{a}$ in the direction of $\underline{b} - \underline{c}$ is equal to

$$(2\underline{i} - 3\underline{j} + \underline{k}) \cdot \frac{1}{\sqrt{13}}(3\underline{j} - 2\underline{k}) = -\frac{11}{\sqrt{13}}$$

Therefore, the vector resolute of $\underline{a}$ in the direction of $\underline{b} - \underline{c}$ is equal to

$$-\frac{11}{\sqrt{13}} \times \frac{1}{\sqrt{13}}(3\underline{j} - 2\underline{k}) = -\frac{11}{13}(3\underline{j} - 2\underline{k})$$

Question 9

Answer: D

Explanation:

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \frac{5}{9}\overrightarrow{AB} \\ &= \overrightarrow{OA} + \frac{5}{9}(\overrightarrow{OB} - \overrightarrow{OA}) \\ &= \frac{4}{9}\overrightarrow{OA} + \frac{5}{9}\overrightarrow{OB} \\ &= \frac{1}{9}(4\overrightarrow{OA} + 5\overrightarrow{OB}) \\ &= -\frac{1}{9}(4\overrightarrow{AO} + 5\overrightarrow{BO})\end{aligned}$$

Question 10

Answer: A

Explanation:

$$3x - 5y + 10 = 0 \Rightarrow y = \frac{3}{5}x + 2$$

A line that is perpendicular to the line $3x - 5y + 10 = 0$ has a gradient of $-\frac{5}{3}$.

Therefore, a vector, $\underline{v}$ that is perpendicular to the line $3x - 5y + 10 = 0$ can be expressed as $\underline{v} = 3\underline{i} - 5\underline{j}$. It follows that a vector perpendicular to the line $3x - 5y + 10 = 0$ with magnitude 12 units is

$$\begin{aligned}12 \times \hat{\underline{v}} &= 12 \times \frac{\underline{v}}{|\underline{v}|} \\ &= 12 \times \frac{\pm 1}{\sqrt{34}}(3\underline{i} - 5\underline{j}) \\ &= \pm \frac{6\sqrt{34}}{17}(3\underline{i} - 5\underline{j})\end{aligned}$$

Question 11*Answer: C**Explanation*

$$x = 1 + \cos 2t$$

$$y = \sin t$$

$$x = 1 + 1 - 2\sin^2 t$$

$$= 2 - 2\sin^2 t$$

$$= 2 - 2y^2$$

$$2y^2 = 2 - x$$

$$y^2 = \frac{2-x}{2}$$

$$y = \pm \sqrt{\frac{2-x}{2}}$$

$$\text{Since } t \in \left[0, \frac{\pi}{2}\right] \Rightarrow y \in [0, 1]$$

$$\text{Therefore, } y = \sqrt{\frac{2-x}{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{4-2x}}{2}$$

Question 12*Answer: E**Explanation:*

At the point of intersection,

$$2 - x^2 = \sqrt{x} \Rightarrow x = 1$$

The volume, V is

$$\begin{aligned} V &= \pi \int_0^1 (2 - x^2)^2 dx - \pi \int_0^1 (\sqrt{x})^2 dx \\ &= \pi \int_0^1 (2 - x^2)^2 - x dx \end{aligned}$$

Question 13*Answer: B**Explanation:*

$$u = 1 - 2x \Rightarrow x = \frac{1-u}{2}$$

$$\frac{du}{dx} = -2$$

$$-\frac{1}{2} \frac{du}{dx} = 1$$

$$x = -1, u = 3$$

$$x = 2, u = -3$$

Therefore,

$$\begin{aligned} \int_{-1}^2 \frac{x}{\sqrt{1-2x}} dx &= \int_3^{-3} \frac{1-u}{2} \times \frac{1}{\sqrt{u}} \times -\frac{1}{2} \frac{du}{dx} dx \\ &= \int_3^{-3} \frac{u-1}{4\sqrt{u}} du \\ &= \frac{1}{4} \int_{-3}^3 \frac{1-u}{\sqrt{u}} du \end{aligned}$$

Question 14*Answer: D**Explanation:*

$$\frac{d}{dx}(x^2y - y^2) = 0$$

$$2xy + x^2 \frac{dy}{dx} - 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x^2 - 2y) = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 - 2y}$$

