The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2012

Trial Written Examination 2 - SOLUTIONS:

SECTION 1: Multiple Choice

ANSWERS

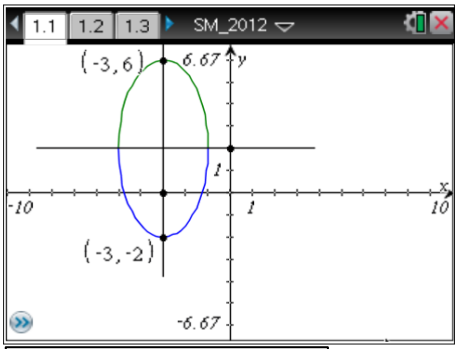
1. D	**2. E**	**3. D**	**4. C**	**5. E**	**6. A**
7. B	**8. C**	**9. B**	**10. A**	**11. E**	**12. C**
13. D	**14. B**	**15.A**	**16. E**	**17. D**	**18. C**
19. B	**20. A**	**21. D**	**22. C**		

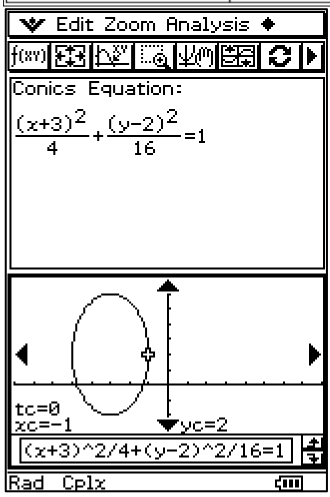
Question 1 **Answer: D**

For range $[-2,6]$ the equation could be of the form $\frac{(x-h)^2}{a^2}+\frac{(y-2)^2}{4^2}=1$. Option D, is of this form:

$$\frac{16(x+3)^2}{64}+\frac{4(y-2)^2}{64}=\frac{64}{64}$$

$$\frac{(x+3)^2}{4}+\frac{(y-2)^2}{16}=1.$$





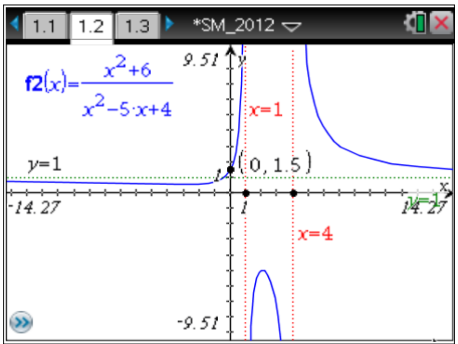
Question 2 **Answer: E**

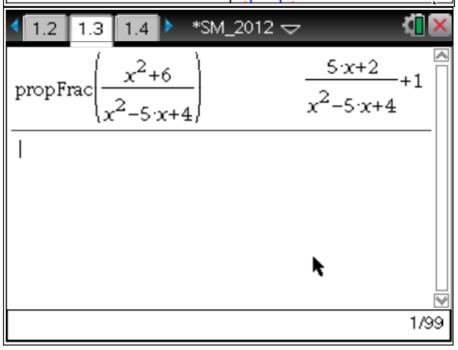
$$y=\frac{x^2+6}{x^2-5x+4}=\frac{x^2+6}{(x-1)(x-4)},\ x\neq 1,4$$

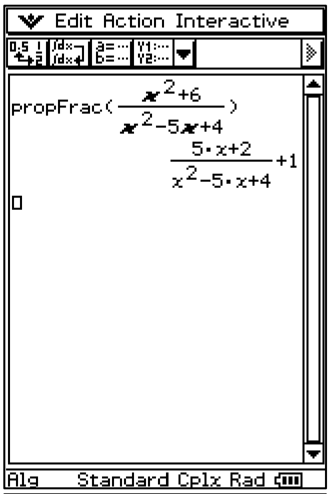
Also, $y=\frac{x^2+6}{x^2-5x+4}=\frac{5x+2}{x^2-5x+4}+1$

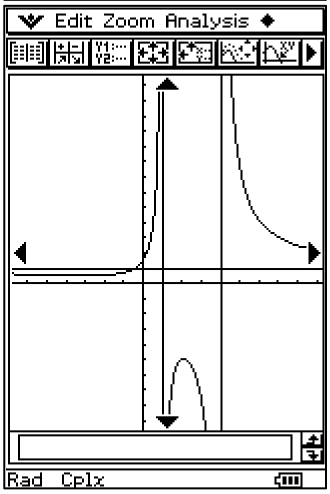
As $x\to\pm\infty$, $y\to 1$

Therefore there are three asymptotes with equations $y=1$, $x=4$ and $x=1$.









Question 3 **Answer: D**

$y = 2\cos(t)$ and

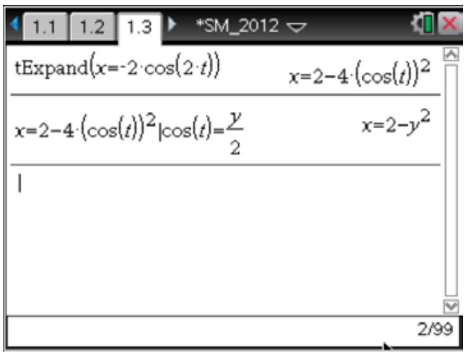
$x = -2\cos(2t)$

$\quad = -2\left(2\cos^2(t) - 1\right)$

$x = -4\cos^2(t) + 2$

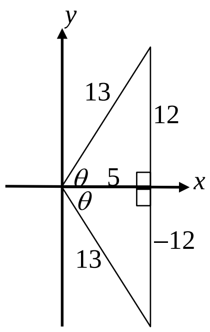
$x = -y^2 + 2$

$y^2 + x - 2 = 0$



Question 4 **Answer: C**

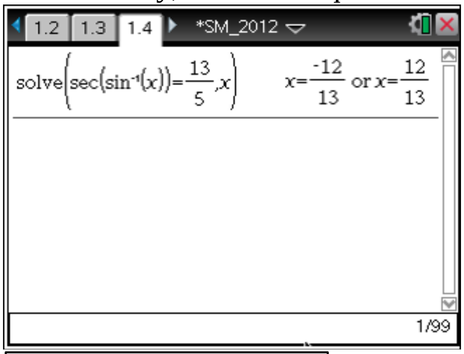
Consider the (5, 12, 13) Pythagorean triangles in the first or fourth quadrants.

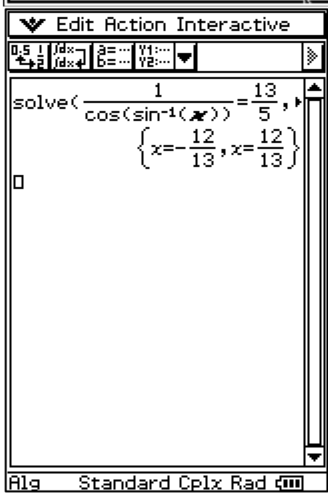


$\sec(\phi) = \frac{hyp}{adj} = \frac{13}{5}$, where

$\phi = \sin(\theta) = \frac{opp}{hyp} = \frac{12}{13}$ or $-\frac{12}{13}$

Alternatively, solve the equation on CAS.





