

SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 1



(TSSM's 2012 trial exam updated for the current study design)

SOLUTIONS

Question 1

$$\left. \begin{aligned} \text{Since } \tan^{-1}(x-20) &\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ then} \\ \frac{1}{2}\tan^{-1}(x-20) + \frac{2\pi}{3} &\in \left(\frac{1}{2}\times -\frac{\pi}{2} + \frac{2\pi}{3}, \frac{1}{2}\times \frac{\pi}{2} + \frac{2\pi}{3}\right) = \left(\frac{5\pi}{12}, \frac{11\pi}{12}\right) \end{aligned} \right\} \text{ [M1]}$$

Therefore,

$$\frac{1}{2}\tan^{-1}(x-20) + \frac{2\pi}{3} = p \quad \text{has no solutions when } p \in \left(-\infty, \frac{5\pi}{12}\right] \cup \left[\frac{11\pi}{12}, \infty\right) \quad \text{[A1]}$$

Question 2

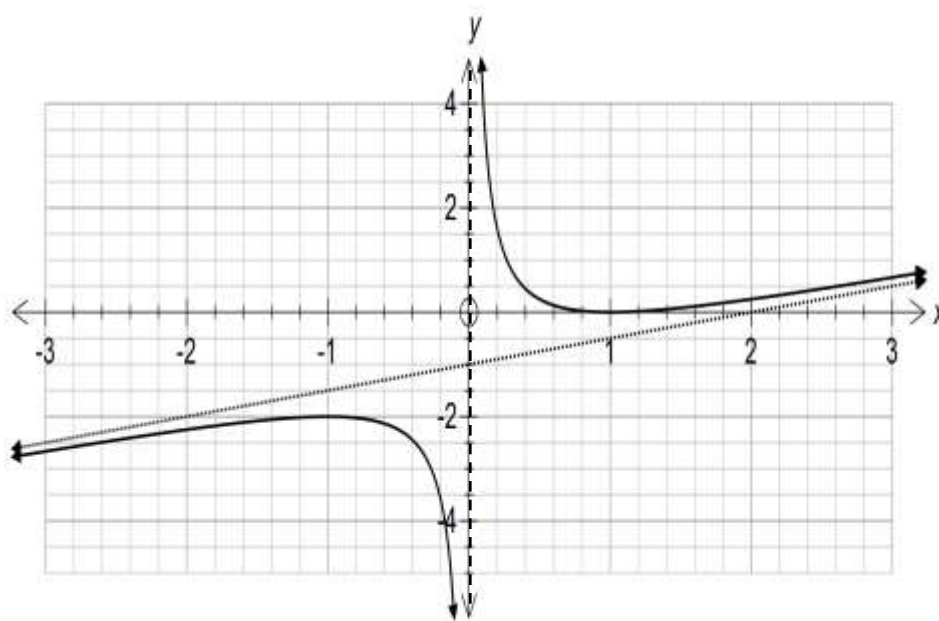
a. $y = \frac{(x-1)^2}{2x} \Rightarrow y = \frac{x^2 - 2x + 1}{2x} = \frac{x}{2} - 1 + \frac{1}{2x} \quad \text{[A1]}$

b. The x -intercept is $(1, 0)$ and the asymptotes are $y = \frac{x}{2} - 1$ and $x = 0 \quad \text{[A1]}$

$$\left. \begin{aligned} \frac{dy}{dx} &= \frac{1}{2} - \frac{1}{2x^2} = \frac{x^2 - 1}{2x^2} \\ \text{At a stationary point, } \frac{x^2 - 1}{2x^2} &= 0 \Rightarrow x = \pm 1 \end{aligned} \right\} \text{ [M1]}$$

Therefore, the minimum and maximum turning points are $(1, 0)$ and $(-1, -2)$ respectively. [A1]

c.



[A2]

Question 3

$$z = 2\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right) \text{ and } w = 4\operatorname{cis}\left(\frac{\pi}{6}\right) \quad [\text{M1}]$$

From de Moivre's theorem,

$$\left. \begin{aligned} \operatorname{Arg}\left(\frac{z}{w}\right)^3 &= \frac{\operatorname{cis}\left(-\frac{3\pi}{4}\right)}{\operatorname{cis}\left(\frac{\pi}{2}\right)} \\ &= \operatorname{cis}\left(-\frac{5\pi}{4}\right) \end{aligned} \right\} [\text{M1}]$$

$$\text{Since } \operatorname{Arg}\left(\frac{z}{w}\right)^3 \in (-\pi, \pi] \Rightarrow \operatorname{Arg}\left(\frac{z}{w}\right)^3 = 2\pi - \frac{5\pi}{4} = \frac{3\pi}{4} \quad [\text{A1}]$$

Question 4

$$\begin{aligned}
 f'(x) &= \frac{1}{\sqrt{1 - \left(\frac{1}{2x}\right)^2}} \times \frac{d}{dx} \left(\frac{1}{2x} \right) \\
 &= -\frac{\frac{1}{2x^2}}{\sqrt{\frac{4x^2 - 1}{4x^2}}} \\
 &= -\frac{1}{\sqrt{x^2(4x^2 - 1)}} \\
 &= \frac{-1}{|x|\sqrt{4x^2 - 1}}, \quad x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} f'(x) &= \frac{1}{\sqrt{1 - \left(\frac{1}{2x}\right)^2}} \times \frac{d}{dx} \left(\frac{1}{2x} \right)} \right\} \text{[M2]}$$