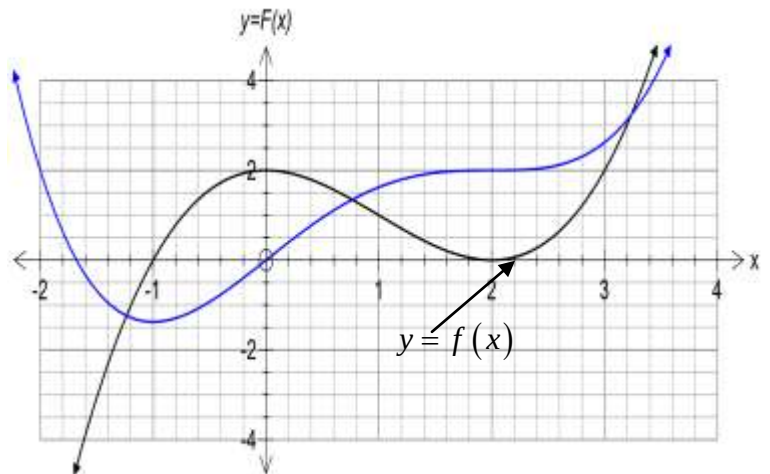
The gradient is undefined when $x^2 - 2y = 0$ This condition is satisfied by the point $(-2\sqrt{2}, 4)$

Question 15

Answer: B

Explanation

The graph at right represents a possible function, $y = f(x)$. It shows that a local minimum occurs at $(-1, 0)$, a non-stationary point of inflection at $(0, 2)$ and a stationary point occurs at $(2, 0)$



Question 16

Answer: A

Explanation

Tabulating, we have

n	x_n	y_n	$\frac{dy}{dx} = 2 \cos^{-1}\left(\frac{x_n}{3}\right)$
0	1.00	2.000000	2.461919
1	1.20	2.492384	2.318559
2	1.40	2.956096	2.170556
3	1.60	3.390207	

Question 17

Answer: E

Explanation:

Inflow = 0

$$\text{Outflow} = \frac{6x}{300 + 2t} = \frac{3x}{150 + t}$$

Therefore, $\frac{dx}{dt} = 0 - \frac{3x}{150 + t} = \frac{-3x}{150 + t}$ where $x = 1200$ grams when $t = 0$

Question 18*Answer: A**Explanation:*

The gradient of the line joining the points $(10, 40)$ and $(20, -20)$ is -6 . Therefore, the equation of the line joining the points $(10, 40)$ and $(20, -20)$ is

$$v - 40 = -6(t - 10) \Rightarrow v = 100 - 6t$$

$$\text{When } v = 0 \Rightarrow 100 - 6t = 0 \Rightarrow t = \frac{50}{3}$$

Therefore, the total distance travelled by the particle is

$$\frac{1}{2} \times \frac{50}{3} \times 40 + \frac{1}{2} \times \left(30 - \frac{50}{3}\right) \times 20 = \frac{1400}{3}$$

Question 19*Answer: E**Explanation:*

If the vertical displacement of the two objects are denoted s_1 and s_2 , and if t represents the time in seconds that have elapsed after the projection of the first object, then

$$s_1 = 20t - 4.9t^2 \text{ and } s_2 = 18(t - 2) - 4.9(t - 2)^2$$

$$\text{At the point of collision, } s_1 = s_2 \Rightarrow 20t - 4.9t^2 = 18(t - 2) - 4.9(t - 2)^2$$

Solving, we have $t = 3.16$ seconds with the collision point $20 \times 3.16 - 4.9 \times 3.16^2 = 14.3$ metres above ground level.

Therefore, the objects will collide approximately 3.16 seconds after the projection of the first object.

Question 20

Answer: B

Explanation:

$$a(v) = v^2 + 4$$

$$v \frac{dv}{dx} = v^2 + 4$$

$$\frac{dv}{dx} = \frac{v^2 + 4}{v}$$

$$\frac{dx}{dv} = \frac{v}{v^2 + 4}$$

$$x = \int \frac{v}{v^2 + 4} dv$$

$$= \frac{1}{2} \log_e |v^2 + 4| + c$$

Since $v(1) = 2$, then

$$1 = \frac{1}{2} \log_e 8 + c \Rightarrow c = 1 - \frac{1}{2} \log_e 8$$

Therefore,

$$x = \frac{1}{2} \log_e \frac{v^2 + 4}{8}$$

$$\log_e \frac{v^2 + 4}{8} = 2(x - 1)$$

$$v^2 = 4(2e^{2(x-1)} - 1) \Rightarrow v = \pm 2\sqrt{2e^{2(x-1)} - 1}$$

$$v = 2\sqrt{2e^{2(x-1)} - 1}$$

Question 21

Answer: B

Explanation:

The resultant force,

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

$$= 2\vec{i} - 3\vec{j} - \vec{k} - \vec{i} + 4\vec{j} + 2\vec{k} + \vec{i} + 4\vec{j} - 2\vec{k}$$

$$= 2\vec{i} + 5\vec{j} - \vec{k}$$

$$|\vec{F}| = \sqrt{2^2 + 5^2 + (-1)^2}$$

$$= \sqrt{30}$$

$$\text{Therefore, the mass of the particle, } m = \frac{\sqrt{30}}{2.5} = \frac{2\sqrt{30}}{5} \text{ kg}$$

Question 22

Answer: D

Explanation:

$$25 \times 9.8 \times \sin(46^\circ) = 25 \times a$$
$$a = 7.05$$

SECTION 2

Question 1

a.

$$\begin{aligned}
 \operatorname{cis}\left(\theta - \frac{3\pi}{2}\right) &= \cos\left(\theta - \frac{3\pi}{2}\right) + i \sin\left(\theta - \frac{3\pi}{2}\right) \\
 &= \cos\left(\frac{3\pi}{2} - \theta\right) - i \sin\left(\frac{3\pi}{2} - \theta\right) \\
 &= -\sin \theta - (-i \cos \theta) \\
 &= -\sin \theta + i \cos \theta
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \operatorname{cis}\left(\theta - \frac{3\pi}{2}\right) &= \cos\left(\theta - \frac{3\pi}{2}\right) + i \sin\left(\theta - \frac{3\pi}{2}\right) \\ &= \cos\left(\frac{3\pi}{2} - \theta\right) - i \sin\left(\frac{3\pi}{2} - \theta\right) \\ &= -\sin \theta - (-i \cos \theta) \\ &= -\sin \theta + i \cos \theta \end{aligned}} \right\} \text{ [M2]}$$

b.

$$\begin{aligned}
 \text{LHS} &= \frac{1}{\operatorname{cis}\left(\theta - \frac{3\pi}{2}\right)} \\
 &= \left(\operatorname{cis}\left(\theta - \frac{3\pi}{2}\right)\right)^{-1} \\
 &= \operatorname{cis}\left(\frac{3\pi}{2} - \theta\right) \\
 &= -\sin \theta - i \cos \theta \\
 &= -(\sin \theta + i \cos \theta) \\
 &= -\left(\cos\left(\frac{\pi}{2} - \theta\right) + i \sin\left(\frac{\pi}{2} - \theta\right)\right) \\
 &= -\operatorname{cis}\left(\frac{\pi}{2} - \theta\right) \\
 &= \text{RHS}
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \text{LHS} &= \frac{1}{\operatorname{cis}\left(\theta - \frac{3\pi}{2}\right)} \\ &= \left(\operatorname{cis}\left(\theta - \frac{3\pi}{2}\right)\right)^{-1} \\ &= \operatorname{cis}\left(\frac{3\pi}{2} - \theta\right) \\ &= -\sin \theta - i \cos \theta \\ &= -(\sin \theta + i \cos \theta) \\ &= -\left(\cos\left(\frac{\pi}{2} - \theta\right) + i \sin\left(\frac{\pi}{2} - \theta\right)\right) \\ &= -\operatorname{cis}\left(\frac{\pi}{2} - \theta\right) \\ &= \text{RHS} \end{aligned}} \right\} \text{ [M3]}$$

c.

Using technology,

$$(-\sin \theta + i \cos \theta)^4 = 1 - 8 \sin^2 \theta \cos^2 \theta + 4 \sin \theta \cos \theta (2 \cos^2 \theta - 1) i \quad \text{[A1]}$$

d.

$$\begin{aligned}
 \text{LHS} &= \left(\text{cis} \left(\theta - \frac{3\pi}{2} \right) \right)^4 \\
 &= \text{cis} \left(4 \left(\theta - \frac{3\pi}{2} \right) \right) \\
 &= \text{cis}(4\theta - 6\pi) \\
 &= \text{cis}(4\theta) \\
 &= \text{RHS}
 \end{aligned}
 \left. \vphantom{\begin{aligned} \text{LHS} &= \left(\text{cis} \left(\theta - \frac{3\pi}{2} \right) \right)^4 \\ &= \text{cis} \left(4 \left(\theta - \frac{3\pi}{2} \right) \right) \\ &= \text{cis}(4\theta - 6\pi) \\ &= \text{cis}(4\theta) \\ &= \text{RHS} \end{aligned}} \right\} \text{ [M2]}$$