Question 5 **Answer: E**

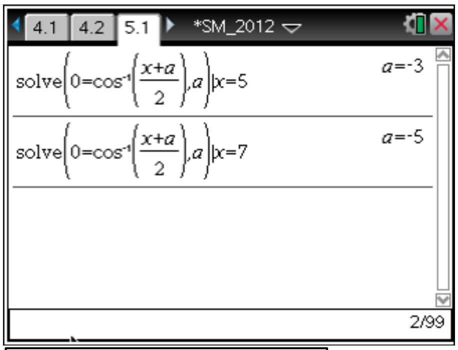
For the circle

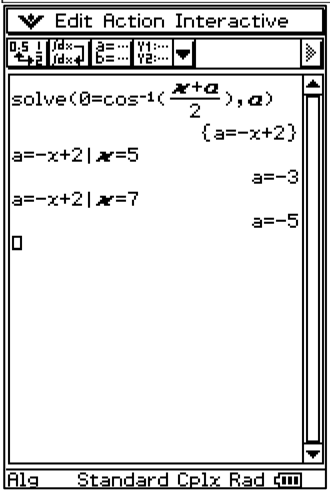
$(x-6)^2 + y^2 = 1$

Centre (6, 0) , radius = 1 and x-intercepts $(5,0)$ and $(7,0)$.

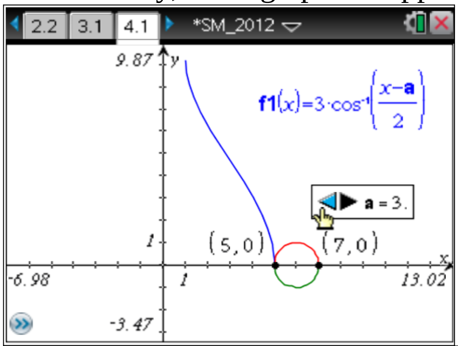
$3\cos^{-1}\left(\frac{x+a}{2}\right) = 0$, with $x = 5$ or $x = 7$.

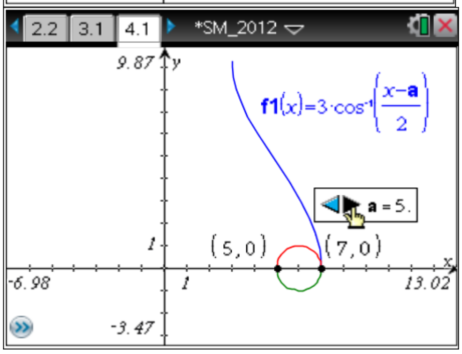
Hence $a = -3$ or $a = -5$.

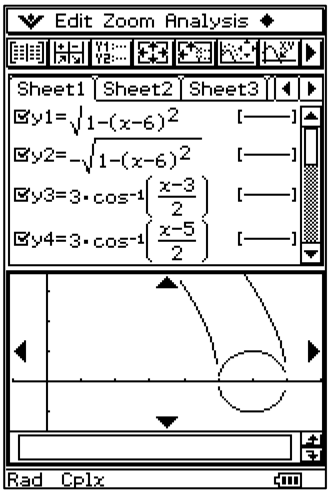




Alternatively, use a graphical approach.







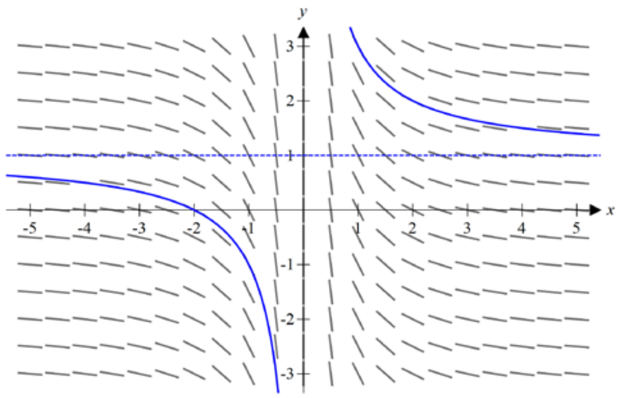
Question 6 **Answer: A**

$f'(b)=0$ and $f''(b)=0$ does **not necessarily** mean a stationary point inflection at $x=b$; consider a function such as $f(x)=(x-b)^4$ to illustrate the point. The additional conditions given in options **D** and **E** also need be met for $f'(b)=0$ and $f''(b)=0$ to signify a stationary point of inflection at $x=b$.

Question 7 **Answer: B**

The family of functions could be of the form

$y=\dfrac{a}{x}+c$, where $c\in R$.



The DE giving rise could therefore be

$\dfrac{dy}{dx}=-\dfrac{a}{x^2}$, because

$y=-a\int x^{-2}\,dx$

$y=ax^{-1}+c=\dfrac{a}{x}+c$

Question 8 **Answer: C**

Euler's method: $y_{n+1} = y_n + h\,f(x_n)$, where

$h = \frac{1}{5}$, $x_0 = 1$ and $y_0 = -3$.

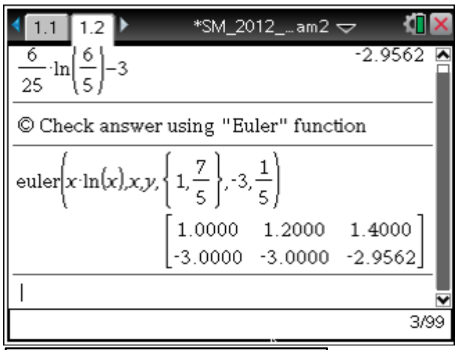
$$y_1 = y_0 + h\,f(x_0)$$

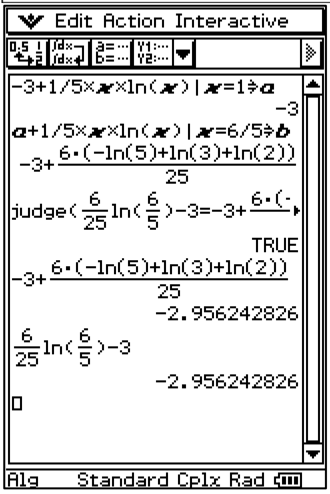
$$y_1 = -3 + \frac{1}{5} \times 1 \times \log_e(1) = -3$$

$$y_2 = y_1 + h\,f(x_1)$$

$$y_2 = -3 + \frac{1}{5} \times \frac{6}{5} \log_e\left(\frac{6}{5}\right)$$

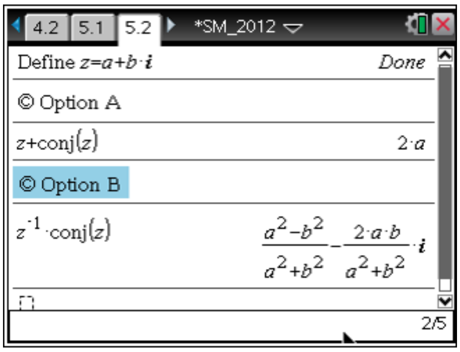
$$y_2 = \frac{6}{25} \log_e\left(\frac{6}{5}\right) - 3$$