Since $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$$f'(x) = \begin{cases} \frac{1}{x\sqrt{4x^2 - 1}}, & x \in \left(-\infty, -\frac{1}{2}\right) \\ \frac{-1}{x\sqrt{4x^2 - 1}}, & x \in \left(\frac{1}{2}, \infty\right) \end{cases} \quad \text{[A2]}$$

Question 5

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{x}{1-x^2}$$

$$\frac{1}{2}v^2 = \int \frac{x}{1-x^2} dx$$

} [M1]

$$u = 1 - x^2$$

$$-\frac{1}{2} \frac{du}{dx} = x$$

$$\frac{1}{2}v^2 = -\frac{1}{2} \int \frac{1}{u} du$$

} [M1]

$$v^2 = -\log_e |1 - x^2| + c$$

Given $v(2) = -1$

$$1 = -\log_e |-3| + c$$

$$c = 1 + \log_e |-3|$$

$$v^2 = 1 + \log_e |-3| - \log_e |1 - x^2|$$

$$= 1 + \log_e \left| \frac{3}{x^2 - 1} \right|$$

} [M1]

And since $v(2) = -1$ then $v(x) = -\sqrt{1 + \log_e \left| \frac{3}{x^2 - 1} \right|}$ [A1]

Question 6

a. $\overrightarrow{OB} = \underline{a} + \underline{b}$ and $\overrightarrow{AC} = \underline{b} - \underline{a}$ [A1]

b. If $\overrightarrow{OB} \perp \overrightarrow{AC}$

It follows that

$$(\underline{a} + \underline{b}) \cdot (\underline{b} - \underline{a}) = 0$$

$$\underline{a} \cdot \underline{b} - \underline{a} \cdot \underline{a} + \underline{b} \cdot \underline{b} - \underline{b} \cdot \underline{a} = 0$$

$$-\underline{a} \cdot \underline{a} + \underline{b} \cdot \underline{b} = 0$$

$$-|\underline{a}|^2 + |\underline{b}|^2 = 0$$

$$|\underline{a}|^2 = |\underline{b}|^2$$

$$|\underline{a}| = |\underline{b}|$$

} [M2]

Therefore, $|\underline{a}| = |\underline{b}|$ as required [A1]

Question 7

$$\left. \begin{aligned} \frac{d}{dx}(x^2y + 2y^2) &= 0 \\ 2xy + x^2 \frac{dy}{dx} + 4y \frac{dy}{dx} &= 0 \\ 2xy + \frac{dy}{dx}(x^2 + 4y) &= 0 \\ \frac{dy}{dx} &= -\frac{2xy}{x^2 + 4y} \end{aligned} \right\} \text{ [M1]}$$

$$\left. \begin{aligned} \text{When } x &= 1, \\ y + 2y^2 &= 6 \\ 2y^2 + y - 6 &= 0 \\ (2y - 3)(y + 2) &= 0 \\ y &= \frac{3}{2} \text{ since } y > 0 \end{aligned} \right\} \text{ [M1]}$$

Therefore, $\frac{dy}{dx} = -\frac{2 \times 1 \times \frac{3}{2}}{1 + 4 \times \frac{3}{2}} = -\frac{3}{7}$ and so the gradient of the normal is $\frac{7}{3}$ [M1]

It follows that the equation of the normal is

$$y - \frac{3}{2} = \frac{7}{3}(x - 1) \Rightarrow y = \frac{7}{3}x - \frac{5}{6} \quad \text{[A1]}$$

Question 8

$$\int_0^{\frac{\pi}{4}} \frac{(\sec^2 \theta - 1)}{\cos^2 \theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{\tan^2 \theta}{\cos^2 \theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^2 \theta \sec^2 \theta d\theta$$

Let $u = \tan \theta$

$$\frac{du}{d\theta} = \sec^2 \theta$$

$$\theta = 0 \Rightarrow u = 0$$

$$\theta = \frac{\pi}{4} \Rightarrow u = 1$$

Therefore,

$$\int_0^{\frac{\pi}{4}} \frac{(\sec^2 \theta - 1)}{\cos^2 \theta} d\theta = \int_0^1 u^2 \frac{du}{d\theta} d\theta$$

$$= \int_0^1 u^2 du$$

$$= \left[\frac{u^3}{3} \right]_0^1$$

$$= \frac{1}{3}$$

Question 9

At the point of intersection, $x = e^2 \Rightarrow y = \log_e e^2 = 2$ [M1]

Also $y = \log_e x \Rightarrow x = e^y$

The required volume, V is calculated as

$$V = \pi \int_0^2 (e^2)^2 dy - \pi \int_0^2 (e^y)^2 dy$$

$$= \pi \int_0^2 e^4 dy - \pi \int_0^2 e^{2y} dy$$

$$= \pi [e^4 y]_0^2 - \frac{\pi}{2} [e^{2y}]_0^2$$

$$= 2\pi e^4 - \frac{\pi}{2} (e^4 - 1)$$

$$= \frac{\pi}{2} (3e^4 + 1) \text{ units}^3$$

Question 10

a.

$$T - 4g \sin(30^\circ) = 4a$$

$$5g - T = 5a \rightarrow T = 5g - 5a$$

$$5g - 5a - 2g = 4a \quad [M2]$$

$$9a = 3g$$

$$a = \frac{g}{3} m/s^2$$

[A1]

b.

$$T = 5g - 5a = 5g - \frac{5g}{3} = \frac{10g}{3} N$$