e.From parts **c.** and **d.**

$$\begin{aligned}
 \text{cis}(4\theta) &= (-\sin \theta + i \cos \theta)^4 \\
 \cos(4\theta) + i \sin(4\theta) &= 1 - 8\sin^2 \theta \cos^2 \theta + 4\sin \theta \cos \theta (2\cos^2 \theta - 1)i \\
 \text{Equating coefficients,} \\
 \cos(4\theta) &= 1 - 8\sin^2 \theta \cos^2 \theta \\
 &= 1 - 8\sin^2 \theta (1 - \sin^2 \theta) \\
 &= 1 - 8\sin^2 \theta + 8\sin^4 \theta
 \end{aligned}
 \left. \vphantom{\begin{aligned} \text{cis}(4\theta) &= (-\sin \theta + i \cos \theta)^4 \\ \cos(4\theta) + i \sin(4\theta) &= 1 - 8\sin^2 \theta \cos^2 \theta + 4\sin \theta \cos \theta (2\cos^2 \theta - 1)i \\ \cos(4\theta) &= 1 - 8\sin^2 \theta \cos^2 \theta \\ &= 1 - 8\sin^2 \theta (1 - \sin^2 \theta) \\ &= 1 - 8\sin^2 \theta + 8\sin^4 \theta \end{aligned}} \right\} \text{ [M1] [A1]}$$

f.

$$\begin{aligned}
 \cot \theta &= -\frac{5}{2} \Rightarrow \sin \theta = \frac{2}{\sqrt{29}} \\
 \cos(4\theta) &= 1 - 8 \left(\frac{2}{\sqrt{29}} \right)^2 + 8 \left(\frac{2}{\sqrt{29}} \right)^4 = \frac{41}{841}
 \end{aligned}
 \left. \vphantom{\begin{aligned} \cot \theta &= -\frac{5}{2} \Rightarrow \sin \theta = \frac{2}{\sqrt{29}} \\ \cos(4\theta) &= 1 - 8 \left(\frac{2}{\sqrt{29}} \right)^2 + 8 \left(\frac{2}{\sqrt{29}} \right)^4 = \frac{41}{841} \end{aligned}} \right\} \text{ [M1] [A1]}$$

Question 2

a.

$$f(x) = \frac{1}{e^x \sec x}$$

$$= e^{-x} \cos x$$

$$f'(x) = -e^{-x} \cos x - e^{-x} \sin x$$

$$= -e^{-x} (\cos x + \sin x)$$

$$f''(x) = -e^{-x} (-\sin x + \cos x) + e^{-x} (\cos x + \sin x)$$

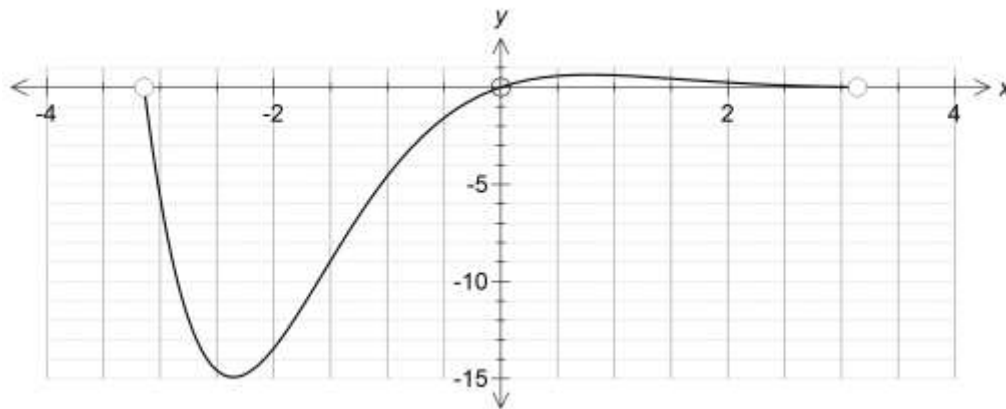
$$= 2e^{-x} \sin x$$

[M2]

At a point of inflection, $f''(x) = 0$

Therefore, $2e^{-x} \sin x = 0 \Rightarrow x = -\pi, 0, \pi$

And sketching $y = f''(x)$, we have



Since $y = f''(x)$ changes sign for $x \in (-\infty, -\pi) \cup (-\pi, \pi) \cup (\pi, \infty)$ then $x = -\pi, 0, \pi$ gives rise to three non-stationary points of inflection.