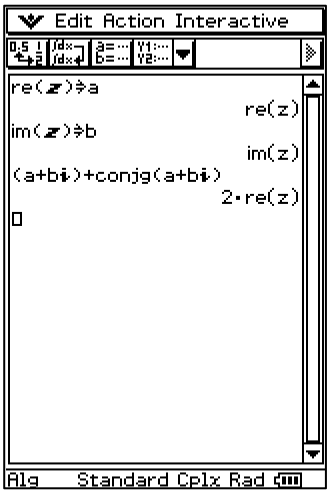




Question 9 **Answer: B**

$$z^{-1}\bar{z} = \frac{a-bi}{a+bi} = \frac{a^2-b^2}{a^2+b^2} - \frac{2ab}{a^2+b^2}i$$

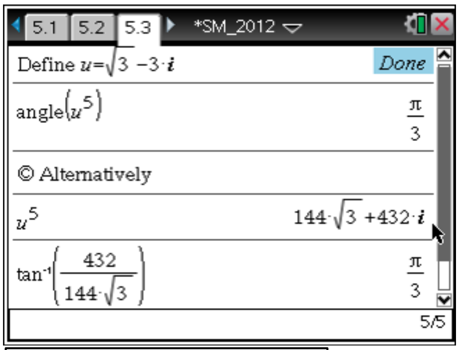


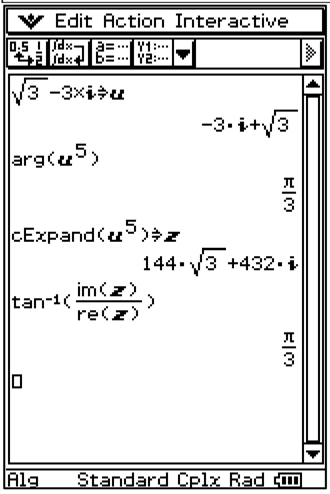


Question 10 **Answer: A**

$$u = \sqrt{3} - 3i$$

$$\text{Arg}\left(u^5\right) = \frac{\pi}{3}$$





Question 11 **Answer: E**

$$w = \left(\sqrt{3}\right)^6 \text{cis}\left(\frac{6\pi}{15}\right)$$

$$\overline{w} = \left(\sqrt{3}\right)^6 \text{cis}\left(-\frac{6\pi}{15}\right)$$

$$\overline{w} = 27\text{cis}\left(-\frac{2\pi}{5}\right)$$

Question 12 **Answer: C**

If the circle were centred at the origin, the region would be

$\{z : |z| > 4\}$

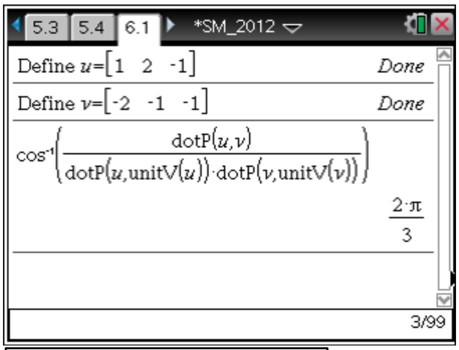
Since the circle is centred at $1-i$, the region is

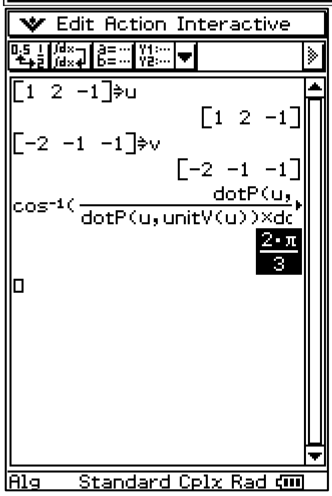
$\{z : |z-(1-i)| > 4\}$, or

$\{z : |z-1+i| > 4\}$

Question 13 **Answer: D**

$$\theta = \cos^{-1}\left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}\right) = \frac{2\pi}{3}$$

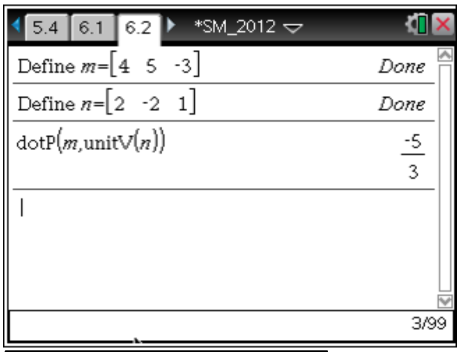


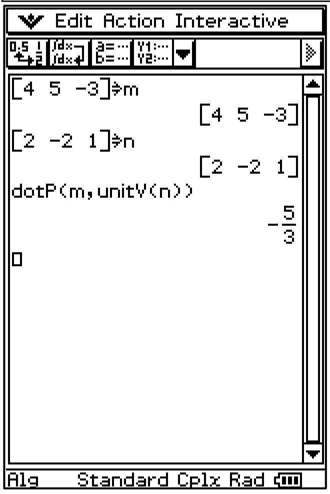


Question 14 **Answer: B**

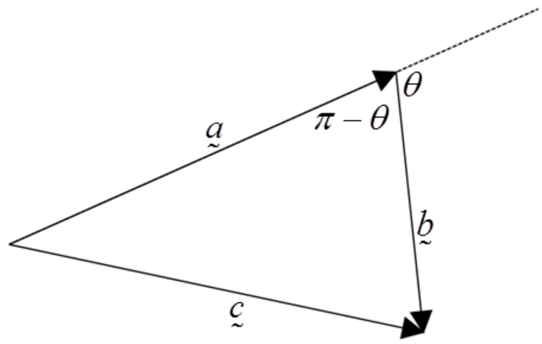
The scalar resolute of the vector $\underset{\sim}{m} = 4\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}$ in the direction of the vector $\underset{\sim}{n} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ is given by

$$\underset{\sim}{m}\cdot\hat{\underset{\sim}{n}} = -\frac{5}{3}$$





Question 15 **Answer: A**



Cosine rule

$$|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos(\pi-\theta)$$

But $\cos(\pi-\theta) = -\cos(\theta)$, therefore

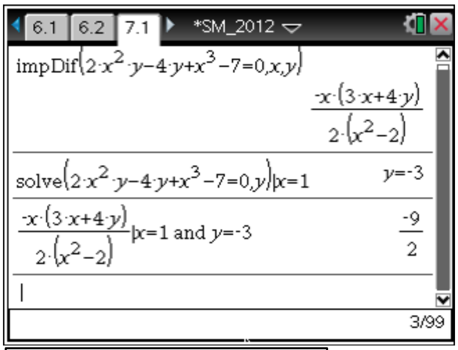
$$|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$$

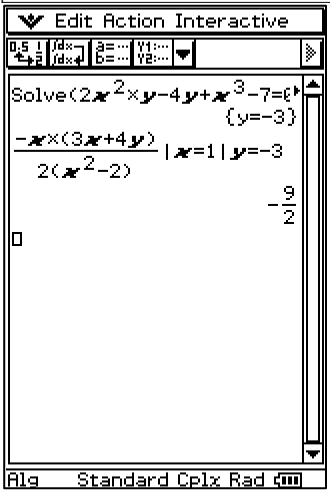
$$\cos(\theta) = \frac{|\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2}{2|\underset{\sim}{a}||\underset{\sim}{b}|}$$

Question 16 **Answer: E**

$2x^2y - 4y + x^3 - 7 = 0$. When $x = 1, y = -3$.

$\dfrac{dy}{dx} = \dfrac{-3x^2 - 4xy}{2x^2 - 4}$. If $x = 1, y = -3$, $\dfrac{dy}{dx} = -\dfrac{9}{2}$.





Question 17 **Answer: D**

$u = \log_e(x)$

$\dfrac{du}{dx} = \dfrac{1}{x}$

When $x = 1$, $u = \log_e(1) = 0$

When $x = e$, $u = \log_e(e) = 1$

$$\int_{x=1}^{x=e} \left(\frac{\left(\log_e\left(x^3\right)\right)^2}{x} \right) dx = \int_{x=1}^{x=e} \left(\left(3\log_e(x)\right)^2 \times \frac{1}{x} \right) dx$$

$$= \int_{u=0}^{u=1} \left((3u)^2 \times \frac{du}{dx} \right) dx$$

$$= \int_0^1 9u^2 \, du$$

Question 18 **Answer: C**

At the instant the rocket runs out of fuel,

$s_1 = h \text{ m}$, $a = -g \text{ ms}^{-2}$, $u = u \text{ ms}^{-1}$

The rocket will reach maximum height when $v = 0 \text{ ms}^{-1}$. From when it runs out of fuel to reaching maximum height,

$v^2 = u^2 + 2as$

$0 = u^2 - 2gs_2$

$s_2 = \dfrac{u^2}{2g}$

Total height reached

$$s_1 + s_2 = h + \frac{u^2}{2g}$$

$$= \frac{u^2 + 2gh}{2g}$$

Question 19 **Answer: B**

The magnitude of area under the acceleration-time graph gives the change in velocity.

From $t = 0$ to $t = 30$, $\Delta v = \dfrac{30 \times 3}{2} = 45 \text{ ms}^{-1}$.

$\Delta v = v - u$

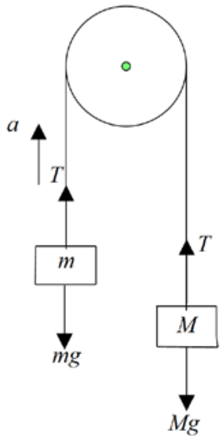
$45 = v - 0$

$v = 45 \text{ ms}^{-1}$

From $t = 30$ to $t = 60$, constant velocity.

Therefore at $t = 60$, $v = 45 \text{ ms}^{-1}$.

Question 20 **Answer: A**



$ma = T - mg$

$T = ma + mg$ … equation(1)

$Ma = Mg - T$ … equation(2)

Substitute equation(1) in equation(2)

$Ma = Mg - ma - mg$

$Ma + ma = Mg - mg$

$a(M + m) = g(M - m)$

$$a = \frac{g(M - m)}{M + m}$$

Question 21 **Answer: D**

$$\frac{dx}{dt} = \cos^2(x)$$

$$\frac{dt}{dx} = \sec^2(x)$$

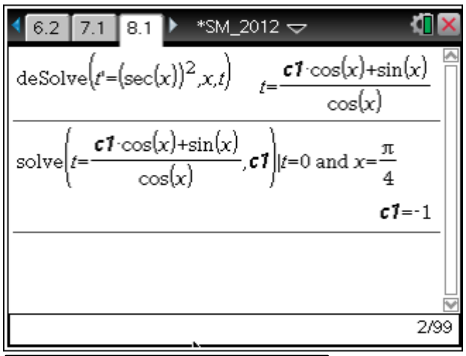
$$t = \int \sec^2(x)\,dx$$

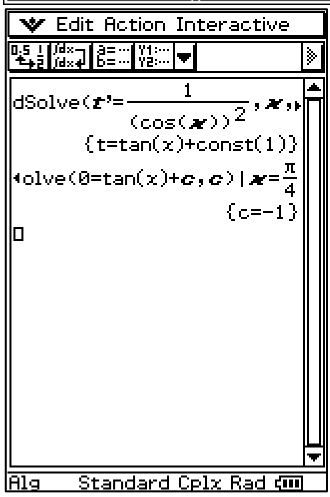
$$t = \tan(x) + C$$

When $t = 0, x = \frac{\pi}{4}$

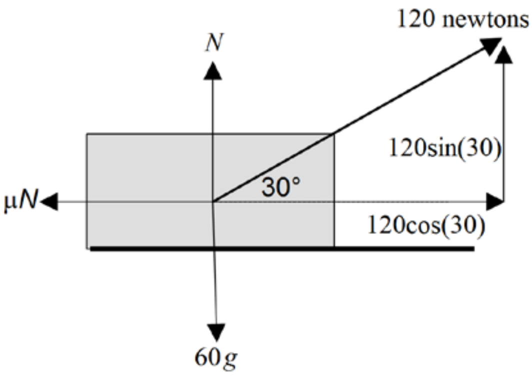
$$c = -\tan\left(\frac{\pi}{4}\right) = -1$$

$$t = \tan(x) - 1$$





Question 22 **Answer: C**

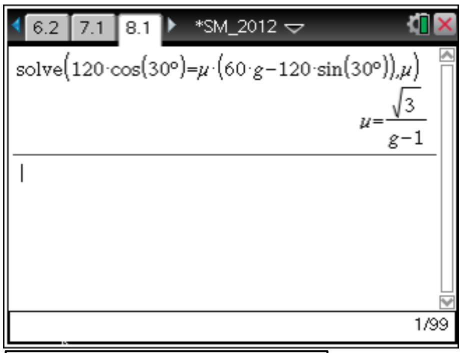


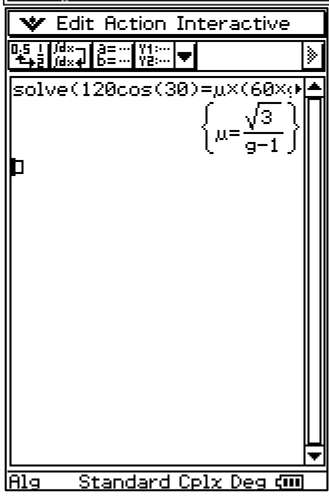
Since the velocity is constant,

$$120\cos(30^\circ) = \mu(60g - 120\sin(30^\circ))$$

Solve for μ,

$$\mu = \frac{\sqrt{3}}{g-1}$$





END OF SECTION 1 SOLUTIONS

SECTION 2: Extended Response SOLUTIONS

Question 1

a.i.