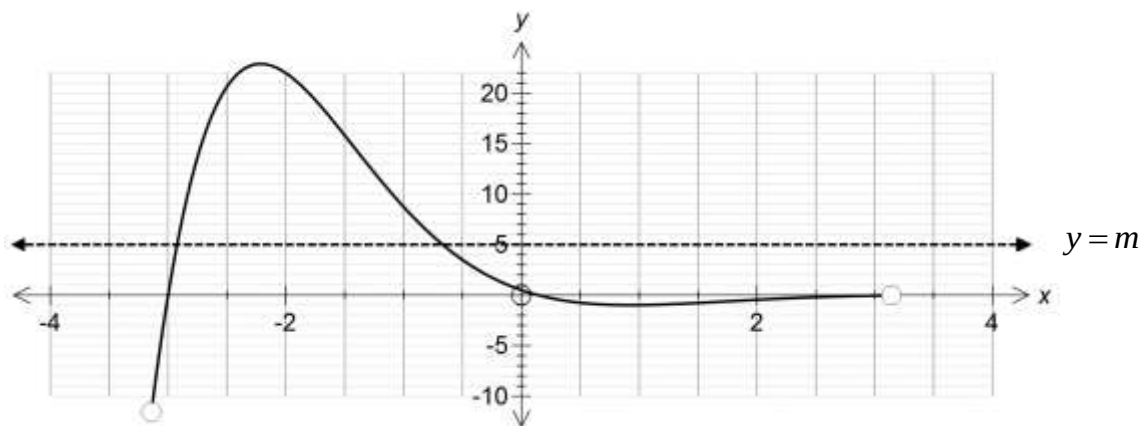
The points of inflection are $(-\pi, -e^\pi)$, $(0, 1)$ and $(\pi, -\frac{1}{e^\pi})$ [A1]

b.

i.

$$\left. \begin{aligned} 2f''(x) + \frac{1}{2}f'(x) + m &= 0 \\ 2(2e^{-x}\sin x) + \frac{1}{2}(-e^{-x}(\cos x + \sin x)) + m &= 0 \\ \frac{1}{2}e^{-x}\cos x - \frac{7}{2}e^{-x}\sin x &= m \end{aligned} \right\} \text{ [M1]}$$

Sketching $y = \frac{1}{2}e^{-x}\cos x - \frac{7}{2}e^{-x}\sin x$, we have



The local minimum is $(0.9273, -0.9891)$. Therefore, there will be three solutions to the differential equation for $m \in (-0.989, 0)$ [M1]

ii

$$\text{Solving } \frac{7}{2}e^{-x}\sin x - \frac{1}{2}e^{-x}\cos x - \frac{1}{4} = 0 \Rightarrow x = -3.00, 0.23, 2.40 \quad \text{[A1]}$$

c.

$$\text{The transformed equation is } g(x) = \frac{-2}{e^{x-\frac{\pi}{8}} \sec\left(x - \frac{\pi}{8}\right)} \quad \text{[A1]}$$

d.

From the calculator,

the points of intersection are $(-1.28, 1.04)$ and $(1.87, -0.04)$ [A2]

e.

From the calculator,

$$\text{The area, } A = \int_{-1.28}^{1.87} \frac{1}{e^x \sec x} - \frac{2}{e^{x-\frac{\pi}{8}} \sec\left(x-\frac{\pi}{8}\right)} dx = 7.3 \text{ square units} \quad [\text{A1}]$$

Question 3

a.

Let the point where the tangent touches the curve be $(a, 2a^2 + 1)$. It follows that the gradient of the line joining the points $(a, 2a^2 + 1)$ and $(1, -3)$ is

$$\frac{2a^2 + 1 - (-3)}{a - 1} = \frac{2a^2 + 4}{a - 1}$$

$$\text{Also, } \frac{dy}{dx} = 4x. \text{ At } (a, 2a^2 + 1), \frac{dy}{dx} = 4a$$

Therefore,

$$\frac{2a^2 + 4}{a - 1} = 4a$$

$$a^2 - 2a - 2 = 0$$

$$a = \frac{2 + \sqrt{12}}{2} \\ = \sqrt{3} + 1$$

$$\text{And } y = 2(\sqrt{3} + 1)^2 + 1 = 4\sqrt{3} + 9$$

Therefore, the point where the tangent touches the curve is $(\sqrt{3} + 1, 4\sqrt{3} + 9)$ [A1]

[M2]

b.