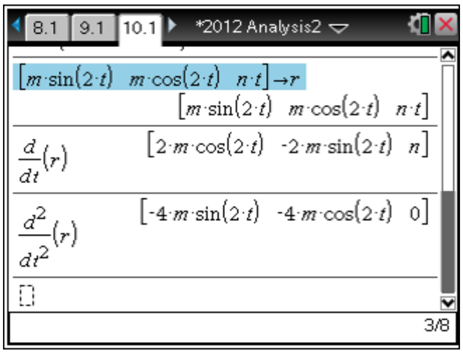
$\underset{\sim}{r} = m\sin(2t)\,\underset{\sim}{i} + m\cos(2t)\,\underset{\sim}{j} + nt\,\underset{\sim}{k}$

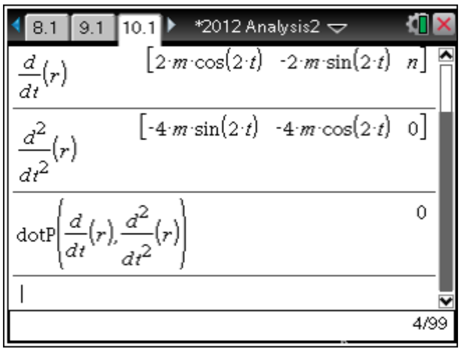
$\dot{\underset{\sim}{r}} = 2m\cos(2t)\,\underset{\sim}{i} - 2m\sin(2t)\,\underset{\sim}{j} + n\,\underset{\sim}{k}$ 1A

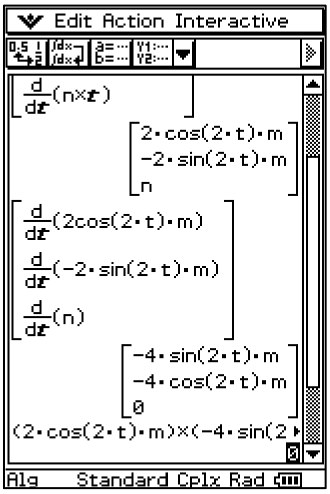
$\ddot{\underset{\sim}{r}} = -4m\sin(2t)\,\underset{\sim}{i} - 4m\cos(2t)\,\underset{\sim}{j}$ 1A

$\dot{\underset{\sim}{r}} \cdot \ddot{\underset{\sim}{r}} = -8m^2\sin(2t)\cos(2t) + 8m^2\sin(2t)\cos(2t) + 0$

$\dot{\underset{\sim}{r}} \cdot \ddot{\underset{\sim}{r}} = 0$, therefore the vectors are perpendicular. 1A







ii.

$|\dot{\underset{\sim}{r}}| = \sqrt{4m^2\left(\cos^2(2t)+\sin^2(2t)\right)+n^2}$

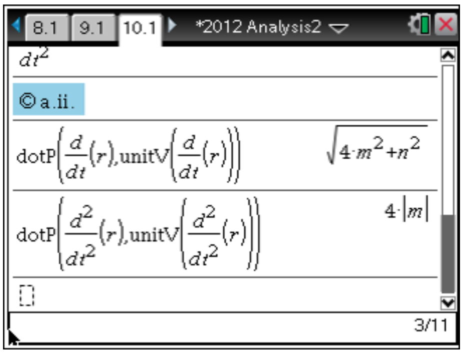
$|\dot{\underset{\sim}{r}}| = \sqrt{4m^2+n^2}$ 1M

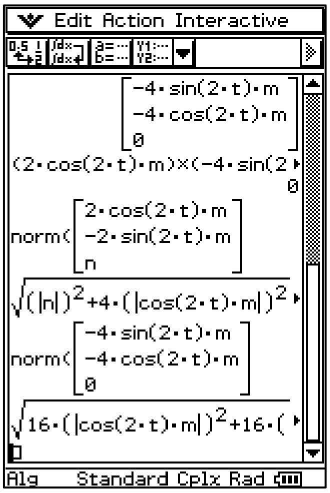
$|\ddot{\underset{\sim}{r}}| = \sqrt{16m^2\left(\sin^2(2t)+\cos^2(2t)\right)}$

$|\ddot{\underset{\sim}{r}}| = 4m$ or $-4m$. Alternatively, $|\ddot{\underset{\sim}{r}}| = 4|m|$ 1A

Both magnitudes are independent of *t* and are therefore constant.

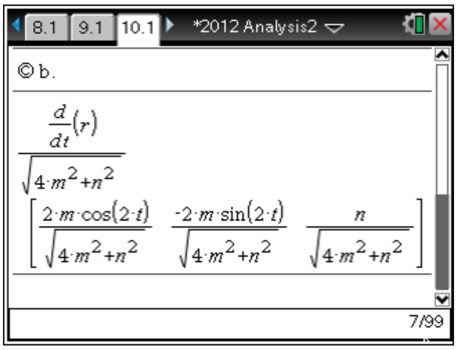
Note that *m* is a non-zero number, which could be negative. That is why the '$-4m$' solution needs to be included.

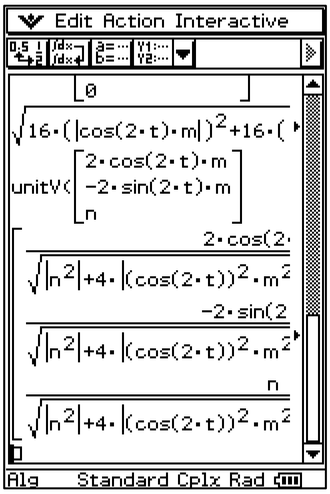


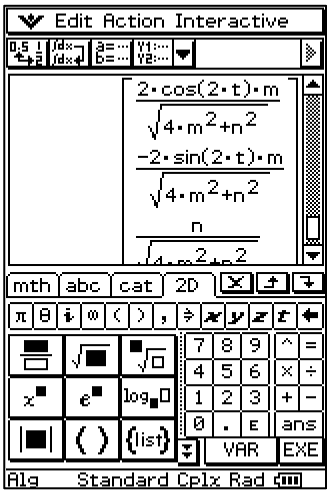


b.i.

$$\hat{\underset{\sim}{r}} = \frac{1}{\sqrt{4m^2+n^2}}\left(2m\cos(2t)\underset{\sim}{i} - 2m\sin(2t)\underset{\sim}{j} + n\underset{\sim}{k}\right)$$ 1A







ii.

Let θ be the angle between the unit vectors $\hat{\underset{\sim}{r}}$ and $\underset{\sim}{k}$.

$\hat{\underset{\sim}{r}} \cdot (0\underset{\sim}{i} + 0\underset{\sim}{j} + \underset{\sim}{k}) = \cos(\theta)$ 1M

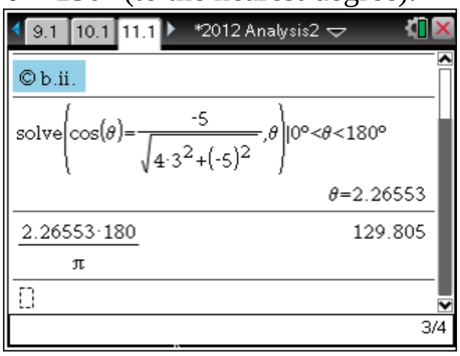
$$\cos(\theta) = \frac{n}{\sqrt{4m^2+n^2}}$$ 1A

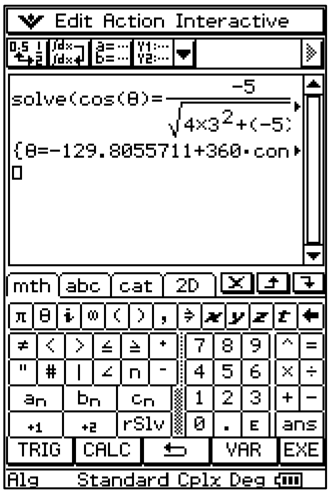
The angle is independent of t and is constant.

For $m = 3$ and $n = -5$, $90^\circ < \theta < 180^\circ$

Solve $\cos(\theta) = \dfrac{-5}{\sqrt{4\times 3^2 + (-5)^2}}$ for θ, $90^\circ < \theta < 180^\circ$

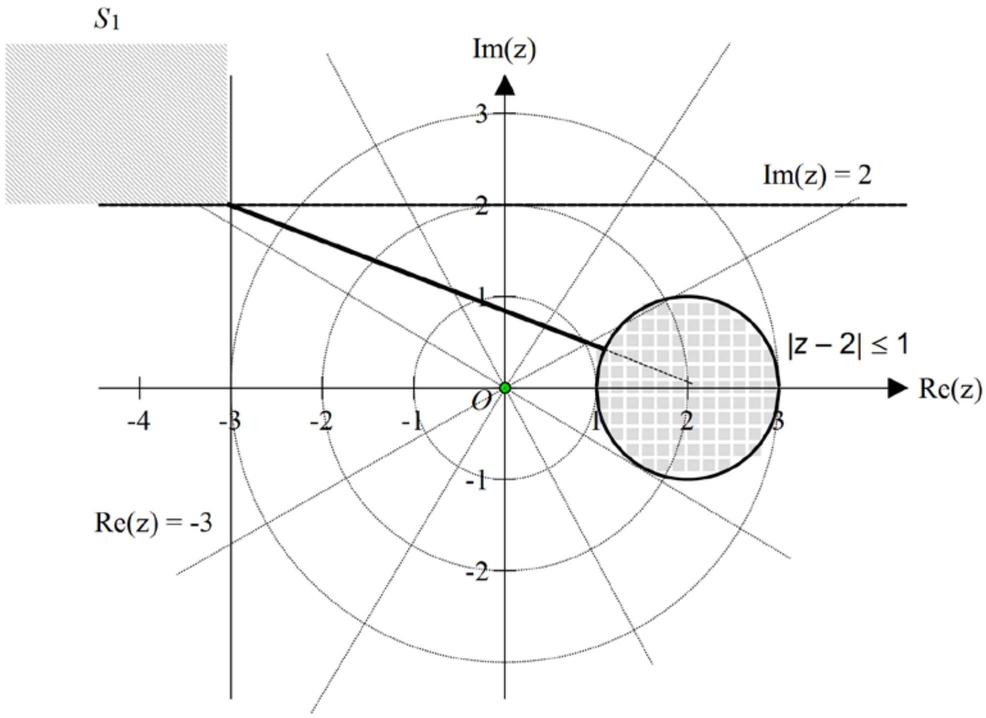
$\theta = 130^\circ$ (to the nearest degree). 1A





Question 2

a.



Correct graphs 1A

Correct region shaded and labels shown. 1A

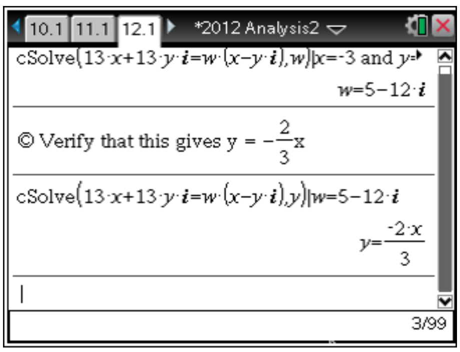
b.

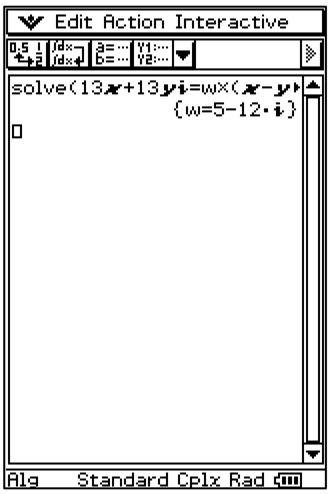
$13z = w\overline{z}$

$13(x+yi) = w(x-yi)$, with $x=-3$ and $y=2$.

$13(-3+2i) = w(-3-2i)$ 1M

$w = \dfrac{13(-3+2i)}{(-3-2i)} = 5-12i$ (or equivalent expression, whether simplified or unsimplified) 1A





c.i.

$|x-2+yi| \le 1$ 1M

$\sqrt{(x-2)^2+y^2} \le 1$

$(x-2)^2+y^2 \le 1$ 1A

The 'method' mark could alternatively be awarded for the student recognising that the region is a translation of $|z| \le 1$, or the like.

ii.

See argand diagram above. Correct centre and shaded inside the circle. 1A

d.i.

Line joining the points with coordinates $(-3,2)$ and $(2,0)$ correctly shown on argand diagram.

Gradient = $-\frac{2}{5}$ and *x*-intercept at $x=2$, therefore 1M

$y=-\frac{2}{5}(x-2)$ or equivalent form 1A

ii.

The minimum distance between S_1 and S_2 is the distance from $(-3,2)$ to the point where the line intersects the boundary of S_2.

Using Pythagoras' theorem 1M

Distance from $(-3,2)$ to the centre of the circle is given by $\sqrt{5^2+2^2}$.

However, since the radius of the circle is 1 unit:

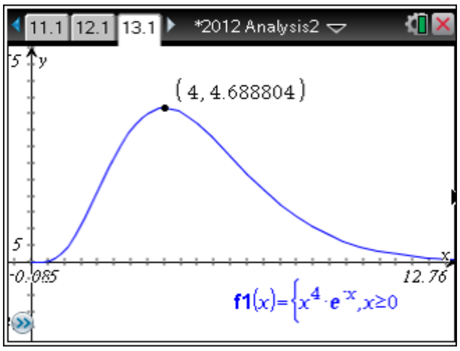
Minimum distance = $\sqrt{5^2+2^2}-1=\sqrt{29}-1$ 1A

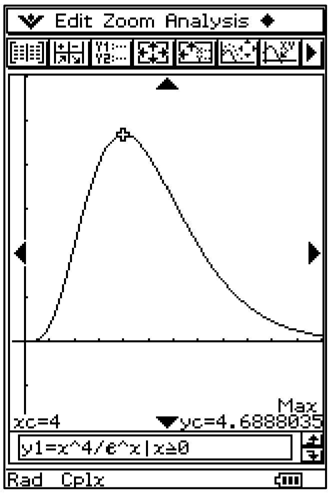
Question 3

a.