At $(1, -3)$,

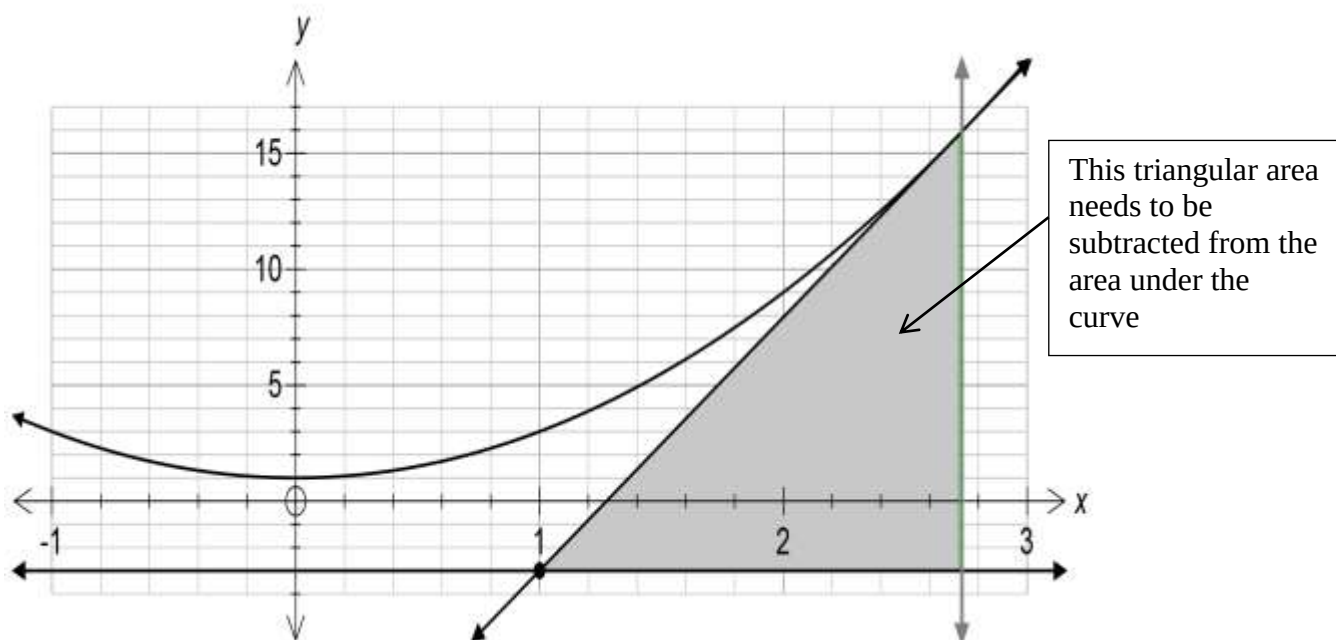
$$y - (-3) = 4(\sqrt{3} + 1)(x - 1)$$

$$y = 4(\sqrt{3} + 1)x - 4\sqrt{3} - 7$$

[A1]

c.

Sketching, we have



The required area, A is calculated as

$$\begin{aligned}
 A &= \int_0^{\sqrt{3}+1} 2x^2 + 1 - (-3) dx - \frac{1}{2} \times (4\sqrt{3} + 9 - (-3)) \times (\sqrt{3} + 1 - 1) && \left. \vphantom{\int_0^{\sqrt{3}+1}} \right\} \text{ [M2]} \\
 &= \int_0^{\sqrt{3}+1} 2x^2 + 4 dx - \frac{1}{2} \times (4\sqrt{3} + 12) \times \sqrt{3} && \left. \vphantom{\int_0^{\sqrt{3}+1}} \right\} \text{ [M1]} \\
 &= 8\sqrt{3} + \frac{32}{3} - (6\sqrt{3} + 6) \\
 &= \frac{2(3\sqrt{3} + 7)}{3} \text{ square units}
 \end{aligned}$$

d.

$$\text{Since } y = 4(\sqrt{3}+1)x - 4\sqrt{3} - 7 \Rightarrow x = \frac{y + 4\sqrt{3} + 7}{4(\sqrt{3}+1)} \quad \left. \vphantom{\frac{y + 4\sqrt{3} + 7}{4(\sqrt{3}+1)}} \right\} \text{ [M1]}$$

$$\text{And since } y = 2x^2 + 1 \Rightarrow x^2 = \frac{y-1}{2}$$

Therefore, the volume, V is calculated as

$$\begin{aligned} V &= \pi \int_{-3}^{4\sqrt{3}+9} \left(\frac{y + 4\sqrt{3} + 7}{4(\sqrt{3}+1)} \right)^2 dy - \pi \int_1^{4\sqrt{3}+9} \frac{y-1}{2} dy \quad \left. \vphantom{\int_1^{4\sqrt{3}+9}} \right\} \text{ [M2]} \\ &= 4\pi(\sqrt{3}+2) \text{ cubic units} \end{aligned}$$