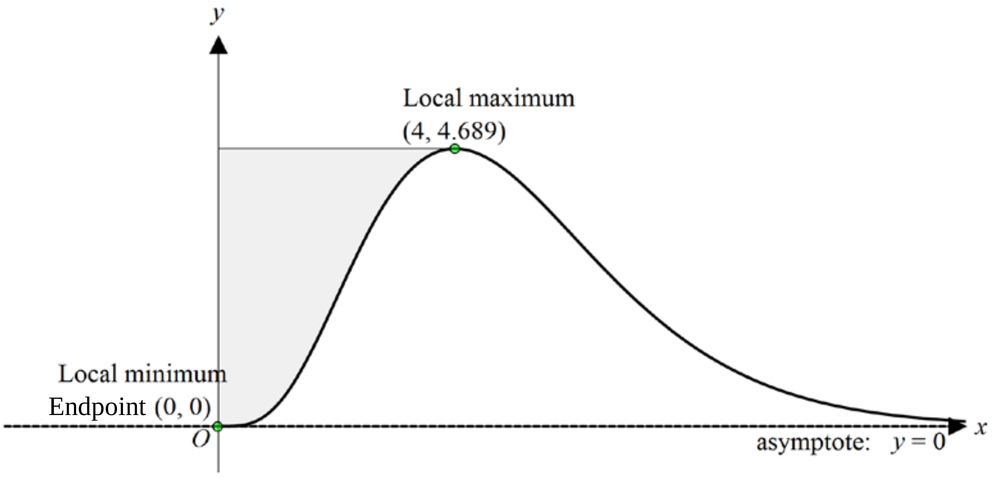
Solve for *x*, $\frac{d\left(f_4(x)\right)}{dx}=0$ or use a graphical approach, or the like. 1M

Maximum value: a = 4.689 (three decimal places) 1A





b.



Correct shape 1A
Correct asymptote 1A
Turning point and endpoint correctly labelled with their coordinates 1A

c.i. and ii.

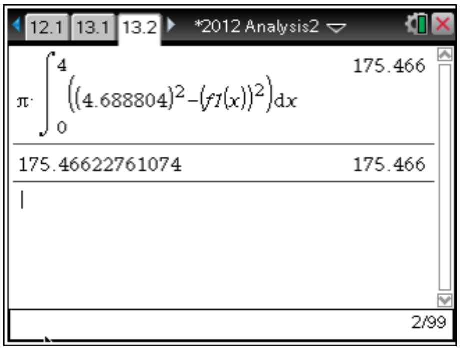
$$V=\pi\int_{x=0}^{4}\left(\left(4.6888...\right)^2-\left(x^4e^{-x}\right)^2\right)dx$$

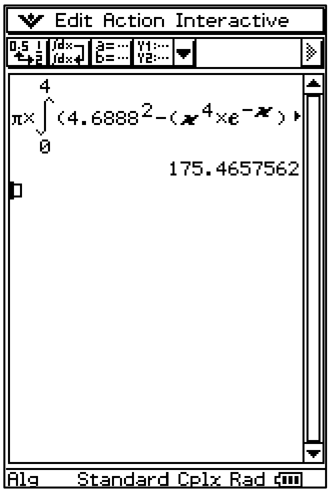
Correct region shaded: **1 A**

Correct integral with correct values 1A

iii.

$V = 175.466\,\text{cm}^3$





1A

d.
i.

Solve for x, $f_k''(x)=0$. 1M

$x=k\pm\sqrt{k}$

$p=k-\sqrt{k}$ and $q=k+\sqrt{k}$ 1A

Alternatively,

$f_k''(x)=x^{2k-2}\left(x^2-2kx+\left(k^2-k\right)\right)e^{-x}=0$

Null factor law: $x=0$ or $x^2-2kx+\left(k^2-k\right)=0$

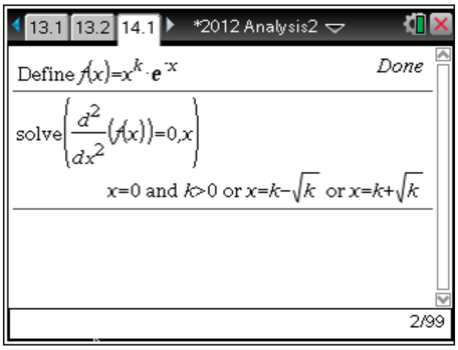
$$x = \frac{2k \pm \sqrt{4k^2 - 4\left(k^2 - k\right)}}{2} \text{ or } x = 0$$

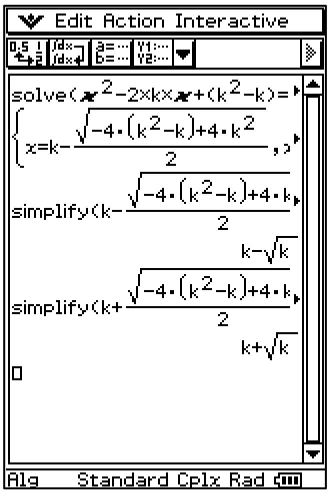
1M

$$x = \frac{2k \pm \sqrt{4k}}{2} \text{ or } x = 0$$

$$x = k + \sqrt{k} \text{ or } x = k - \sqrt{k} \text{ or } x = 0$$

1A





ii.

Midpoint of PQ

$$x_m = \frac{\left(k - \sqrt{k}\right) + \left(k + \sqrt{k}\right)}{2}$$ 1M

$$x_m = \frac{2k}{2} = k$$ 1A

iii.

Maximum value of f_k, $f_k'(x) = 0$

$$f_k'(x) = kx^{k-1}e^{-x} - x^k e^{-x} = 0$$ 1M

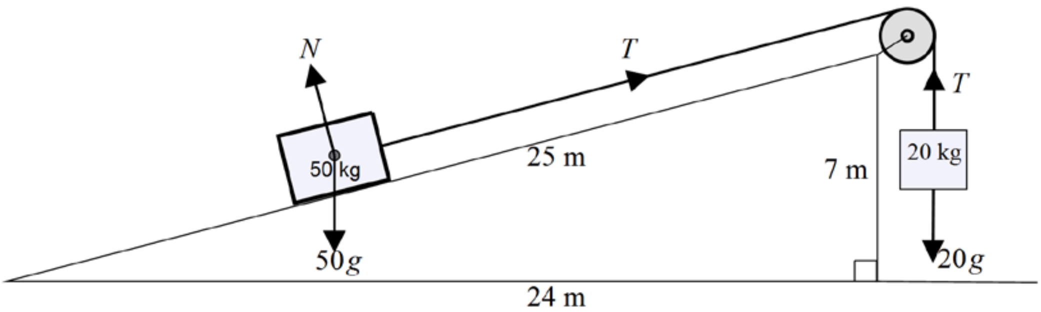
$$x^{k-1}e^{-x}\left(k - x\right) = 0$$

$$x = 0, k$$

For maximum, $x = k = x_m$, as required. 1A

Question 4

a.



$\underset{\sim}{R} = m\underset{\sim}{a}$ for 20 kg mass: $20a = 20g - T$ … eq. (1)

$\underset{\sim}{R} = m\underset{\sim}{a}$ for 50 kg block: $50a = T - 50g \times \frac{7}{25}$, or $50a = T - 14g$ … eq. (2) 1M

Adding equations (1) and (2),

$70a = 6g$

$a = \frac{3g}{35}$, as required. 1M

b.

From equation (1),

$20 \times \frac{3g}{35} = 20g - T$ 1M

$T = 20g - \frac{12g}{7}$

$T = \frac{128g}{7}$ 1A

c.

$s = ut + \frac{1}{2}at^2$

$7 = 0 + \frac{1}{2} \times \frac{3g}{35}t^2$ 1M

$t = \sqrt{\frac{490}{3g}} \approx 4.08$ seconds 1A

d.

i.

The speed when the string first becomes slack:

$v = u + at$

$v = 0 + \frac{3g}{35} \times \sqrt{\frac{490}{3g}}$ 1M

$v \approx 3.43 \text{ m s}^{-1}$ 1A

ii.

The acceleration of the block just after the string becomes slack:

$\underset{\sim}{R} = m\underset{\sim}{a}$

$50a = -50g \times \frac{7}{25}$ 1M

$a = -\frac{7g}{25} \approx -2.74 \text{ m s}^{-2}$ 1A

Question 5

a.

$$\frac{dv}{dt}=\frac{mg-kv}{m}$$

$$t=m\int\left(\frac{1}{mg-kv}\right)dv \qquad \text{1M}$$

$t=-\frac{m}{k}\log_e\left(|mg-kv|\right)+C$, where C is an integration constant. 1M

The absolute value is not required because m and k are both positive constants.

When $t=0$, $v=0$, therefore

$$0=-\frac{m}{k}\log_e(mg)+C$$

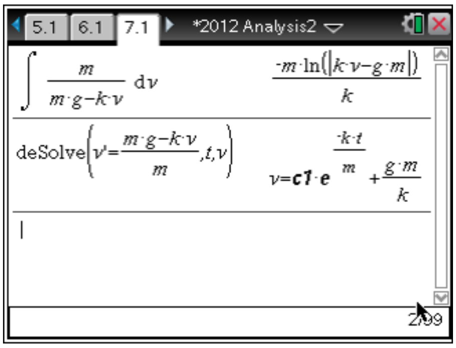
$$C=\frac{m}{k}\log_e(mg) \qquad \text{1M}$$

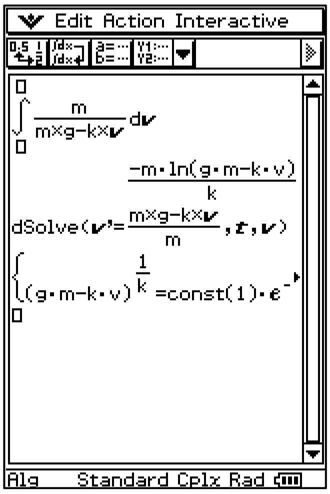
$$t=-\frac{m}{k}\left(\log_e(mg-kv)-\log_e(mg)\right)$$

$$t=-\frac{m}{k}\log_e\left(\frac{mg-kv}{mg}\right)$$

$$-kv=mg\times e^{-\frac{k}{m}t}-mg$$

$v=\frac{mg}{k}\left(1-e^{-\frac{k}{m}t}\right)$, as required. 1M





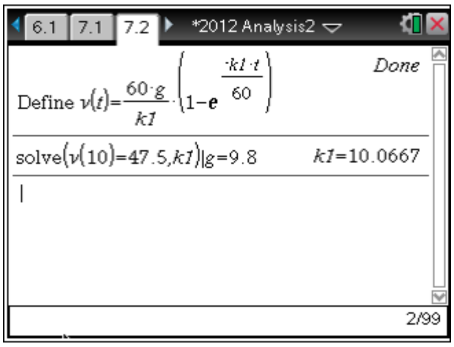
b.

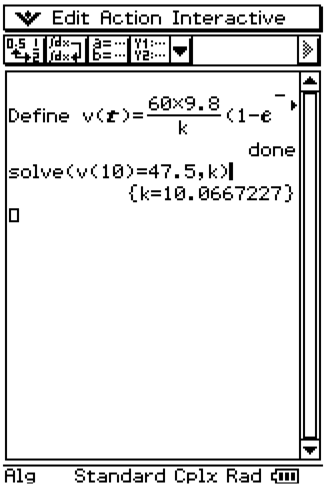
$$v(t)=\frac{60g}{k}\left(1-e^{-\frac{k}{60}t}\right)$$

When $t=10$, $v=47.5$

Solve for k, $v(10)=47.5$ 1M

$k=10$





c.

$$v(t)=6g\left(1-e^{-\frac{t}{6}}\right)$$

As $t\to\infty$, $e^{-\frac{t}{6}}\to 0, v(t)\to 6g$ 1M

Terminal speed = $6g\approx 58.8\,\mathrm{m\,s^{-1}}$

Percentage of terminal speed $=\dfrac{47.5}{58.8}\times 100=81\%$ 1A

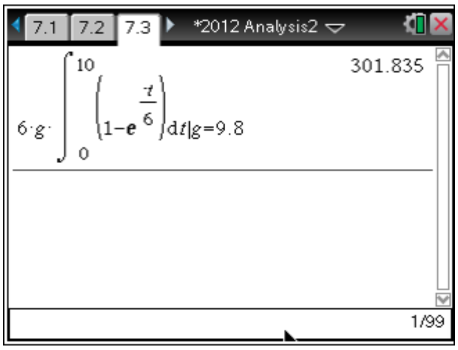
d.

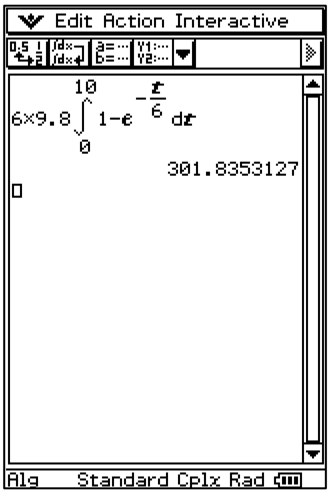
$$\frac{dx}{dt}=6g\left(1-e^{-\frac{t}{6}}\right)$$

$$x=6g\int_0^{10}\left(1-e^{-\frac{t}{6}}\right)dt$$ 1M

$$x=302$$

Distance is 302 m. 1A





e.
i.
After the parachute opens,

$v_t = \frac{mg}{k} = 6$, and $m = 60$

Solve for *k*, $\frac{60g}{k} = 6$ 1M

$k = 10g$, as required

ii.
t seconds after the parachute opens,

$v = Ae^{-\frac{k}{m}t} + \frac{mg}{k}$

$v(t) = Ae^{-\frac{10g}{60}t} + \frac{60g}{10g} = Ae^{-\frac{98}{60}t} + 6$

When the parachute opens, $t = 0$ and $v = 47.5$

$v(0) = 47.5$

$47.5 = Ae^{0} + 6$ 1M

$A = 47.5 - 6 = 41.5$

iii.

$v = Ae^{-\frac{k}{m}t} + \frac{mg}{k}$

$\frac{dx}{dt} = 41.5e^{-\frac{98}{60}t} + 6$

$\frac{dx}{dt} = 41.5e^{-\frac{98}{60}t} + 6$ 1M

After the parachute opens, Yun falls for a further 110 seconds (2 minutes – 10 seconds)

$x = \int_0^{110} \left(41.5e^{-\frac{98}{60}t} + 6\right) dt = 685.408...$ 1M

The distance that Yun falls **after the parachute opens** ≈ 685 m.

Total distance from the balloon to the ground
685.408 … + 301.835 … = 987 m (correct to nearest metre) 1A

END OF SECTION 2 SOLUTIONS