Question 4

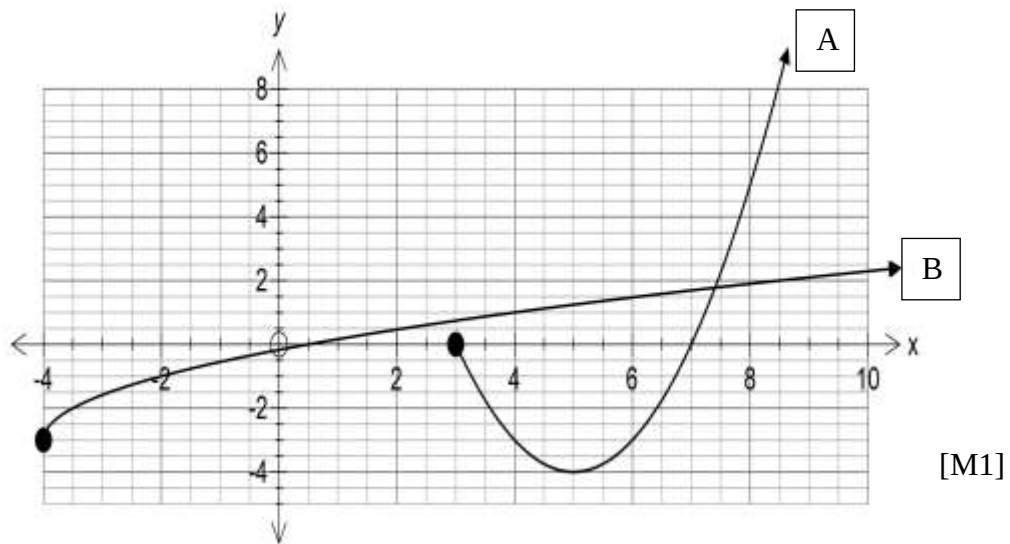
a.

For car B, the parametric equations are $x = \frac{t^2-8}{2}$ and $y = t-3$

Therefore,

$$\begin{aligned} x &= \frac{(y+3)^2 - 8}{2} \\ (y+3)^2 &= 2x+8 \\ y+3 &= \pm\sqrt{2x+8} \\ y &= -3 + \sqrt{2x+8} \text{ since } y \in [-3, \infty) \end{aligned} \quad \left. \vphantom{\frac{(y+3)^2 - 8}{2}} \right\} \text{ [M1]}$$

$$\text{where } 2x+8 \geq 0 \Rightarrow x \in [-4, \infty) \quad \text{[M1]}$$

b.**c.**

Equating the x -coordinates for both cars, $\frac{t^2 - 8}{2} = t + 3 \Rightarrow t = 4.873$.

Therefore, $y_A = 4.873^2 - 4 \times 4.873 = 4.254$ and $y_B = 4.873 - 3 = 1.873$

Since the y -coordinates for both cars do not coincide at the same point in time, the cars do not collide

} [M2]

d.

Note that since the cars do not collide, we cannot use vector methods to find the required angle. Instead, we need to use coordinate geometry.

At the point of intersection,

$$x^2 - 10x + 21 = -3 + \sqrt{2x + 8} \Rightarrow x = 7.403$$

$$\text{For car A, } \frac{dy}{dx} = 2x - 10 \text{ and for car B, } \frac{dy}{dx} = \frac{1}{\sqrt{2x + 8}}$$

At $x = 7.403$,

For car A, the gradient, $m_A = 4.81$ and for car B, $m_B = 0.209$

Therefore, at the point of intersection, the acute angle between the two paths,

$$\theta \text{ is given by } \theta = \tan^{-1}(4.81) - \tan^{-1}(0.21) = 66^\circ \quad [\text{A1}]$$

(Alternatively, you could find the times for both position vectors corresponding to $x = 7.403$, and then find the angle between their respective velocity vectors at those particular times.)

e.

The distance between the cars, d at any time, $t \geq 0$ is given by

$$\sqrt{\left(\frac{t^2 - 2t - 14}{2}\right)^2 + (t^2 - 5t + 3)^2} \quad [\text{M1}]$$

f.

$$\sqrt{\left(\frac{t^2 - 2t - 14}{2}\right)^2 + (t^2 - 5t + 3)^2} = 5 \Rightarrow t = 3.47 \text{ and } 5.31 \text{ seconds} \quad [\text{A1}]$$

g.

$$\left. \begin{aligned} \underline{v}_A &= \underline{i} + (2t - 4)\underline{j} \\ \underline{v}_B &= t\underline{i} + \underline{j} \end{aligned} \right\} \quad [\text{A1}]$$

h.

$$\text{Require } \underline{v}_A \cdot \underline{v}_B = 0 \Rightarrow (\underline{i} + (2t - 4)\underline{j}) \cdot (t\underline{i} + \underline{j}) = 0 \quad [\text{M1}]$$

$$\text{i.e. } t + 2t - 4 = 0 \Rightarrow 3t - 4 = 0 \Rightarrow t = \frac{4}{3} \text{ seconds} \quad [\text{A1}]$$

Question 5

a.

$$v^2 = 2g \times 90 \Rightarrow v = \sqrt{2g \times 90} = 42 \text{ ms}^{-1} \quad [\text{A1}]$$

b.

The equation of motion is

$$5v^2 - 85g = -85a \quad [\text{M1}]$$

Therefore,

$$\left. \begin{aligned} a &= \frac{85g - 5v^2}{85} \\ &= \frac{17g - v^2}{17} \end{aligned} \right\} [\text{M1}]$$

c.

$$\begin{aligned}
 v \frac{dv}{dx} &= \frac{17g - v^2}{17} \\
 \frac{dv}{dx} &= \frac{17g - v^2}{17v} \\
 \frac{dx}{dv} &= \frac{17v}{17g - v^2} \\
 x &= \int \frac{17v}{17g - v^2} dv
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} v \frac{dv}{dx} &= \frac{17g - v^2}{17} \\ \frac{dv}{dx} &= \frac{17g - v^2}{17v} \\ \frac{dx}{dv} &= \frac{17v}{17g - v^2} \end{aligned}} \right\} \text{[M2]}$$

Let $u = 17g - v^2$

$$\frac{du}{dv} = -2v \Rightarrow -\frac{1}{2} \frac{du}{dv} = v$$

Therefore,

$$\begin{aligned}
 x &= \int \frac{17v}{17g - v^2} dv \\
 &= -\frac{17}{2} \int \frac{1}{u} du \\
 &= -\frac{17}{2} \log_e |u| + c \\
 &= -\frac{17}{2} \log_e |17g - v^2| + c
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} x &= \int \frac{17v}{17g - v^2} dv \\ &= -\frac{17}{2} \int \frac{1}{u} du \\ &= -\frac{17}{2} \log_e |u| + c \end{aligned}} \right\} \text{[M1]}$$

When $x = 0$, $v = 42$

Therefore,

$$0 = -\frac{17}{2} \log_e |-1597.4| + c$$

$$c = \frac{17}{2} \log_e |-1597.4|$$

It follows that

$$\begin{aligned}
 x &= \frac{17}{2} \log_e \left| \frac{1597.4}{v^2 - 17g} \right| \\
 e^{\frac{2x}{17}} &= \frac{v^2 - 17g}{1597.4} \\
 v^2 &= 1597.4 e^{\frac{2x}{17}} + 17g \\
 v &= \sqrt{1597.4 e^{\frac{2x}{17}} + 166.6}
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} x &= \frac{17}{2} \log_e \left| \frac{1597.4}{v^2 - 17g} \right| \\ e^{\frac{2x}{17}} &= \frac{v^2 - 17g}{1597.4} \\ v^2 &= 1597.4 e^{\frac{2x}{17}} + 17g \end{aligned}} \right\} \text{[M1]}$$

d.

$$\text{As } x \rightarrow \infty, e^{-\frac{2x}{17}} \rightarrow 0 \Rightarrow v \rightarrow \sqrt{166.6} = 12.91 \text{ms}^{-1} \quad [\text{A1}]$$

e.

$$\text{Solving } \sqrt{1597.4e^{-\frac{2x}{17}} + 166.6} = 25.8 \Rightarrow x = 9.89 \quad [\text{M1}]$$

The distance that Alex is from ground level at this point is $738 - 9.89 \approx 728 \text{ m}$ [A1]

f.

